A MAXDROP STATISTIC FOR STANDARD YOUNG TABLEAUX

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Abstract. In this paper we introduce a new statistic on standard Young tableaux that is closely related to the *maxdrop* permutation statistic that was introduced by the first author. We prove that the value of the statistic must be attained at one of the corners of the standard Young tableau. We determine the coefficients of the generating function of this statistic over two-row standard Young tableaux having n cells. We prove several results for this new statistic that include unimodality of the coefficients for the two-row case.

1. Introduction

The study of permutation statistics has received a lot of attention in the last few decades. Two popular classical permutation statistics are *the number of descents* and *the number of excedances* in a permutation, see MacMahon [9]. Let S_n be the set of permutations $\pi = \pi_1\pi_2\cdots\pi_n$ of the set $[n] := \{1, 2, \ldots, n\}$. The descent set of a permutation $\pi \in S_n$ is the set of indices i for which $\pi_i > \pi_{i+1}$ and the excedance set of $\pi \in S_n$ is the set of indices i for which $\pi_i > i$. It is well known (see e.g. [4, 8]) that

$$\sum_{n\geq 0} x^n \sum_{\pi\in S_n} q^{\#\text{descents in }\pi} = \sum_{n\geq 0} x^n \sum_{\pi\in S_n} q^{\#\text{excedances in }\pi} = \frac{1-q}{e^{x(q-1)}-q},$$

More recently, Chung, Claesson, Dukes and Graham [1] introduced the statistic $\mathsf{maxdrop}(\pi) = \max_{i\in[n]}(\pi_i - i)$. They showed that the number of permutations $\pi \in S_n$ with $\mathsf{maxdrop}(\pi) = k$ is given by $A_{n,k}$, where $A_{n,0} = 1$, $A_{n,n-1} = (n-1)!$ and $A_{n,k} = k!(k+1)^{n-k} - (k-1)!k^{n-k+1}$ for $k = 1, 2, \ldots, n-2$.

A partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ of the integer n is a sequence of positive integers such that $\lambda_1 \geq \cdots \geq \lambda_k$ and $n = \lambda_1 + \ldots + \lambda_k$. We will write $|\lambda|$ for n. The set $\{(i, j) \in \mathbb{N}_0^2 \mid \lambda_i \geq j\}$ is called the Young diagram of shape λ. In this context (i, j) is the cell in row i and column j. A Young tableau of shape λ is obtained by inserting integers $1, 2, \ldots, n = |\lambda|$ into the n cells of λ without repetitions. A standard Young tableau (SYT) of shape λ is a Young tableau of shape λ whose entries strictly increase along rows and columns.

It is well known that the number of SYT having shape $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ is given by hook length formula (Frame, Robinson and Thrall [5]):

$$f^\lambda = \frac{|\lambda|!}{\prod_{c\in\lambda} h_c},$$

where h_c is the *hook length* of the cell c at position (i, j). The hook length of c is the number of cells to the right of c in row i, plus the number of cells beneath c in column j, plus 1.

Several statistics have been studied on SYT. For example, a descent in a SYT T is an entry i such that $i + 1$ is strictly beneath (and hence weakly west) of i. Let $\mathsf{Des}(T)$ be the set of all descents of the tableau T. Then the number of descents of T is $|\mathsf{Des}(T)|$ and the major index of T is $\sum_{i\in\mathsf{Des}(T)} i$ (for a discussion of some other statistics see [7, 10, 11]).

In this paper we will introduce and study a new statistic, *the maximal drop*, for standard Young tableaux. This statistic bears a resemblance to the maxdrop statistic for permutations and is further motivated by similar statistics for other combinatorial structures [4, 10, 8, 1]. We are able to provide exact expressions for the case of two-row standard Young tableaux, but the three (and more) row cases appear to be far more difficult.

2. A statistic on standard Young tableaux

Let SYT_n be the set of all top-left justified standard Young tableaux on the set $\{1, \ldots, n\}$. Given $T \in \mathsf{SYT}_n$, let T_{ij} refer to the entry in row i and column j so that T_{11} is the top-leftmost entry in T. Let us define the statistic $\mathsf{maxdrop} : \mathsf{SYT}_n \to \mathbb{Z}$ as

$$\mathsf{maxdrop}(T) \;:=\; \max_{i,j}(T_{ij} - (i + j - 1)).$$

This statistic can also be seen as the largest value that appears in the tableau $\phi(T)$ where $\phi(T)$ is the tableau whose (i, j) entry is $T_{ij} - (i + j - 1)$.

Example 1. *Consider the following standard Young tableaux T in SYT_9:*

$$T = \begin{array}{|c|c|c|}\hline 1 & 3 & 8 \\ \hline 2 & 4 \\ \cline{1-2} 5 & 6 \\ \cline{1-2} 7 & 9 \\ \cline{1-2}\end{array} .$$

The transformed tableau $\phi(T)$ is

$$\begin{array}{|c|c|c|}\hline 1 & 3 & 8 \\ \hline 2 & 4 \\ \cline{1-2} 5 & 6 \\ \cline{1-2} 7 & 9 \\ \cline{1-2}\end{array} \quad - \quad \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 2 & 3 \\ \cline{1-2} 3 & 4 \\ \cline{1-2} 4 & 5 \\ \cline{1-2}\end{array} \quad = \quad \begin{array}{|c|c|c|}\hline 0 & 1 & 5 \\ \hline 0 & 1 \\ \cline{1-2} 2 & 2 \\ \cline{1-2} 3 & 4 \\ \cline{1-2}\end{array}$$

The maximum of these values is 5, which in this case is uniquely attained at position $(1, 3)$, so $\mathsf{maxdrop}(T) = 5$.

Let us call an entry T_{ij} of a tableau T a *corner entry* if there is no entry to its right or beneath it in T. For the tableau T in Example 1, the entry 9 at position $(4, 2)$ is a corner entry, and so too is the entry $T_{31} = 8$.

Proposition 2. *Given $T \in \mathsf{SYT}_n$, the statistic $\mathsf{maxdrop}$ is realized at one of the corner entries of T.*

Proof. Since T is a standard Young tableau, we have that $T_{ij} < T_{i+1\,j}$ and $T_{ij} < T_{i\,j+1}$ for all those i, j for which these values are defined. Let $T' = \phi(T)$. Then $T'_{ij} = T_{ij} - (i + j - 1) < T_{i+1\,j} - (i + j - 1) = T'_{i+1\,j} + 1$, which is equivalent to $T'_{ij} \leq T'_{i+1\,j}$. Similarly, we have $T'_{ij} = T_{ij} - (i + j - 1) < T_{i\,j+1} - (i + j - 1) = T'_{i\,j+1} + 1$, which is equivalent to $T'_{ij} \leq T'_{i\,j+1}$. In other words, the defining property of standard Young tableaux having entries strictly increasing along rows and down columns in T translates into T' having rows and columns that are *weakly increasing*. Using this observation, the largest values to be found in $T' = \phi(T)$ are those values that are both the rightmost end of rows and at the bottom of columns, which are precisely the *corner* tableau entries. $\square$

Lemma 3. *Given $T \in \mathsf{SYT}_n$, we have $0 \leq \mathsf{maxdrop}(T) \leq n - 2$.*

Proof. Every entry T_{ij} of a standard Young tableau $T \in \mathsf{SYT}_n$ is at least one more than the entry above and the entry to its left (should they exist). Since $T_{11} = 1$ (always), we have that T_{12} and T_{21} are both at least $T_{11} + 1 = 2$. By the same reasoning, we have that

$$\begin{aligned} T_{ij} \;&\geq\; T_{(i-1)\,j} + 1 \;\geq\; \cdots \;\geq\; T_{1\,j} + (i - 1) \\ &\geq\; T_{1\,(j-1)} + 1 + (i - 1) \;\geq\; \cdots \;\geq\; T_{11} + (j - 1) + (i - 1) \;=\; i + j - 1. \end{aligned}$$

n	$F_n(x)$
2	$2(1)$
3	$2(1+x)$
4	$2(1+3x+x^2)$
5	$2(1+2x+9x^2+x^3)$
6	$2(1+2x+17x^2+17x^3+x^4)$
7	$2(1+2x+12x^2+65x^3+35x^4+x^5)$
8	$2(1+2x+12x^2+73x^3+227x^4+66x^5+x^6)$
9	$2(1+2x+12x^2+59x^3+395x^4+707x^5+133x^6+x^7)$
10	$2(1+2x+12x^2+59x^3+395x^4+1923x^5+2102x^6+253x^7+x^8)$
11	$2(1+2x+12x^2+59x^3+353x^4+1987x^5+8833x^6+6093x^7+507x^8+x^9)$
12	$2(1+2x+12x^2+59x^3+353x^4+2041x^5+12106x^6+36958x^7+17570x^8+973x^9+x^{10})$

FIGURE 1. The first few generating functions for $F_n(x)$

Therefore $\mathsf{maxdrop}(T) \geq 0$. We can see that $\mathsf{maxdrop}(T)$ attains the value 0 when T consists of a single row (or column).

As $T'_{ij} = T_{ij} - (i+j-1)$, we may observe that the maximum value that T'_{ij} can attain is certainly bounded by the maximum value that T_{ij} can take, which is n. However, for any standard Young tableau, we must always have $T_{11} = 1$, and so it is not possible for n to occupy the position $(i,j) = (1,1)$ that minimizes $(i+j-1)$. It is therefore not possible to have $T'_{ij} = n$ or $n-1$, and so it must be that $T'_{ij} < n-1$. The entry n may appear in position $(2,1)$ or $(1,2)$ so long as T without n is a row or column tableau, respectively. For both these cases we have $(i+j-1) = 2$. Thus we may have $T'_{ij} = n-2$. □

Let $F_n(x)$ be the generating function of $\mathsf{maxdrop}$ over SYT_n:

$$F_n(x) \ := \sum_{T \in \mathsf{SYT}_n} x^{\mathsf{maxdrop}(T)}.$$

The first few instances of $F_n(x)$ are listed in Figure 1.

3. Two-row tableaux

In this section we will look at the statistic $\mathsf{maxdrop}$ on a restricted class of standard Young tableaux, those that have precisely two rows. This class is 'solvable' in the sense that we can give exact expressions for the number of such tableaux that have a prescribed value of $\mathsf{maxdrop}$, but also illustrate that the expressions which appear are very case specific. This suggests that the expressions for other (larger) classes will also be heavily case dependent and contain no 'unifying' expressions.

Let $\mathsf{SYT}_n^{(k)}$ be the set of $T \in \mathsf{SYT}_n$ that have exactly k rows, and let $\mathsf{SYT}_n^{(\leq k)}$ be the set of $T \in \mathsf{SYT}_n$ that have at most k rows. In this section we will focus our attention on $\mathsf{SYT}_n^{(2)}$. Define the generating function

$$F_n^{(2)}(x) \ := \sum_{T \in \mathsf{SYT}_n^{(2)}} x^{\mathsf{maxdrop}(T)}.$$

The first few instances of $F_n^{(2)}(x)$ are listed in Figure 2. Let us now look at some enumerative results for this class of tableaux. Let $\mathsf{SYT}(a,b)$ be the set of standard Young tableaux having shape $\lambda = (a,b)$ where $a \geq b \geq 1$. The hook-length formula provides the following formula for the size of $\mathsf{SYT}(a,b)$:

Lemma 4. *For all $a \geq b \geq 0$, $|\mathsf{SYT}(a,b)| = \frac{a+1-b}{a+1}\binom{a+b}{a} = \binom{a+b}{b} - \binom{a+b}{b-1}$.*

n	$F_n^{(2)}(x)$
2	1
3	$2x$
4	$4x + x^2$
5	$2x + 6x^2 + x^3$
6	$2x + 11x^2 + 5x^3 + x^4$
7	$2x + 6x^2 + 19x^3 + 6x^4 + x^5$
8	$2x + 6x^2 + 33x^3 + 20x^4 + 7x^5 + x^6$
9	$2x + 6x^2 + 19x^3 + 62x^4 + 27x^5 + 8x^6 + x^7$
10	$2x + 6x^2 + 19x^3 + 104x^4 + 75x^5 + 35x^6 + 9x^7 + x^8$
11	$2x + 6x^2 + 19x^3 + 62x^4 + 207x^5 + 110x^6 + 44x^7 + 10x^8 + x^9$
12	$2x + 6x^2 + 19x^3 + 62x^4 + 339x^5 + 275x^6 + 154x^7 + 54x^8 + 11x^9 + x^{10}$
13	$2x + 6x^2 + 19x^3 + 62x^4 + 207x^5 + 704x^6 + 429x^7 + 208x^8 + 65x^9 + 12x^{10} + x^{11}$
14	$2x + 6x^2 + 19x^3 + 62x^4 + 207x^5 + 1133x^6 + 1001x^7 + 637x^8 + 273x^9 + 77x^{10} + 13x^{11} + x^{12}$

FIGURE 2. The first few generating functions for $F_n^{(2)}(x)$

Let us define

$$G_{a,b}(x)(x) \;:=\; \sum_{T \in \mathsf{SYT}(a,b)} x^{\mathsf{maxdrop}(T)}.$$

Proposition 5. *For all $1 \le b < a$ we have*

$$G_{a,b}(x) \;=\; \left(\binom{a+b-1}{b-1} - \binom{a+b-1}{b-2}\right) x^{a-1} + \frac{2}{b+1}\binom{2b-1}{b-1} x^b + \sum_{i=b}^{a-2} \left(\binom{b+i}{b-1} - \binom{b+i}{b-2}\right) x^i.$$

Note the sum on the right is empty if $a-2 < b$. Furthermore for all $a \ge 1$ we have

$$G_{a,a}(x) = \frac{1}{a+1}\binom{2a}{a} x^{a-1}.$$

Proof. Given $T \in \mathsf{SYT}(a,b)$ with $a > b \ge 1$, the largest entry $a+b$ is either T_{1a} or T_{2b}. Let $T' = \phi(T)$.

- Suppose first that $T_{2b} = a+b$. The inequality
$$T'_{1a} = T_{1a} - (a+1-1) \le a+b-(2+b-1) = T'_{2b}$$
is equivalent to $T_{1a} \le 2a-1$, and this latter inequality holds true since $T_{1a} \le a+b-1 \le 2a-1$. Thus every $T \in \mathsf{SYT}(a,b)$ with $T_{2b} = a+b$ is such that $\mathsf{maxdrop}(T) = a-1$. The number of these tableaux is the number of standard Young tableaux of shape $(a, b-1)$, hence we have a contribution to $G_{a,b}(x)$ of
$$|\mathsf{SYT}(a,b-1)|\, x^{a-1} \;=\; \frac{a-b+2}{a+1}\binom{a+b-1}{b-1} x^{a-1}.$$
- Alternatively, suppose that $T_{1a} = a+b$ and let $d := T_{2b}$. As d is largest in its row and column, it must be greater than all of the $2b-1$ distinct entries $T_{11}, \ldots, T_{1b}, T_{21}, \ldots, T_{2,b-1}$, i.e. it is at least $2b$. Since the largest entry is in the first row, we must have that $d \le a+b-1$, so d must satisfy
$$2b \le d \le a+b-1.$$
Let us split the values d may take into two cases:

  Case $d = 2b$: This ensures that $\mathsf{maxdrop}$ is $T'_{1a} = a+b-a = b \;>\; b-1 = 2b-b-1 = T'_{2b}$, representing a contribution (to $G_{a,b}(x)$) of
$$|\mathsf{SYT}(a,b;d=2b)|\, x^b,$$

where $SYT(a,b;d)$ is the number of $T \in \mathsf{SYT}(a,b)$ with $a > b$ and $T_{2b} = d$.

Case $2b+1 \le d \le a+b-1$**:** In this case $\mathsf{maxdrop}$ will take the value T'_{2b} since $T'_{1a} = a+b-a = b \ \le\ d-b-1 = T'_{2b}$. This case gives a contribution (to $G_{a,b}(x)$) of

$$\sum_{d=2b+1}^{a+b-1} |\mathsf{SYT}(a,b;d)|\, x^{d-b-1}.$$

Combining both of the above cases yields

$$G_{a,b}(x) \;=\; \frac{a-b+2}{a+1}\binom{a+b-1}{b-1}x^{a-1} + |\mathsf{SYT}(a,b;2b)|x^b + \sum_{d=2b+1}^{a+b-1}|\mathsf{SYT}(a,b;d)|x^{d-b-1}.$$

The value $|\mathsf{SYT}(a,b;d)| = \frac{d-2b+2}{d-b+1}\binom{d-1}{b-1}$, as is seen by noticing that one may remove the contiguous entries $(d+1,\ldots,a+b)$ from the end of the first row to have $a-(a+b-(d+1)+1) = d-b$ remaining cells in that first row, and also remove d from the end of the second row to have $b-1$ cells in that second row. This reduced structure must be a SYT on the elements $\{1,\ldots,d-1\}$, and the number of these is given by Lemma 3. Thus

$$\begin{aligned}
G_{a,b}(x) \;&=\; \frac{a-b+2}{a+1}\binom{a+b-1}{b-1}x^{a-1} + |\mathsf{SYT}(a,b;2b)|x^b + \sum_{d=2b+1}^{a+b-1}|\mathsf{SYT}(a,b;d)|x^{d-b-1}\\
&= \frac{a-b+2}{a+1}\binom{a+b-1}{b-1}x^{a-1} + \frac{2}{b+1}\binom{2b-1}{b-1}x^b + \sum_{d=2b+1}^{a+b-1}\frac{d-2b+2}{d-b+1}\binom{d-1}{b-1}x^{d-b-1}\\
&= \frac{a-b+2}{a+1}\binom{a+b-1}{b-1}x^{a-1} + \frac{2}{b+1}\binom{2b-1}{b-1}x^b + \sum_{i=b}^{a-2}\frac{i+3-b}{i+2}\binom{i+b}{i+1}x^i.
\end{aligned}$$

The quantity $\frac{i+3-b}{i+2}\binom{i+b}{i+1} = \binom{b+i}{b-1} - \binom{b+i}{b-2}$, and the latter expression will prove more useful in reducing summations that we will meet. For the second statement in the proposition, in the event that $a=b$ then one has, by Proposition 2, that $\mathsf{maxdrop}$ is attained at the only corner entry $T_{2a} = 2a$. In this case the statistic of any such rectangular T is $\mathsf{maxdrop}(T) = 2a-(2+a-1) = a-1$. This gives

$$G_{a,a}(x) = \frac{1}{a+1}\binom{2a}{a}x^{a-1}. \quad \square$$

While the expression for $G_{a,b}(x)$ does not lend itself to determining a closed form for $F_n^{(2)}(x)$, we may give a closed form expression for the coefficient of x^k in $F_n^{(2)}(x)$.

Proposition 6. *For all $n \ge 6$, the coefficient $t_{n,k} := [x^k]F_n^{(2)}(x)$ is*

$$t_{n,k} = \begin{cases} \binom{2k}{k} - \binom{2k}{k-3} + [n \text{ even and } k = n/2-1]\cdot\left(\binom{n-1}{k} - \binom{n-1}{k+2}\right) & \text{if } k \le \lfloor \frac{n-1}{2} \rfloor \\ \binom{n-1}{k+1} - \binom{n-1}{k+3} & \text{if } k \ge \lfloor \frac{n-1}{2}\rfloor + 1. \end{cases}$$

where $[P]$ is the Iverson bracket that takes the value 1 if P is true, and is otherwise 0.

Proof. Let us write $G_{a,b}(x) = \sum_{\ell=b}^{a-1} g_{a,b}(\ell)x^\ell$. The coefficients $g_{a,b}(\ell)$ are

$$g_{a,a}(\ell) = \begin{cases} \frac{1}{a+1}\binom{2a}{a} = \binom{2a}{a} - \binom{2a}{a-1} & \text{if } \ell = a-1 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

$$g_{b+1,b}(\ell) = \begin{cases} \binom{2b}{b} - \binom{2b}{b-2} & \text{if } \ell = b \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

$$g_{a,b}(\ell) = \begin{cases} \binom{2b}{b} - \binom{2b}{b-2} & \text{if } \ell = b \\ \binom{b+\ell}{b-1} - \binom{b+\ell}{b-2} & \text{if } \ell \in \{b+1,\ldots,a-1\} \\ 0 & \text{otherwise} \end{cases} \qquad \text{if } a \geq b+2 \tag{3}$$

Suppose that $n \geq 3$ and $1 \leq k \leq n-2$. Then

$$[x^k]F_n^{(2)}(x) \;=\; [x^k]\sum_{b=1}^{\lfloor n/2 \rfloor} G_{n-b,b}(x) \;=\; \sum_{b=1}^{\lfloor n/2 \rfloor} g_{n-b,b}(k). \tag{4}$$

Notice that if $a > b \geq 1$, then $\mathsf{maxdrop}$ may take values in the range $\{b,\ldots,a-1\}$. However, if $a = b \geq 1$, then $\mathsf{maxdrop}$ may only take the value $a-1$. Phrased slightly differently, if $n-b > b \geq 1$, then the value $\mathsf{maxdrop} \in \{b,\ldots,n-b-1\}$. If n is even then we will see that there is an extra term to be included due to the rectangular tableau that cannot occur in the n is odd case. In order to present this, let us condition on the parity of n.

Case n odd: Let $n = 2m+1$ and consider equation 4. Notice that those values of b for which there is a non-zero $g_{n-b,b}(k)$ correspond to $b \leq k \leq n-b-1$. Since $k \leq n-b-1$ is equivalent to $b \leq n-k-1$, and since $b \leq k$, we must consider precisely those b for which $b \leq \min(k, n-k-1)$. Replacing the range of the sum in (4) with this gives

$$t_{2m+1,k} \;=\; \sum_{b=1}^{\min(k,2m-k)} g_{2m+1-b,b}(k).$$

Consider now the value of k relative to $2m+1$. Note that there will be non-zero values of $t_{2m+1,k}$ for $k = 1,\ldots,2m-1$. The quantity $g_{2m+1-b,b}(k)$ will always be evaluated using equation 3 for all $b \leq m-1$. Since b in the above sum must always be $\leq m$, the case $b = m$ is special and dealt with using equation 2 and this happens precisely when $k = m$. Thus:

(i) If $k > m$ then $\min(k, 2m-k) = 2m-k$ and

$$\begin{aligned} t_{2m+1,k} &= \sum_{b=1}^{2m-k} g_{2m+1-b,b}(k) \;=\; \sum_{b=1}^{2m-k} \binom{b+k}{b-1} - \binom{b+k}{b-2} \\ &= \binom{2m+1}{2m-k-1} - \binom{2m+1}{2m-k-2} \;=\; \binom{n-1}{k+1} - \binom{n-1}{k+3}. \end{aligned}$$

(ii) If $k < m$ then $\min(k, 2m-k) = k$ and

$$\begin{aligned} t_{2m+1,k} &= \sum_{b=1}^{k} g_{2m+1-b,b}(k) \;=\; \left(\sum_{b=1}^{k-1} \binom{b+k}{b-1} - \binom{b+k}{b-2}\right) + \binom{2k}{k} - \binom{2k}{k-2} \\ &= \binom{2k}{k} - \binom{2k}{k-3}. \end{aligned}$$

(iii) If $k = m$ then $\min(k, 2m-k) = m$ and

$$\begin{aligned} t_{2m+1,m} = \sum_{b=1}^{m} g_{2m+1-b,b}(k) &= \left(\sum_{b=1}^{m-1} \binom{b+k}{b-1} - \binom{b+k}{b-2}\right) + g_{m+1,m}(m) \\ &= \left(\sum_{b=1}^{m-1} \binom{b+k}{b-1} - \binom{b+k}{b-2}\right) + \binom{2m}{m} - \binom{2m}{m-2} \\ &= \binom{2m}{m} - \binom{2m}{m-3}. \end{aligned}$$

In reducing the binomial sums, we have omitted several repeated steps of using the identity $-\binom{A+1}{B} + \binom{A}{B} = -\binom{A}{B-1}$ and using the hockey-stick identity.

Case n even: Let $n = 2m$. Just as in the previous case we can write the required coefficient as We have

$$\begin{aligned} t_{2m,k} &= [x^k]F^{(2)}_{2m}(x) \\ &= \left(\sum_{b=1}^{m-1} g_{2m-b,b}(k)\right) + g_{m,m}(k). \end{aligned}$$

Notice that we have an extra (potentially non-zero) term due to the case $b = m = 2m - b$. Further notice that those values of $b < m$ for which there is a non-zero coefficient is when $k \in \{b, \ldots, 2m-b-1\}$, i.e. $b \le k$ and $k \le 2m-b-1$, the latter of which is equivalent to $b \le 2m-k-1$. Thus the last sum can be restricted to a sum over all $b \le \min(k, 2m-k-1) \le m-1$ and

$$t_{2m,k} = \left(\sum_{b=1}^{\min(k,2m-k-1)} g_{2m-b,b}(k)\right) + g_{m,m}(k).$$

As before, we condition on the value of k relative to $2m$. There will be non-zero values for all $k = 1, \ldots, 2m-2$.

(iv) If $k \le m-2$ then $t_{2m,k} = \left(\sum_{b=1}^{m-2} g_{2m-b,b}(k)\right) + g_{m,m}(k) = \left(\binom{2k}{k} - \binom{2k}{k-3}\right) + 0$.

(v) If $k = m-1$ then $t_{2m,m-1} = \left(\sum_{b=1}^{m-2} g_{2m-b,b}(k)\right) + g_{m,m}(m-1)$. Set $k = m-1$ in (iv) to find that the term in the parentheses is $\binom{2(m-1)}{m-1} - \binom{2(m-1)}{m-1-3} = \binom{2m-2}{m-1} - \binom{2m-2}{m-4}$. Thus

$$t_{2m,m-1} = \binom{2m-2}{m-1} - \binom{2m-2}{m-4} + \binom{2m}{m} - \binom{2m}{m-1}.$$

Let us remark that since $n = 2m$ and $k = m-1$ for this case, the term above admits the expression:

$$t_{n,k} = \binom{2k}{k} - \binom{2k}{k-3} + \binom{n-1}{k} - \binom{n-1}{k+2}.$$

(vi) If $k \ge m$ then $t_{2m,k} = \left(\sum_{b=1}^{k} g_{2m-b,b}(k)\right) + g_{m,m}(k) = \binom{2m-1}{2m-k-2} - \binom{2m-1}{2m-k-4}$. The latter is equal to

$$\binom{2m-1}{k+1} - \binom{2m-1}{k+3} = \binom{n-1}{k+1} - \binom{n-1}{k+3}.$$

The answers from these two different cases match that stated in the theorem when the appropriate value, $n = 2m+1$ or $n = 2m$, is substituted. $\square$

4. Properties of $\mathsf{maxdrop}$

We collect together some results and conjectures about this new statistic $\mathsf{maxdrop}$. A glance at Figure 1 will reveal a factor 2 in all expressions for $F_n(x)$. This observation is not difficult to prove.

Proposition 7. *For all $n \geq 2$, every coefficient of $F_n(x)$ is an even number.*

Proof. Suppose that $T \in \mathsf{SYT}_n$ is a tableau with $\mathsf{maxdrop}(T) = k$. Then the tableau $T'' = \mathsf{reflect}(T)$, the reflection of T about its main diagonal also has $\mathsf{maxdrop}(T) = k$. Since this reflection operation is an involution on SYT_n, tableau having the same value of $\mathsf{maxdrop}$ appear in pairs, hence every coefficient of $F_n(x)$ is an even number. $\square$

The reflection operation just used was used in Dukes [3] to establish a symmetry result for the descent statistic on standard Young tableaux. We can make us of this fact once again to give the following identity for the bistatistic $(\mathsf{maxdrop}, \mathsf{des})$. Let $H_n(x,y) = \sum_{T \in \mathsf{SYT}_n} x^{\mathsf{maxdrop}(T)} y^{\mathsf{des}(T)}$. Then $H_n(x,y) = y^{n-1} H_n(x, y^{-1})$.

Looking at the sequence of coefficients that appear in both Figures 1 and 2, the coefficients seem to be strictly increasing to a maximum value and then strictly decreasing.

Conjecture 8.

(a) *The sequence of coefficients of $F_n(x)$ is unimodal.*
(b) *The sequence of coefficients of $F_n^{(a)}(x)$ is unimodal.*

Proposition 9. *The sequence of coefficients of $F_n^{(2)}(x)$ is unimodal.*

Proof. Proposition 6 provides exact expressions for each of the coefficients and this allows us to verify unimodality. Instead of performing an exhaustive verification of these inequalities, let us present for the first 'half' of the sequence. Let us suppose that $1 \leq k < \lfloor (n-1)/2 \rfloor$. We wish to show that $t_{n,k} \leq t_{n,k+1}$, which is true iff

$$\binom{2k}{k} - \binom{2k}{k-3} \;\leq\; \binom{2(k+1)}{k+1} - \binom{2(k+1)}{k+1-3}.$$

The binomial coefficient $\binom{2k+2}{k+1} = \binom{2k}{k+1} + 2\binom{2k}{k} + \binom{2k}{k-1}$ and $\binom{2k+2}{k-2} = \binom{2k}{k-2} + 2\binom{2k}{k-3} + \binom{2k}{k-4}$. Substituting these in the above inequality, and rearranging gives

$$\binom{2k}{k-2} + \binom{2k}{k-3} + \binom{2k}{k-4} \leq \binom{2k}{k+1} + \binom{2k}{k} + \binom{2k}{k-1}.$$

For all $k \geq 1$ we have $\binom{2k}{k-2} \leq \binom{2k}{k+1}$, $\binom{2k}{k-3} \leq \binom{2k}{k}$, and $\binom{2k}{k-4} \leq \binom{2k}{k-1}$. Adding these three inequalities completes the proof of this part. $\square$

4.1. Involutions and RSK.

Standard Young tableaux are in 1-1 correspondence with involutions via the RSK correspondence. It is therefore natural to ask whether there is an interpretation of the statistic $\mathsf{maxdrop}$ in terms of a permutation statistic, ustat say, on involutions. For example, it is known that the permutation statistic 'number of fixed points' for an involution is equal to the number of columns having odd length in the pair of SYT to which it corresponds via the RSK correspondence.

Question 10. *What is the permutation statistic* ustat *that corresponds to* $\mathsf{maxdrop}$*? What is the distribution of this statistic* ustat *over S_n?*

Proposition 2 tells us that it is the corner entries of a SYT that contain those extreme values from which $\mathsf{maxdrop}$ is calculated. In the search for a permutation statistic interpretation of $\mathsf{maxdrop}$, identifying and evaluating such entries would seem a natural route to consider. As a first step in this direction, we have the following proposition that gives an expression for the number of corners in a SYT in terms of subsequence-properties of the involution to which it corresponds.

n	$F_n^{(3)}(x)$
3	1
4	$2x + x^2$
5	$11x^2$
6	$11x^2 + 20x^3$
7	$6x^2 + 52x^3 + 34x^4$
8	$6x^2 + 20x^3 + 178x^4 + 49x^5$
9	$6x^2 + 20x^3 + 234x^4 + 359x^5 + 90x^6$
10	$6x^2 + 20x^3 + 136x^4 + 885x^5 + 736x^6 + 153x^7$
11	$6x^2 + 20x^3 + 136x^4 + 753x^5 + 2743x^6 + 1391x^7 + 287x^8$
12	$6x^2 + 20x^3 + 136x^4 + 549x^5 + 4162x^6 + 6336x^7 + 2872x^8 + 506x^9$
13	$6x^2 + 20x^3 + 136x^4 + 549x^5 + 3757x^6 + 14452x^7 + 14598x^8 + 5623x^9 + 978x^{10}$
14	$6x^2 + 20x^3 + 136x^4 + 549x^5 + 2998x^6 + 15926x^7 + 45640x^8 + 31562x^9 + 11584x^{10} + 1781x^{11}$
15	$6x^2 + 20x^3 + 136x^4 + 549x^5 + 2998x^6 + 15816x^7 + 71787x^8 + 115284x^9 + 71176x^{10} + 22856x^{11} + 3509x^{12}$

FIGURE 3. The first few generating functions for $F_n^{(3)}(x)$

Proposition 11. *Let $\pi \in S_n$ be an involution and let (P, P) be the pair of SYT corresponding to π via the RSK correspondence where P has shape λ. Let $I_j(\pi)$ be the maximal number of elements in a union of j increasing subsequences of π. Then the number of corners in P is equal to the number of indices $i \in \{1, \ldots, n\}$ such that*

$$\frac{I_{i+1}(\pi) + I_{i-1}(\pi)}{2} \; < \; I_i(\pi). \tag{5}$$

Equivalently, there is a corner at entry (i, λ_i) iff inequality 5 holds.

Proof. Let $\pi \in S_n$ be an involution and let (P, P) be the pair of SYT corresponding to π via the RSK correspondence. Suppose that P has shape $\lambda = (\lambda_1, \ldots, \lambda_k)$ and define $\lambda_{k+1} = 0$. Let $I_j(\pi)$ be the maximal number of element in a union of j increasing subsequences of π.

Greene's theorem [6] tells us that $I_j(\pi) = \lambda_1 + \ldots + \lambda_j$. In terms of λ's, we have $\lambda_j = I_j(\pi) - I_{j-1}(\pi)$ where $I_0(\pi) := 0$. The corner entries of λ are identified in the following way: (i, λ_i) is a corner in the shape λ if $\lambda_{i+1} < \lambda_i$ (true for all $i \in [1, k]$). As $\lambda_{i+1} = I_{i+1}(\pi) - I_i(\pi)$ and $\lambda_i = I_i(\pi) - I_{i-1}(\pi)$, the previous inequality translates into

$$I_{i+1}(\pi) - I_i(\pi) \; < \; I_i(\pi) - I_{i-1}(\pi)$$

which is equivalent to

$$\frac{I_{i+1}(\pi) + I_{i-1}(\pi)}{2} \; < \; I_i(\pi).$$

□

A natural next step would be to find an answer to the following:

Question 12. *Suppose that a standard Young tableau T corresponds to an involution π via the RSK algorithm. Is it possible to determine the corner entries of T in terms of properties of the involution π ?*

Question 13. *If is possible to determine closed forms for the coefficients of $F_n^{(3)}(x)$? The first few instances of these generating functions are listed in Figure 3.*

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