Sorting networks and staircase Young tableaux

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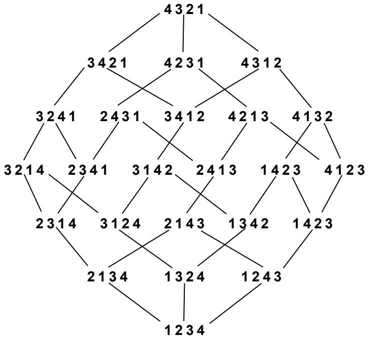
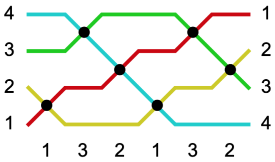
In the *permutohedron* of order n:

- the vertices are the elements of the symmetric group S_n
- edges connect permutations that differ by an adjacent swap
- edges can be *directed* in the direction of increasing inversions





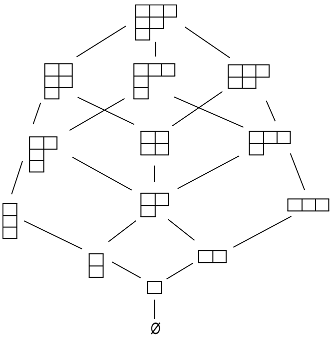
A *sorting network* of order n is equivalently:

- a minimal sequence of adjacent swaps that brings $\mathrm{id}_n = (1, \dots, n)$ to $\mathrm{rev}_n = (n, \dots, 1)$
- a directed walk on the permutohedron of order n from id_n to rev_n

Staircase (standard) Young tableaux

In the *Young graph* of order n:

- the vertices are the subdiagrams of the staircase Young diagram $\delta_n = (n-1, n-2, \ldots, 1)$
- edges connect diagram that differ by one box
- edges can be *directed* in the direction of increasing number of boxes



1	4	5
2	6	
3		

A *staircase Young tableau* of order n is equivalently:

- a staircase Young diagram of shape δ_n filled with the numbers $1, 2, \ldots, \binom{n}{2}$ so that columns and rows are increasing
- a directed walk on the Young lattice of order n from $\varnothing$ to δ_n

Edelman-Green bijection

- SN_n = sorting networks of order n
- SYT_n = staircase Young tableaux of order n

Theorem [Stanley, 1984]

$$|\mathrm{SN}_n| = |\mathrm{SYT}_n|$$

Theorem [Edelman-Greene, 1987]

Explicit combinatorial bijection

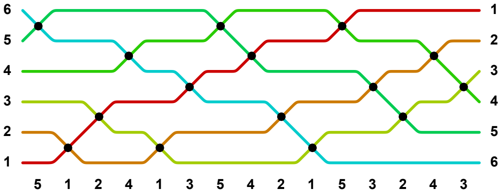
$$\mathrm{SN}_n \longleftrightarrow \mathrm{SYT}_n$$

$t \in \mathrm{SYT}_6$

1	3	4	7	11
2	6	8	14	
5	12	15		
9	13			
10				

$\mapsto$

$s \in \mathrm{SN}_6$



Combinatorial identity

Conjecture [B.-Cunden-Gibbons-Romik, 2020]

$$\sum_{t \in \mathrm{SYT}_n} f_t(x_1,\dots,x_{n-1})\sigma_t = \sum_{s \in \mathrm{SN}_n} g_s(x_1,\dots,x_{n-1})\pi_s$$

- f_t and g_s are rational functions
- σ_t and π_s are permutations associated with tableau t and sorting network s
- to be interpreted as an equality in $\mathbb{C}(x_1,\dots,x_{n-1})\,S_{n-1}$

Theorem [B.-Cunden-Gibbons-Romik, 2020]

The conjecture is true for $n \leq 6$.

Proof: by Mathematica symbolic calculus.

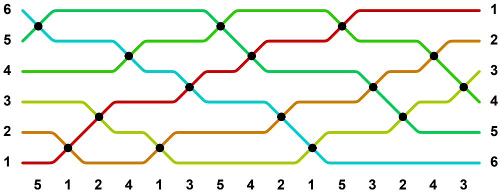
Example

$t \in \mathrm{SYT}_6$

1	3	4	7	11
2	6	8	14	
5	12	15		
9	13			
10				

$\mapsto$

$s \in \mathrm{SN}_6$



$\sigma_t = (1,3,5,4,2) = \pi_s \quad \longrightarrow \quad$ encodes "finishing boxes/swaps"

$$f_t = \frac{1}{(x_1+1)(x_1+2)^2(x_1+3)^3(x_1+4)^4} \cdot \frac{1}{x_2+3} \cdot \frac{1}{(x_3+2)(x_3+3)} \cdot \frac{1}{x_4+2} \cdot \frac{1}{x_5+1}$$

$$g_s = \frac{1}{(x_1+5)(x_1+4)(x_1+3)^5(x_1+2)^3} \cdot \frac{1}{x_2+2} \cdot \frac{1}{(x_3+1)(x_3+2)} \cdot \frac{1}{x_4+1} \cdot \frac{1}{x_5+1}$$

In general: $\qquad \sigma_t \equiv \pi_s \qquad$ BUT $\qquad f_t \not\equiv g_s$.

Probabilistic motivation

Recall we can view

- a sorting network as a directed walk on the permutohedron;
- a Young tableau as a directed walk on the Young graph

Considering now *continuous-time random walks* on these directed graphs, we obtain, respectively:

- a random sorting network: the *oriented swap process*
- a random Young tableau: the *corner growth process*

Theorem [B.-Cunden-Gibbons-Romik, 2020]

The combinatorial identity is equivalent to an equality in distribution between vectors of "finishing times" of the oriented swap process and the corner growth process.

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