Riordan arrays and Lattice paths

Aoife Hennessy, WIT

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The Riordan group

Let $\mathbb{F}_n$ for $n = 0, 1, 2\ldots$ be the set of formal power series defined by

$$\mathbb{F}_n = \{f(x) = f_n x^n + f_{n+1}x^{n+1} + f_{n+2}x^{n+2} + \ldots | f_n \neq 0, f_i \in \mathbb{C}\}$$

A *Riordan array* denoted $(g(x), f(x))$ is an infinite lower triangular matrix where the generating function of the k^{th} column for $k = 0, 1, 2, \ldots$ is $g(x)f^k(x)$, $g(x) \in \mathbb{F}_0$ and $f(x) \in \mathbb{F}_1$. The set of all Riordan arrays forms a group under the following operation of matrix multiplication

$$(g(x), f(x))(h(x), l(x)) = (g(x).h(f(x)), l(f(x))).$$

Example

The Riordan matrix $(\frac{1}{1-x}, \frac{x}{1-x})$ is known as the Binomial matrix B.

$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 1 & & 1 & & & \\ & & 1 & & 2 & & 1 & & \\ & 1 & & 3 & & 3 & & 1 & \\ 1 & & 4 & & 6 & & 4 & & 1 \\ & & & & \vdots & & & & \end{array} \longrightarrow \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 0 & 0 & 0 & \cdots \\ 1 & 2 & 1 & 0 & 0 & \cdots \\ 1 & 3 & 3 & 1 & 0 & \cdots \\ 1 & 4 & 6 & 4 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

A and Z sequences

A Riordan array $R = (g(x), f(x)) = [r_{n,k}]_{n,k \geq 0}$ can be characterized by two sequences $A = (a_0, a_1 \ldots)$ and $Z = (z_0, z_1 \ldots)$ such that

$$r_{n+1,0} = \sum_{j \geq 0} z_j r_{n,j} \qquad r_{n+1,k+1} = \sum_{j \geq 0} a_j r_{n,k+j}$$

for $n, k \geq 0$.
If $A(x)$ and $Z(x)$ are the generating functions for the A and Z sequences respectively, then it follows that

$$Z(x) = \frac{1}{\bar{f}(x)}\left(1 - \frac{1}{g(\bar{f}(x))}\right) \qquad A(x) = \frac{x}{\bar{f}(x)}$$

Example

Let $C(x) = \frac{1-\sqrt{1-4x}}{2x}$, the generating function of the Catalan numbers.
The Riordan array $(C(x), xC(x))$ has first few rows that expand as

$$\begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 0 & 0 & \cdots \\ 2 & 2 & 1 & 0 & \cdots \\ 5 & 5 & 3 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

The corresponding A and Z sequences are $(1, 1, 1 \ldots)$.

Production matrices

A Riordan matrix R, has production matrix P_R that satisfies the equation $RP_R = \hat{R}$, where $\hat{R}$ is the matrix R with its first row removed.

The first column of P_R is generated by $Z(x)$, while the k^{th} column is generated by $x^{k-1}A(x)$.

Example

$$(C(x), xC(x)) = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 0 & 0 & \cdots \\ 2 & 2 & 1 & 0 & \cdots \\ 5 & 5 & 3 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \quad P_R = \begin{pmatrix} 1 & 1 & 0 & 0 & \cdots \\ 1 & 1 & 1 & 0 & \cdots \\ 1 & 1 & 1 & 1 & \cdots \\ 1 & 1 & 1 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Motzkin paths have well established links to orthogonal polynomials. The entries of the tridiagonal production(also known as Stieltjes) matrix represent the weights of the possible steps in Motzkin paths.

Example

$(M(x), M(x))$, $M(x)$ the generating function of paths with two coloured motzkin steps has production matrix

$$P_R = \begin{pmatrix} 2 & 1 & 0 & 0 & \cdots \\ 1 & 2 & 1 & 0 & \cdots \\ 0 & 1 & 2 & 1 & \cdots \\ 0 & 0 & 1 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Lattice paths

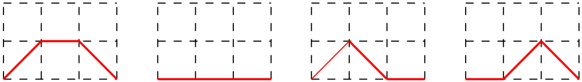
Definition

A Motzkin path $\pi = (\pi(0), \pi(1), \ldots, \pi(n))$, of length n, is a lattice path in the first quadrant starting at $(0,0)$ and ending at $(n,0)$, with possible steps $(1,1)$, $(1,0)$ and $(1,-1)$.

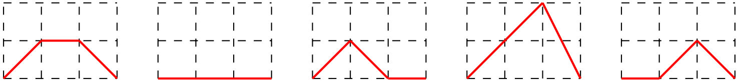
Definition

A Łukasiewicz path $\pi = (\pi(0), \pi(1), \ldots, \pi(n))$, of length n, is a lattice path in the first quadrant starting at $(0,0)$ and ending at $(n,0)$ with elementary steps $(1,1)$, $(1,0)$, $(1,-1)$ and $L = (1,-k)$ for $k \geq 2$.

Example The four Motzkin paths for $n = 3$ are



The five Łukasiewicz paths for $n = 3$ are



Shifted Binomial decomposition

The Riordan array R_M with associated tridiagonal production matrix P_{R_M} can be decomposed as

$$R_M = R_L \cdot B,$$

where B is the 'shifted' Binomial Riordan array with the first column 0^n, the first few entries of which expand as

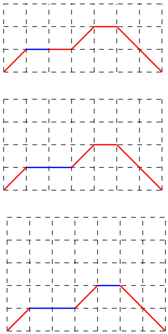
$$B = \left(1, \frac{x}{1-x}\right) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & 0 & 0 & \dots \\ 0 & 1 & 1 & 0 & 0 & \dots \\ 0 & 1 & 2 & 1 & 0 & \dots \\ 0 & 1 & 3 & 3 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \tag{1}$$

$$P_{R_L} = \begin{pmatrix} \beta_0 & 1 & 0 & 0 \dots \\ \alpha_0 & \beta_1 & 1 & 0 \dots \\ \alpha_0 & \beta_1 & \beta_1 & 1 \dots \\ \alpha_0 & \beta_1 & \beta_1 & \beta_1 \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \quad P_{R_M} = \begin{pmatrix} \beta_0 & 1 & 0 & 0 \dots \\ \alpha_0 & \beta_1 + 1 & 1 & 0 \dots \\ 0 & \beta_1 & \beta_1 + 1 & 1 \dots \\ 0 & 0 & \beta_1 & \beta_1 + 1 \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Let $R_L = (C(x), xC(x)), (\alpha_0 = \beta_0 = \beta_1 = 1)$

$$\mathbf{P_{R_M}} = \begin{pmatrix} 1 & 1 & 0\dots \\ 1 & 2 & 1\dots \\ 0 & 1 & 2\dots \\ \vdots & \vdots & \ddots \end{pmatrix} \quad \mathbf{P_{R_L}} = \begin{pmatrix} 1 & 1 & 0\dots \\ 1 & 1 & 1\dots \\ 1 & 1 & 1\dots \\ \vdots & \vdots & \ddots \end{pmatrix}$$

A bijection exists between Łukasiewicz paths with weight 1 for all east steps and south-east steps and Motzkin paths with weight of 2 for all east steps, except east steps on the x-axis and all south-east steps which have weight 1.



Resulting paths are Łukasiewicz paths where east steps and Łukasiewicz step have 1 possible colour.

Let $R_L = (C(x), xC(x)), (\alpha_0 = \beta_0 = \beta_1 = 1)$

$$\mathbf{P_{R_M}} = \begin{pmatrix} 1 & 1 & 0\dots \\ 1 & 2 & 1\dots \\ 0 & 1 & 2\dots \\ \vdots & \vdots & \ddots \end{pmatrix} \quad \mathbf{P_{R_L}} = \begin{pmatrix} 1 & 1 & 0\dots \\ 1 & 1 & 1\dots \\ 1 & 1 & 1\dots \\ \vdots & \vdots & \ddots \end{pmatrix}$$

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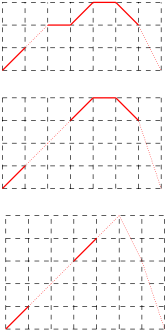

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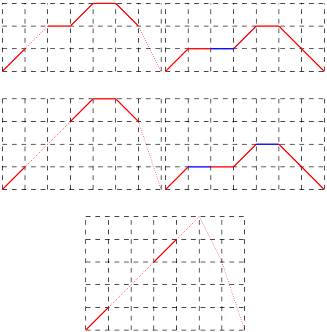

Resulting paths are Łukasiewicz paths where east steps and Łukasiewicz step have 1 possible colour.

Let $R_L = (C(x), xC(x)), (\alpha_0 = \beta_0 = \beta_1 = 1)$

$$\mathbf{P}_{\mathbf{R_M}} = \begin{pmatrix} 1 & 1 & 0\dots \\ 1 & 2 & 1\dots \\ 0 & 1 & 2\dots \\ \vdots & \vdots & \ddots \end{pmatrix} \quad \mathbf{P}_{\mathbf{R_L}} = \begin{pmatrix} 1 & 1 & 0\dots \\ 1 & 1 & 1\dots \\ 1 & 1 & 1\dots \\ \vdots & \vdots & \ddots \end{pmatrix}$$

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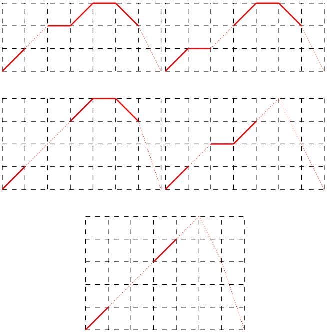

Resulting paths are Łukasiewicz paths where east steps and Łukasiewicz step have 1 possible colour.

Let $R_L = (C(x), xC(x)), (\alpha_0 = \beta_0 = \beta_1 = 1)$

$$\mathbf{P_{R_M}} = \begin{pmatrix} 1 & 1 & 0 \dots \\ 1 & 2 & 1 \dots \\ 0 & 1 & 2 \dots \\ \vdots & \vdots & \ddots \end{pmatrix} \quad \mathbf{P_{R_L}} = \begin{pmatrix} 1 & 1 & 0 \dots \\ 1 & 1 & 1 \dots \\ 1 & 1 & 1 \dots \\ \vdots & \vdots & \ddots \end{pmatrix}$$

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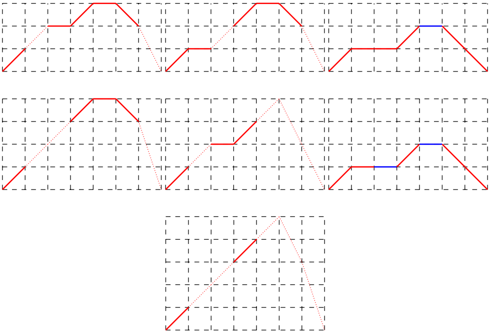

Resulting paths are Łukasiewicz paths where east steps and Łukasiewicz step have 1 possible colour.

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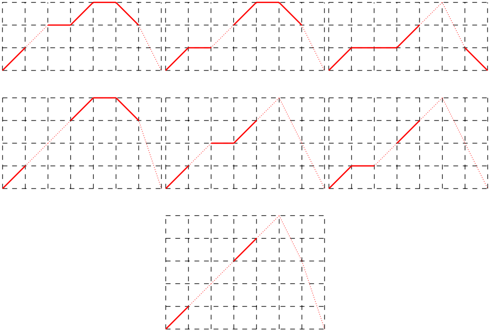

Resulting paths are Łukasiewicz paths where east steps and Łukasiewicz step have 1 possible colour.

Fibonnaci weighted Łukasiewicz paths

The Riordan array R_M with associated tridiagonal production matrix P_{R_M} can be decomposed as $R_M = R_L \cdot S$, where S be the inverse of the shifted negated Catalan numbers, $S = \left(1, \frac{-1+\sqrt{4x+1}}{2}\right)^{-1} = \left(1, x+x^2\right)$ with the first column 0^n.

$$P_{R_L} = \begin{pmatrix} \beta_0 & 1 & 0 & 0\ldots \\ \beta_0 & \beta_1 - 1 & 1 & 0\ldots \\ 2\beta_0 & \beta_1 + 1 & \beta_1 - 1 & 1\ldots \\ 3\beta_0 & 2\beta_1 - 1 & \beta_1 + 1 & \beta_1 - 1\ldots \\ 5\beta_0 & 3\beta_1 + 1 & 2\beta_1 - 1 & \beta_1 + 1\ldots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \quad P_{R_M} = \begin{pmatrix} \beta_0 & 1 & 0 & 0\ldots \\ \alpha_0 & \beta_1 + 1 & 1 & 0\ldots \\ 0 & \beta_1 & \beta_1 + 1 & 1\ldots \\ 0 & 0 & \beta_1 & \beta_1 + 1\ldots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Example

When S is the inverse of the shifted negated Catalan Riordan matrix, associated Łukasiewicz paths have Fibonnacci weighted steps ($\beta_0 = 1, \beta_1 = 1, \alpha_0 = 1$).

$$P_{R_M} = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & \ldots \\ 1 & 2 & 1 & 0 & 0 & \ldots \\ 0 & 1 & 2 & 1 & 0 & \ldots \\ 0 & 0 & 1 & 2 & 1 & \ldots \\ 0 & 0 & 0 & 1 & 2 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, P_{R_L} = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & \ldots \\ 1 & 0 & 1 & 0 & 0 & \ldots \\ 2 & 2 & 0 & 1 & 0 & \ldots \\ 3 & 1 & 2 & 0 & 1 & \ldots \\ 5 & 4 & 1 & 2 & 0 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Natural number weighted Łukasiewicz paths

The Riordan array R_M with associated tridiagonal production matrix P_{R_M} can be decomposed as $R_M = R_L \cdot S$, where S is the inverse of the shifted negated Motzkin Riordan array

$$S = \left(1, \frac{1+x-\sqrt{-3x^2+2x+1}}{2x^2}\right)^{-1} = \left(1, \frac{x+x^2}{1+x^3}\right)$$ with the first column 0^n.

$$P_{R_L} = \begin{pmatrix} \beta_0 & 1 & 0 & 0 \dots \\ \beta_0 & \beta_1 - 1 & 1 & 0 \dots \\ 2\beta_0 & \beta_1 + 1 & \beta_1 - 1 & 1 \dots \\ 3\beta_0 & 2\beta_1 & \beta_1 + 1 & \beta_1 - 1 \dots \\ 4\beta_0 & 3\beta_1 & 2\beta_1 & \beta_1 + 1 \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \quad P_{R_M} = \begin{pmatrix} \beta_0 & 1 & 0 & 0 \dots \\ \alpha_0 & \beta_1 + 1 & 1 & 0 \dots \\ 0 & \beta_1 & \beta_1 + 1 & 1 \dots \\ 0 & 0 & \beta_1 & \beta_1 + 1 \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Example

When S is the inverse of the shifted negated Motzkin Riordan matrix, associated Łukasiewicz paths have steps weighted with the natural numbers ($\beta_0 = 1, \beta_1 = 1, \alpha_0 = 1$).

$$P_{R_M} = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & \dots \\ 1 & 2 & 1 & 0 & 0 & \dots \\ 0 & 1 & 2 & 1 & 0 & \dots \\ 0 & 0 & 1 & 2 & 1 & \dots \\ 0 & 0 & 0 & 1 & 2 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, P_{R_L} = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & \dots \\ 1 & 0 & 1 & 0 & 0 & \dots \\ 2 & 2 & 0 & 1 & 0 & \dots \\ 3 & 2 & 2 & 0 & 1 & \dots \\ 4 & 3 & 2 & 2 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$