Riordan arrays and Lattice paths

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The Riordan group

Let $F_n$ for $n = 0, 1, 2 \ldots$ be the set of formal power series defined by

$$F_n = \{ f(x) = f_n x^n + f_{n+1} x^{n+1} + f_{n+2} x^{n+2} + \ldots | f_n \neq 0, f_i \in \mathbb{C} \}$$

A Riordan array denoted $(g(x), f(x))$ is an infinite lower triangular matrix where the generating function of the $k^{th}$ column for $k = 0, 1, 2, \ldots$ is $g(x) f^k(x)$, $g(x) \in F_0$ and $f(x) \in F_1$. The set of all Riordan arrays forms a group under the following operation of matrix multiplication

$$(g(x), f(x))(h(x), l(x)) = (g(x). h(f(x)), l(f(x))).$$

Example

The Riordan matrix $(\frac{1}{1-x}, \frac{x}{1-x})$ is known as the Binomial matrix $B$.

$$\begin{pmatrix} 1 \\ 1 & 1 \\ 1 & 2 & 1 \\ 1 & 3 & 3 & 1 \\ 1 & 4 & 6 & 4 & 1 \\ \vdots \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \ldots \\ 1 & 1 & 0 & 0 & 0 & \ldots \\ 1 & 2 & 1 & 0 & 0 & \ldots \\ 1 & 3 & 3 & 1 & 0 & \ldots \\ 1 & 4 & 6 & 4 & 1 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$
A Riordan array \( R = (g(x), f(x)) = [r_{n,k}]_{n,k \geq 0} \) can be characterized by two sequences \( A = (a_0, a_1 \ldots) \) and \( Z = (z_0, z_1 \ldots) \) such that

\[
\begin{align*}
r_{n+1,0} &= \sum_{j \geq 0} z_j r_{n,j} & r_{n+1,k+1} &= \sum_{j \geq 0} a_j r_{n,k+j}
\end{align*}
\]

for \( n, k \geq 0 \).

If \( A(x) \) and \( Z(x) \) are the generating functions for the \( A \) and \( Z \) sequences respectively, then it follows that

\[
Z(x) = \frac{1}{\bar{f}(x)} \left(1 - \frac{1}{g(\bar{f}(x))}\right) \quad A(x) = \frac{x}{\bar{f}(x)}
\]

**Example**

Let \( C(x) = \frac{1 - \sqrt{1 - 4x}}{2x} \), the generating function of the Catalan numbers.

The Riordan array \((C(x), xC(x))\) has first few rows that expand as

\[
\begin{pmatrix}
1 & 0 & 0 & 0 & \ldots \\
1 & 1 & 0 & 0 & \ldots \\
2 & 2 & 1 & 0 & \ldots \\
5 & 5 & 3 & 1 & \ldots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}
\]

The corresponding \( A \) and \( Z \) sequences are \((1, 1, 1 \ldots)\).
A Riordan matrix $R$, has production matrix $P_R$ that satisfies the equation $RP_R = \hat{R}$, where $\hat{R}$ is the matrix $R$ with its first row removed.

The first column of $P_R$ is generated by $Z(x)$, while the $k^{th}$ column is generated by $x^{k-1}A(x)$.

**Example**

\[
(C(x), xC(x)) = \begin{pmatrix}
1 & 0 & 0 & 0 & \ldots \\
1 & 1 & 0 & 0 & \ldots \\
2 & 2 & 1 & 0 & \ldots \\
5 & 5 & 3 & 1 & \ldots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}
\]

\[
P_R = \begin{pmatrix}
1 & 1 & 0 & 0 & \ldots \\
1 & 1 & 1 & 0 & \ldots \\
1 & 1 & 1 & 1 & \ldots \\
1 & 1 & 1 & 1 & \ldots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}
\]

Motzkin paths have well established links to orthogonal polynomials. The entries of the tridiagonal production (also known as Stieltjes) matrix represent the weights of the possible steps in Motzkin paths.

**Example**

\[(M(x), M(x)), M(x)\] the generating function of paths with two coloured motzkin steps has production matrix

\[
P_R = \begin{pmatrix}
2 & 1 & 0 & 0 & \ldots \\
1 & 2 & 1 & 0 & \ldots \\
0 & 1 & 2 & 1 & \ldots \\
0 & 0 & 1 & 1 & \ldots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}
\]
Lattice paths

**Definition**

A Motzkin path $\pi = (\pi(0), \pi(1), \ldots, \pi(n))$, of length $n$, is a lattice path in the first quadrant starting at $(0, 0)$ and ending at $(n, 0)$, with possible steps $(1, 1)$, $(1, 0)$ and $(1, -1)$.

**Definition**

A Łukasiewicz path $\pi = (\pi(0), \pi(1), \ldots, \pi(n))$, of length $n$, is a lattice path in the first quadrant starting at $(0, 0)$ and ending at $(n, 0)$ with elementary steps $(1, 1)$, $(1, 0)$, $(1, -1)$ and $L = (1, -k)$ for $k \geq 2$.

**Example** The four Motzkin paths for $n = 3$ are

![Motzkin paths for n = 3](image)

The five Łukasiewicz paths for $n = 3$ are

![Łukasiewicz paths for n = 3](image)
Shifted Binomial decomposition

The Riordan array $R_M$ with associated tridiagonal production matrix $P_{RM}$ can be decomposed as

$$R_M = R_L \cdot B,$$

where $B$ is the ‘shifted’ Binomial Riordan array with the first column $0^n$, the first few entries of which expand as

$$B = \left( 1, \frac{x}{1-x} \right) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \ldots \\ 0 & 1 & 0 & 0 & 0 & \ldots \\ 0 & 1 & 1 & 0 & 0 & \ldots \\ 0 & 1 & 2 & 1 & 0 & \ldots \\ 0 & 1 & 3 & 3 & 1 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \quad (1)$$

$$P_{RL} = \begin{pmatrix} \beta_0 & 1 & 0 & 0 & \ldots \\ \alpha_0 & \beta_1 & 1 & 0 & \ldots \\ \alpha_0 & \beta_1 & \beta_1 & 1 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \quad P_{RM} = \begin{pmatrix} \beta_0 & 1 & 0 & 0 & \ldots \\ \alpha_0 & \beta_1 + 1 & 1 & 0 & \ldots \\ 0 & \beta_1 & \beta_1 + 1 & 1 & \ldots \\ 0 & 0 & \beta_1 & \beta_1 + 1 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$
Let $R_L = (C(x), xC(x)), (\alpha_0 = \beta_0 = \beta_1 = 1)$

$$P_{RM} = \begin{pmatrix}
1 & 1 & 0 & \ldots \\
1 & 2 & 1 & \ldots \\
0 & 1 & 2 & \ldots \\
\vdots & \vdots & \vdots & \ddots
\end{pmatrix} \quad P_{RL} = \begin{pmatrix}
1 & 1 & 0 & \ldots \\
1 & 1 & 1 & \ldots \\
1 & 1 & 1 & \ldots \\
\vdots & \vdots & \vdots & \ddots
\end{pmatrix}$$

A bijection exists between Łukasiewicz paths with weight 1 for all east steps and south-east steps and Motzkin paths with weight of 2 for all east steps, except east steps on the x-axis and all south-east steps which have weight 1.

Resulting paths are Łukasiewicz paths where east steps and Łukasiewicz step have 1 possible colour.
Let $R_L = (C(x), xC(x)), (\alpha_0 = \beta_0 = \beta_1 = 1)$

$$
P_{RM} = \begin{pmatrix}
1 & 1 & 0 & \ldots \\
1 & 2 & 1 & \ldots \\
0 & 1 & 2 & \ldots \\
\vdots & \vdots & \ddots & \ddots
\end{pmatrix} \quad P_{RL} = \begin{pmatrix}
1 & 1 & 0 & \ldots \\
1 & 1 & 1 & \ldots \\
1 & 1 & 1 & \ldots \\
\vdots & \vdots & \ddots & \ddots
\end{pmatrix}
$$

A bijection exists between Łukasiewicz paths with weight 1 for all east steps and south-east steps and Motzkin paths with weight of 2 for all east steps, except east steps on the $x$-axis and all south-east steps which have weight 1.

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$$P_{RM} = \begin{pmatrix} 1 & 1 & 0 \ldots \\
1 & 2 & 1 \ldots \\
0 & 1 & 2 \ldots \\
\vdots & \vdots & \vdots 
\end{pmatrix} \quad P_{RL} = \begin{pmatrix} 1 & 1 & 0 \ldots \\
1 & 1 & 1 \ldots \\
1 & 1 & 1 \ldots \\
\vdots & \vdots & \vdots 
\end{pmatrix}$$

A bijection exists between Łukasiewicz paths with weight 1 for all east steps and south-east steps and Motzkin paths with weight of 2 for all east steps, except east steps on the x-axis and all south-east steps which have weight 1.

Resulting paths are Łukasiewicz paths where east steps and Łukasiewicz step have 1 possible colour.
Let $R_L = (C(x), xC(x)), (\alpha_0 = \beta_0 = \beta_1 = 1)$

$$P_{RM} = \begin{pmatrix} 1 & 1 & 0 & \ldots \\ 1 & 2 & 1 & \ldots \\ 0 & 1 & 2 & \ldots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \quad P_{RL} = \begin{pmatrix} 1 & 1 & 0 & \ldots \\ 1 & 1 & 1 & \ldots \\ 1 & 1 & 1 & \ldots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

A bijection exists between Łukasiewicz paths with weight 1 for all east steps and south-east steps and Motzkin paths with weight of 2 for all east steps, except east steps on the x-axis and all south-east steps which have weight 1.

Resulting paths are Łukasiewicz paths where east steps and Łukasiewicz step have 1 possible colour.
Let $R_L = (C(x), xC(x)), (\alpha_0 = \beta_0 = \beta_1 = 1)$

$\begin{pmatrix}
1 & 1 & 0 & \ldots \\
1 & 2 & 1 & \ldots \\
0 & 1 & 2 & \ldots \\
\vdots & \vdots & \vdots & \ddots
\end{pmatrix}$

$\begin{pmatrix}
1 & 1 & 0 & \ldots \\
1 & 1 & 1 & \ldots \\
\vdots & \vdots & \vdots & \ddots
\end{pmatrix}$

A bijection exists between Łukasiewicz paths with weight 1 for all east steps and south-east steps and Motzkin paths with weight of 2 for all east steps, except east steps on the $x$-axis and all south-east steps which have weight 1.

Resulting paths are Łukasiewicz paths where east steps and Łukasiewicz step have 1 possible colour.
Let $R_L = (C(x), \alpha x C(x))$, $(\alpha_0 = \beta_0 = \beta_1 = 1)$

$$P_{RM} = \begin{pmatrix} 1 & 1 & 0 \ldots \\ 1 & 2 & 1 \ldots \\ 0 & 1 & 2 \ldots \\ \vdots \; \vdots \; \vdots \end{pmatrix}, \quad P_{RL} = \begin{pmatrix} 1 & 1 & 0 \ldots \\ 1 & 1 & 1 \ldots \\ 1 & 1 & 1 \ldots \\ \vdots \; \vdots \; \vdots \end{pmatrix}$$

A bijection exists between Łukasiewicz paths with weight 1 for all east steps and south-east steps and Motzkin paths with weight of 2 for all east steps, except east steps on the x-axis and all south-east steps which have weight 1.

Resulting paths are Łukasiewicz paths where east steps and Łukasiewicz step have 1 possible colour.
Fibonacci weighted Łukasiewicz paths

The Riordan array $R_M$ with associated tridiagonal production matrix $P_{RM}$ can be decomposed as $R_M = R_L \cdot S$, where $S$ be the inverse of the shifted negated Catalan numbers, $S = \left(1, \frac{-1 + \sqrt{4x+1}}{2}\right)^{-1} = \left(1, x + x^2\right)$ with the first column $0^n$.

$$P_{RL} = \begin{pmatrix}
\beta_0 & 1 & 0 & 0 & 0 & \ldots \\
\beta_0 & \beta_1 - 1 & 1 & 0 & 0 & \ldots \\
2\beta_0 & \beta_1 + 1 & \beta_1 - 1 & 1 & 0 & \ldots \\
3\beta_0 & 2\beta_1 - 1 & \beta_1 + 1 & \beta_1 - 1 & 1 & \ldots \\
5\beta_0 & 3\beta_1 + 1 & 2\beta_1 - 1 & \beta_1 + 1 & \beta_1 - 1 & \ldots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}$$

$$P_{RM} = \begin{pmatrix}
\alpha_0 & \beta_0 & 1 & 0 & 0 & 0 & \ldots \\
0 & \beta_1 + 1 & 1 & 0 & 0 & 0 & \ldots \\
0 & 0 & \beta_1 + 1 & 1 & 0 & 0 & \ldots \\
0 & 0 & 0 & \beta_1 + 1 & 1 & 0 & \ldots \\
0 & 0 & 0 & 0 & \beta_1 + 1 & 1 & \ldots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}$$

**Example**

When $S$ is the inverse of the shifted negated Catalan Riordan matrix, associated Łukasiewicz paths have Fibonacci weighted steps ($\beta_0 = 1, \beta_1 = 1, \alpha_0 = 1$).

$$P_{RM} = \begin{pmatrix}
1 & 1 & 0 & 0 & 0 & \ldots \\
1 & 2 & 1 & 0 & 0 & \ldots \\
0 & 1 & 2 & 1 & 0 & \ldots \\
0 & 0 & 1 & 2 & 1 & \ldots \\
0 & 0 & 0 & 1 & 2 & \ldots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}, P_{RL} = \begin{pmatrix}
1 & 1 & 0 & 0 & 0 & \ldots \\
1 & 0 & 1 & 0 & 0 & \ldots \\
2 & 2 & 0 & 1 & 0 & \ldots \\
3 & 1 & 2 & 0 & 1 & \ldots \\
5 & 4 & 1 & 2 & 0 & \ldots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}$$
Natural number weighted Łukasiewicz paths

The Riordan array $R_M$ with associated tridiagonal production matrix $P_{R_M}$ can be decomposed as $R_M = R_L \cdot S$, where $S$ is the inverse of the shifted negated Motzkin Riordan array

$$S = \left(1, \frac{1+x-\sqrt{-3x^2+2x+1}}{2x^2}\right)^{-1} = \left(1, \frac{x+x^2}{1+x^3}\right)$$

with the first column $0^n$.

$$P_{R_L} = \begin{pmatrix}
\beta_0 & 1 & 0 & 0 & \cdots \\
\beta_0 & \beta_1 - 1 & 1 & 0 & \cdots \\
2\beta_0 & \beta_1 + 1 & \beta_1 - 1 & 1 & \cdots \\
3\beta_0 & 2\beta_1 & \beta_1 + 1 & \beta_1 - 1 & \cdots \\
4\beta_0 & 3\beta_1 & 2\beta_1 & \beta_1 + 1 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}$$

$$P_{R_M} = \begin{pmatrix}
\beta_0 & 1 & 0 & 0 & \cdots \\
\alpha_0 & \beta_1 + 1 & 1 & 0 & \cdots \\
0 & \beta_1 & \beta_1 + 1 & 1 & \cdots \\
0 & 0 & \beta_1 & \beta_1 + 1 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}$$

Example

When $S$ is the inverse of the shifted negated Motzkin Riordan matrix, associated Łukasiewicz paths have steps weighted with the natural numbers ($\beta_0 = 1$, $\beta_1 = 1$, $\alpha_0 = 1$).

$$P_{R_M} = \begin{pmatrix}
1 & 1 & 0 & 0 & 0 & \cdots \\
1 & 2 & 1 & 0 & 0 & \cdots \\
0 & 1 & 2 & 1 & 0 & \cdots \\
0 & 0 & 1 & 2 & 1 & \cdots \\
0 & 0 & 0 & 1 & 2 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}$$

$$P_{R_L} = \begin{pmatrix}
1 & 1 & 0 & 0 & 0 & \cdots \\
1 & 0 & 1 & 0 & 0 & \cdots \\
2 & 2 & 0 & 1 & 0 & \cdots \\
3 & 2 & 2 & 0 & 1 & \cdots \\
4 & 3 & 2 & 2 & 0 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}$$

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