Odd diagrams of permutations

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(joint with Francesco Brenti and Bridget Tenner)

Workshop on Enumerative Combinatorics 2021
University College Dublin

Permutations and odd inversions

Let $\sigma \in S_n$.

The inversion number of σ is $\operatorname{inv}(\sigma) = |\{(i,j) \in [n] : i < j,\ \sigma(i) > \sigma(j)\}|$.

$$\operatorname{inv}(2413) = |\{(1,3),(2,3),(2,4)\}| = 3$$

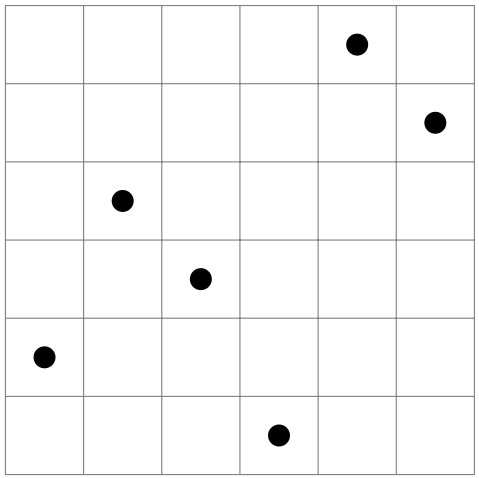
The **odd** inversion number of σ is $\operatorname{oddinv}(\sigma) = |\{(i,j) \in [n] : i < j,\ i-j \equiv 1,\ \sigma(i) > \sigma(j)\}|$.

$$\operatorname{oddinv}(2413) = |\{(2,3)\}| = 1$$

Odd length...

- was introduced in the context of zeta functions in algebra (Klopsch-Voll '09)
- has interesting applications to the enumeration of matrices over finite fields
- has been generalised to all Weyl groups (Brenti-C. '19)

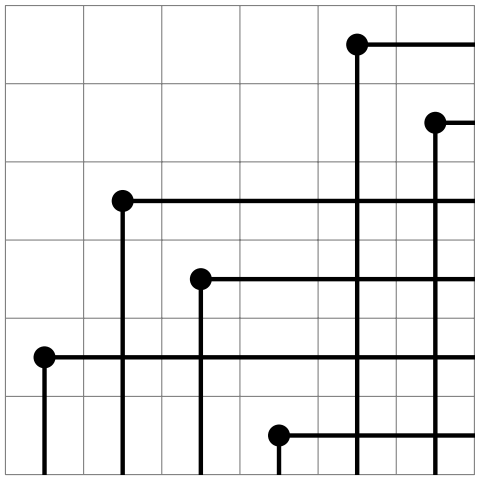
Odd diagrams



$\sigma = 562314$

$\bullet$ = graph of σ

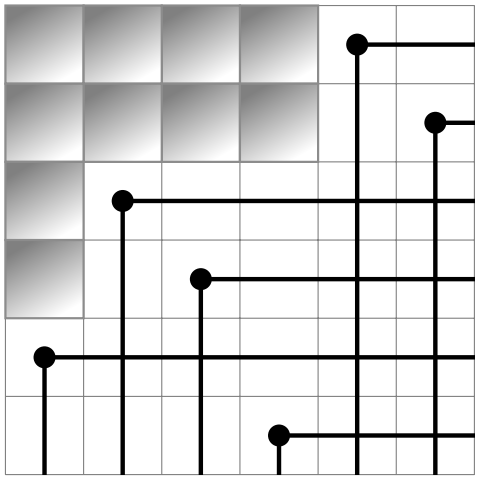
Odd diagrams



$\sigma = 562314$

- $=$ graph of σ

Odd diagrams



$\sigma = 562314$

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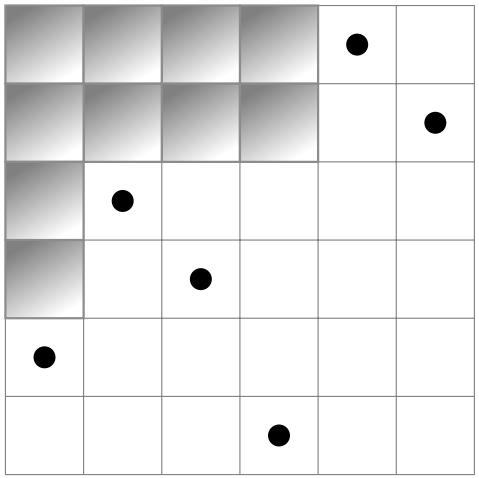
Odd diagrams

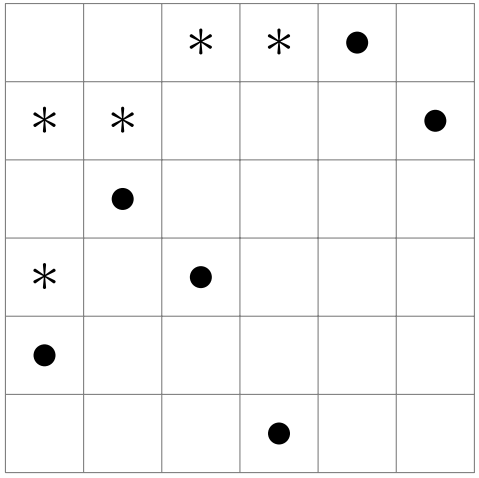


$\sigma = 562314$

● = graph of σ

▩ = diagram of σ

Odd diagrams

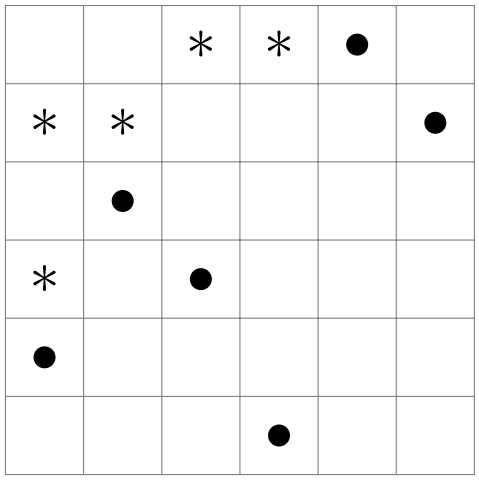


$\sigma = 562314$

$\bullet$ = graph of σ

$*$ = odd diagram of σ

Odd diagrams



$\sigma = 562314$

$\bullet$ = graph of σ

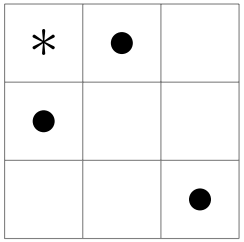
$*$ = odd diagram of σ

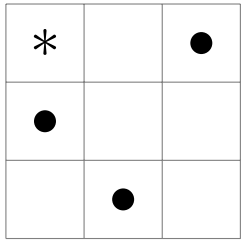
$$\text{oddinv}(\sigma) = |\text{odd diagram of } \sigma|$$

Note: the diagram of a permutation "knows everything" about the permutation...

Odd diagrams

…how much does an odd diagram know about a permutation? Not so much!



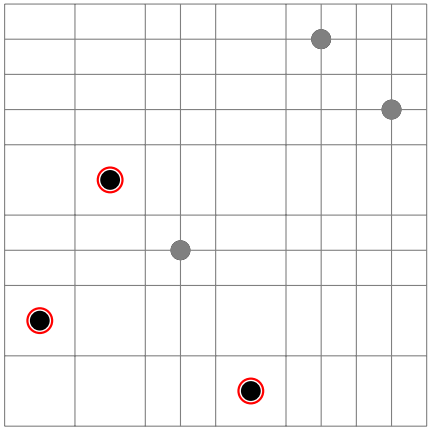


For instance, $213 \in S_3$ and $312 \in S_3$ have the same odd diagram.

Questions:

- How many odd diagrams are there?
- How do "odd diagram classes" look like?

Odd diagrams and permutation patterns



$\sigma = 562314$

σ contains the pattern 213

Theorem (Brenti - C. - Tenner '20)

Every odd diagram class contains at most one permutation avoiding the pattern 213 and at most one avoiding 312.

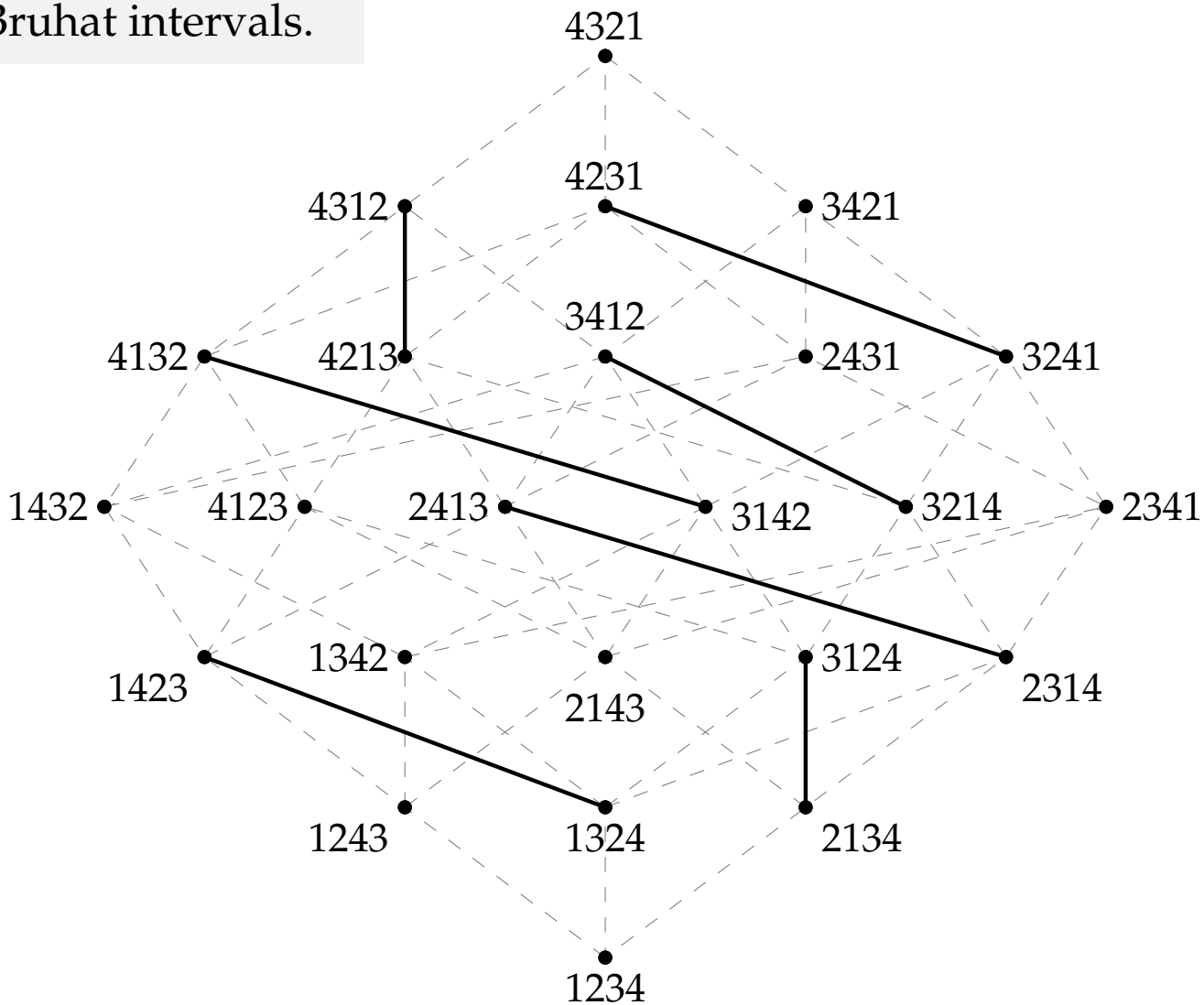
Corollary. There are at least n-th-Catalan-many (and in fact, at least n-th-Bell-many) odd diagrams arising from permutations in S_n.

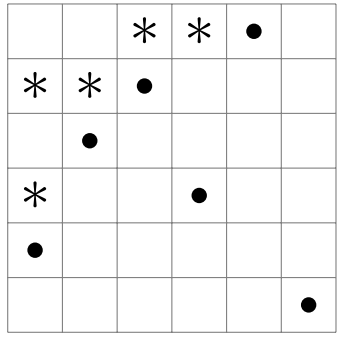
The first values of the sequence $|\{\text{odd diagram of } \sigma : \sigma \in S_n\}|$ are: 1, 2, 5, 17, 70, 351, 2041, 13732, 103873, 882213.

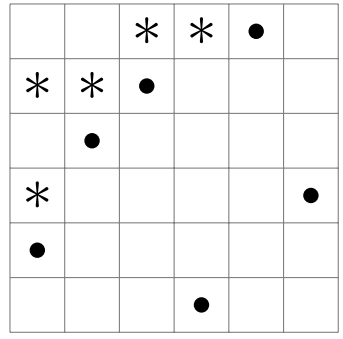
Odd diagrams and Bruhat order

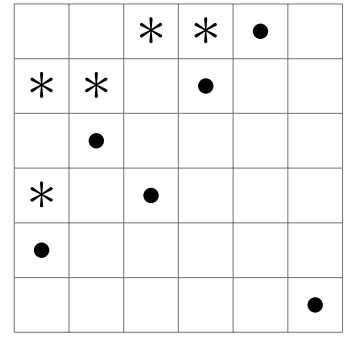
Theorem (Brenti - C. - Tenner ′20)

Odd diagram classes are Bruhat intervals.









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