SEQUENCES OF INTEGER PAIRS RELATED TO THE GAME OF TCHOUKAILLON SOLITAIRE

MARK DUKES

Abstract. In this paper we examine a procedure (due to Colm Fagan) that, on starting with an integer n, results in a pair of equal integers that are no greater than n. We call the resulting value the *strange root* of n and we show how this strange root finding procedure is intimately linked to the game of Tchoukaillon solitaire. We analyse the strange root finding procedure in reverse to be able to give a method for determining when a prescribed value is the strange root of at most two integers.

1. Introduction

In this paper we will present some results relating an algorithmic procedure on integer pairs due to Colm Fagan [5], an actuary with a keen interest in mathematics, to the game of Tchoukaillon solitaire. For every integer n, the procedure results in a pair of equal integers that are no greater than n, and we will call this value the *strange root* of n. We analyse the strange root finding procedure in reverse to be able to give a method for determining when a prescribed value is the strange root of at most two integers.

Fagan's Construction: Choose a positive multiple of four, $4m$ say. It may be written as the product $2 \times 2m$ where m is a positive integer. Record this as the pair $(2, 2m)$. If the current pair is (i, y) and $y > i$, then construct the pair $(i + 1, z)$ where z is the smallest integer such that $(i+1)z > iy$ and $i+1+z$ is even. The outcome of this procedure seems to yield a pair of equal positive integers that we denote $(\mathsf{cf}(m), \mathsf{cf}(m))$.

To illustrate this choose 16 as the multiple of four. We begin with the pair $(2, 8)$ which produces the next pair $(3, 7)$. Applying the rule once again, we construct the pair $(4, 6)$ followed by $(5, 5)$. As both entries in this pair are equal we are done and $\mathsf{cf}(4) = 5$. Let us use $\mathsf{CF}(m)$ to refer to the resulting sequence of pairs in this case, i.e.,

$$\mathsf{CF}(4): \ (2, 8) \to (3, 7) \to (4, 6) \to (5, 5).$$

Fagan's Question: Are there an infinite number of positive integers n for which $\{m \in \mathbb{N} : \mathsf{cf}(m) = n\}$ is a singleton set? The values of n for which this is known to be true are 1, 2, 3, 6, 30, 493080, and 242650650.

m	$\mathsf{cf}(m)$
1	2
2	3
3	4
6	6
30	14
493080	1760
242650650	39046

To analyse this construction and approach the question, in Section 2 we will consider the above procedure on a larger set of integers. This will then allow us, in Section 3, to show a relationship between this procedure on integers and winning configurations of Tchoukaillon solitaire. In Section 4 we will characterise the inverse step of the main construction in order to provide a characterisation of those integers that map to an (almost) unique value, and discuss Fagan's question in light of this characterisation.

2. Strange roots

In this section we will consider two constructions on the natural numbers. First we will define and explain the sequence of integer pairs that arise in the original construction [5] in a slightly more general setting. Following this we will give some examples and prove that the terminating pairs that have been observed are as they claim to be. After this we recast the sequence of pairs of integers by performing a linear transform so that the parity condition of the original problem is absorbed into the recursion. Let $\mathbb{N} = \{1, 2, \dots\}$ be the set of natural numbers.

Definition 2.1. Let $n \in \mathbb{N}$. Let Blist_n be the sequence of pairs produced by the following algorithm: Begin with the pair $(1, 2n-1)$. Given a pair (i, y_i) with $y_i > i$, construct the subsequent pair $(i+1, y_{i+1})$ where y_{i+1} is the smallest integer such that

$$(i+1)y_{i+1} > iy_i \quad \underline{\text{and}} \quad i+1+y_{i+1} \text{ if even.}$$

Let $(\mathsf{sr}(n), y_{\mathsf{sr}(n)})$ be the final pair in this sequence. We will find it convenient to call the value $\mathsf{sr}(n)$ the *strange root* of n.

Example 2.2.

(i) Suppose $n = 2$. We begin with $(1, 2n-1 = 3)$. As $3 > 1$ we let y_2 be the smallest integer greater than $1(3)/2 = 1.5$ such that $2 + y_2$ is even, and this gives $y_2 = 2$ and the pair $(2, 2)$. Since $2 \leq 2$ We are done and $(2, 2) = (\mathsf{sr}(2), y_{\mathsf{sr}(2)})$. Thus Blist_2 is $(1, 3) \to (2, 2)$.

(ii) Suppose $n = 8$. We start with $(1, 2n-1 = 15)$. As $15 > 1$ we let y_2 be the smallest integer greater than $1(15)/2 = 7.5$ such that $2 + y_2$ is even, and this gives $y_2 = 8$. We now have the pair $(2, 8)$ and since $8 > 2$ we let y_3 be the smallest integer greater than $2(8)/3 = 5.33$ such that $3 + y_3$ is even. This is $y_3 = 7$ and we have the pair $(3, 7)$. As $7 > 3$ we let y_4 be the smallest integer greater than $3(7)/4 = 5.25$ such that $4 + y_4$ is even, which is $y_4 = 6$, and we have the pair $(4, 6)$. As $6 > 4$ we let y_5 be the smallest integer greater than $4(6)/5 = 4.8$ such that $5 + y_5$ is even. This is $y_5 = 5$ and the pair $(5, 5)$. Since $5 \leq 5$ this is the final pair and so $(\mathsf{sr}(8), y_{\mathsf{sr}(8)}) = (5, 5)$. Thus Blist_8 is $(1, 15) \to (2, 8) \to (3, 7) \to (4, 6) \to (5, 5)$.

(iii) The Blist sequences for the first few integers are illustrated in Figure 1.

Proposition 2.3. For every $n \in \mathbb{N}$, we have $\mathsf{sr}(n) = y_{\mathsf{sr}(n)}$.

Proof. Consider the sequence of pairs produced by Definition 2.1. If $(\mathsf{sr}(n), y_{\mathsf{sr}(n)})$ is the final entry of Blist_n, then it must be the case that $y_{\mathsf{sr}(n)} \leq \mathsf{sr}(n)$. We will now show the following:

(i) If we have pairs (i, y_i) and $(i+1, y_{i+1})$ as part of this process, then $y_{i+1} \leq y_i - 1$.

(ii) If we have pairs (i, y_i) and $(i+1, y_{i+1})$ as part of this process, then $y_{i+1} \geq i + 1$.

(i) Note that since they are pairs defined by Definition 2.1, both $i + y_i$ and $i + 1 + y_{i+1}$ must be even. Since $(i, y_i) \to (i+1, y_{i+1})$, by assumption it must be the case that $y_i > i$. Because of this, the curve $H_1 := \{(x, y) : xy = i(y_i)\}$ that passes through the point (i, y_i) (which is above the diagonal line $x = y$) is such that the slope of the H_1 is always less than the slope of the line $L := \{(x, y) : x + y = i + y_i\}$ (which also passes through the point (i, y_i)), for all $x \in [i, i+1] \subset \mathbb{R}$ (since $i+1+y_{i+1}$ is even). Thus the point $(i+1, y_i - 1)$ that is on L and whose sum of coordinates is even is above the point $(i+1, iy_i/(i+1))$ that is on H. This necessarily means that $y_{i+1} \leq y_i - 1$.

(ii) Notice that since (i, y_i) is such that $y_i > i$ and $i + y_i$ is even, it is not possible to have $y_i = i+1$ since then $i + y_i$ would be odd. This means $y_i \geq i + 2$. Furthermore, the curve H_1 that passes through the point (i, y_i) also passes through the point $(i+1, z_{i+1})$ where $z_{i+1} := iy_i/(i+1)$. Since $y_i \geq i + 2$ and $z_{i+1} := iy_i/(i+1)$ we have $z_{i+1} \geq i(i+2)/(i+1) = i + (i/(i+1))$. The value $y_{i+1} \geq \lceil i + (i/(i+1)) \rceil = i + 1$.

n	Blist_n
1	$(1,1)$
2	$(1,3) \to (2,2)$
3	$(1,5) \to (2,4) \to (3,3)$
4	$(1,7) \to (2,4) \to (3,3)$
5	$(1,9) \to (2,6) \to (3,5) \to (4,4)$
6	$(1,11) \to (2,6) \to (3,5) \to (4,4)$
7	$(1,13) \to (2,8) \to (3,7) \to (4,6) \to (5,5)$
8	$(1,15) \to (2,8) \to (3,7) \to (4,6) \to (5,5)$
9	$(1,17) \to (2,10) \to (3,7) \to (4,6) \to (5,5)$
10	$(1,19) \to (2,10) \to (3,7) \to (4,6) \to (5,5)$
11	$(1,21) \to (2,12) \to (3,9) \to (4,8) \to (5,7) \to (6,6)$
12	$(1,23) \to (2,12) \to (3,9) \to (4,8) \to (5,7) \to (6,6)$
13	$(1,25) \to (2,14) \to (3,11) \to (4,10) \to (5,9) \to (6,8) \to (7,7)$
14	$(1,27) \to (2,14) \to (3,11) \to (4,10) \to (5,9) \to (6,8) \to (7,7)$
15	$(1,29) \to (2,16) \to (3,11) \to (4,10) \to (5,9) \to (6,8) \to (7,7)$
16	$(1,31) \to (2,16) \to (3,11) \to (4,10) \to (5,9) \to (6,8) \to (7,7)$
17	$(1,33) \to (2,18) \to (3,13) \to (4,10) \to (5,9) \to (6,8) \to (7,7)$
18	$(1,35) \to (2,18) \to (3,13) \to (4,10) \to (5,9) \to (6,8) \to (7,7)$
19	$(1,37) \to (2,20) \to (3,15) \to (4,12) \to (5,11) \to (6,10) \to (7,9) \to (8,8)$
20	$(1,39) \to (2,20) \to (3,15) \to (4,12) \to (5,11) \to (6,10) \to (7,9) \to (8,8)$

FIGURE 1. The first few Blist sequences as defined in Definition 2.1

The implication of (i) is that the sequence of pairs must terminate since the second value is strictly decreasing. Part (ii), with $i = \mathsf{sr}(n) - 1$ gives us the inequality $y_{\mathsf{sr}(n)} \geq \mathsf{sr}(n)$. Since $y_{\mathsf{sr}(n)} \leq \mathsf{sr}(n)$ we must have that there is a final pair, and this final pair is $(\mathsf{sr}(n), y_{\mathsf{sr}(n)} = \mathsf{sr}(n))$, as claimed. □

The strange roots for the first few integers are given in Figure 2. A first observation is that the sequence seems to be weakly increasing.

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
$\mathsf{sr}(n)$	1	2	3	3	4	4	5	5	5	5	6	6	7	7	7	7	7	7	8	8

FIGURE 2.

We will now offer the following alternative definition for strange roots that consists of the simple linear transformation $(x, y) \mapsto (x, (x+y)/2)$. The purpose of the alternative definition is that it does not require the additional 'is even' proposition that is stated in Definition 2.1. We will change the notation for the new pairs so that it is clear that the operation at the heart of it differs to that of the original. In what follows $\langle x, y\rangle$ will correspond to the pair $(x, 2y - x)$ of Definition 2.1, and equivalently $\langle x, (x+y)/2\rangle$ will correspond to the pair (x, y) of Definition 2.1.

Definition 2.4. Let $n \in \mathbb{N}$. Let Alist_n be the sequence of pairs produced by the following algorithm: Begin with the pair $\langle 1, n\rangle$. Given a pair $\langle i, u\rangle$ with $u > i$, construct the subsequent pair $\langle i+1, v\rangle$ where v is the smallest integer such that $(i+1)v > i(u+1)$. Equivalently, v is the unique integer such that

$$v > \frac{i(u+1)}{i+1} \geq v - 1.$$

Proposition 2.5. Let $n \in \mathbb{N}$. The final pair of the sequence Alist_n is $\langle \mathsf{sr}(n), \mathsf{sr}(n)\rangle$

n	Alist_n
1	$\langle 1,1\rangle$
2	$\langle 1,2\rangle \to \langle 2,2\rangle$
3	$\langle 1,3\rangle \to \langle 2,3\rangle \to \langle 3,3\rangle$
4	$\langle 1,4\rangle \to \langle 2,3\rangle \to \langle 3,3\rangle$
5	$\langle 1,5\rangle \to \langle 2,4\rangle \to \langle 3,4\rangle \to \langle 4,4\rangle$
6	$\langle 1,6\rangle \to \langle 2,4\rangle \to \langle 3,4\rangle \to \langle 4,4\rangle$
7	$\langle 1,7\rangle \to \langle 2,5\rangle \to \langle 3,5\rangle \to \langle 4,5\rangle \to \langle 5,5\rangle$
8	$\langle 1,8\rangle \to \langle 2,5\rangle \to \langle 3,5\rangle \to \langle 4,5\rangle \to \langle 5,5\rangle$
9	$\langle 1,9\rangle \to \langle 2,6\rangle \to \langle 3,5\rangle \to \langle 4,5\rangle \to \langle 5,5\rangle$
10	$\langle 1,10\rangle \to \langle 2,6\rangle \to \langle 3,5\rangle \to \langle 4,5\rangle \to \langle 5,5\rangle$
11	$\langle 1,11\rangle \to \langle 2,7\rangle \to \langle 3,6\rangle \to \langle 4,6\rangle \to \langle 5,6\rangle \to \langle 6,6\rangle$
12	$\langle 1,12\rangle \to \langle 2,7\rangle \to \langle 3,6\rangle \to \langle 4,6\rangle \to \langle 5,6\rangle \to \langle 6,6\rangle$
13	$\langle 1,13\rangle \to \langle 2,8\rangle \to \langle 3,7\rangle \to \langle 4,7\rangle \to \langle 5,7\rangle \to \langle 6,7\rangle \to \langle 7,7\rangle$
14	$\langle 1,14\rangle \to \langle 2,8\rangle \to \langle 3,7\rangle \to \langle 4,7\rangle \to \langle 5,7\rangle \to \langle 6,7\rangle \to \langle 7,7\rangle$
15	$\langle 1,15\rangle \to \langle 2,9\rangle \to \langle 3,7\rangle \to \langle 4,7\rangle \to \langle 5,7\rangle \to \langle 6,7\rangle \to \langle 7,7\rangle$
16	$\langle 1,16\rangle \to \langle 2,9\rangle \to \langle 3,7\rangle \to \langle 4,7\rangle \to \langle 5,7\rangle \to \langle 6,7\rangle \to \langle 7,7\rangle$
17	$\langle 1,17\rangle \to \langle 2,10\rangle \to \langle 3,8\rangle \to \langle 4,7\rangle \to \langle 5,7\rangle \to \langle 6,7\rangle \to \langle 7,7\rangle$
18	$\langle 1,18\rangle \to \langle 2,10\rangle \to \langle 3,8\rangle \to \langle 4,7\rangle \to \langle 5,7\rangle \to \langle 6,7\rangle \to \langle 7,7\rangle$
19	$\langle 1,19\rangle \to \langle 2,11\rangle \to \langle 3,9\rangle \to \langle 4,8\rangle \to \langle 5,8\rangle \to \langle 6,8\rangle \to \langle 7,8\rangle \to \langle 8,8\rangle$
20	$\langle 1,20\rangle \to \langle 2,11\rangle \to \langle 3,9\rangle \to \langle 4,8\rangle \to \langle 5,8\rangle \to \langle 6,8\rangle \to \langle 7,8\rangle \to \langle 8,8\rangle$

FIGURE 3. The first few Alist sequences as defined in Definition 2.4

Proof. Let $n \in \mathbb{N}$. Consider the sequence of pairs Alist_n and Blist_n (produced by Definitions 2.1 and 2.4, respectively). Initially we have $\langle 1, n\rangle = (1, 2n-1)$. Given $\langle i, u\rangle = (i, y_i)$, if $u > i$ then define the next pair $\langle i+1, v\rangle = (i+1, y_{i+1})$ where v is the unique integer such that:

$$2(i+1)v \ > \ 2i(u+1)+1 \ \geq \ 2(i+1)(v-1).$$

By Definition 2.1 this is equivalent to finding the smallest y_{i+1} such that $(i+1)y_{i+1} > iy_i$ and $i+1+y_{i+1}$ is even. This procedure will, by Proposition 2.3, result in the pair of equal integers $(\mathsf{sr}(n), \mathsf{sr}(n)) = \langle \mathsf{sr}(n), \mathsf{sr}(n)\rangle$.

Let us make the following further simple observations on the inequalities: $2(i+1)v > 2i(u+1)+1 \iff 2(i+1)v \geq 2i(u+1)+2 \iff (i+1)v \geq i(u+1)+1 \iff (i+1)v > i(u+1)$. Also, $2i(u+1)+1 \geq 2(i+1)(v-1) \iff 2i(u+1)+2 > 2(i+1)(v-1) \iff i(u+1)+1 > (i+1)(v-1) \iff i(u+1) \geq (i+1)(v-1)$. Thus the above inequality holds true iff v is the smallest integer such that $(i+1)v > i(u+1)$. □

Proposition 2.6. Let $n \in \mathbb{N}$ and define $y_1 = n$. Then the sequence of numbers $(y_1, \ldots, y_r)$ are such that Alist_n is $\langle 1, y_1\rangle \to \langle 2, y_2\rangle \to \cdots \to \langle r, r = y_r\rangle$ iff the sequence of numbers w_i defined by $w_i = y_i - y_{i+1} + 1$ satisfies

$$w_i = \left\lceil \frac{n-1-(w_1+\cdots+w_{i-1})}{i+1} \right\rceil,$$

for all $i = 1, 2, \ldots, r-1$. (Note that the sum of w_j's is zero when $i = 1$.)

Proof. Let $n \in \mathbb{N}$ and set $y_1 := n$. By Definition 2.4, a sequence of numbers $(y_1, \ldots, y_r)$ is such that $\langle 1, y_1\rangle \to \langle 2, y_2\rangle \to \cdots \to \langle r, y_r\rangle$ with $r = \mathsf{sr}(y_1) = y_r$ iff $y_{i+1} > i(y_i+1)/(i+1) \geq y_{i+1} - 1$ for all $i \in \{1, \ldots, r-1\}$ and $y_r = r = \mathsf{sr}(y_1)$.

Let us translate this last statement into one that concerns only the differences between the y_i's. For a general sequence of numbers $(y_1, \ldots, y_r)$, we will consider the sequence $(z_1, \ldots, z_{r-1})$ of differences where $z_i := y_i - y_{i+1}$ for all $1 \leq i < r$.

The value z_i comes from the transition $\langle i, y_i\rangle \to \langle i+1, y_{i+1}\rangle$. Since there is a transition we must have $y_i > i$ and, by Definition 2.4, y_{i+1} is the unique integer such that

$$y_{i+1} > \frac{i(y_i+1)}{i+1} \geq y_{i+1} - 1.$$

Subtracting every term in this inequality from y_i gives

$$y_i - y_{i+1} < y_i - \frac{i(y_i+1)}{i+1} = \frac{y_i+1}{i+1} - 1 \leq y_i - y_{i+1} + 1,$$

which is equivalent to

$$z_i < \frac{y_i+1}{i+1} - 1 \leq z_i + 1.$$

This, in turn, is equivalent to $\left\lceil \frac{y_i+1}{i+1} - 1 \right\rceil = z_i + 1$, i.e., $\left\lceil \frac{y_i+1}{i+1} \right\rceil = z_i + 2$. We can now provide an expression for the z_i values without the y_i values by noticing that the sum of the first $i-1$ z values is $z_1 + \cdots + z_{i-1} = y_1 - y_i = n - y_i$. This gives

$$z_i = \left\lceil \frac{1 + n - (z_1 + z_2 + \cdots + z_{i-1})}{i+1} \right\rceil - 2$$

for all $i = 1, \ldots, r-1$. Note that when $i = 1$ the sum of the z terms in the expression is empty, and is consequently 0.

Substituting $z_i = w_i - 1$ into the above expression, and simplifying, gives:

$$w_1 = \left\lceil \frac{n-1}{2} \right\rceil$$

and for all $i = 2, \ldots, r-1$,

$$w_i = \left\lceil \frac{n - 1 - (w_1 + \cdots + w_{i-1})}{i+1} \right\rceil.$$

□

3. Tchoukaillon solitaire

Let us now introduce the board game Tchoukaillon solitaire and detail some of its properties. The board for this game is a sequence of holes numbered 0,1,2,.... We will assume that hole 1 is to the right of hole 0, and hole 2 is to the right of hole 1, and so on. The game is played as follows: n stones are placed in these holes, but hole 0 is special and does not initially contain any stones.

The aim of the game is to move all the stones in holes 1 and above to hole 0 through some sequence of valid moves. A valid move consists of selecting a hole, i say, that currently contains s_i stones, and then re-distributing these s_i stones by placing one stone into each of the s_i holes $i-1$, $i-2$, ..., $i-s_i$. If $i - s_i < 0$ then we have no holes left in which to place the remaining $s_i - i$ stones, and we immediately *lose* the game. One should therefore never select a hole that currently has more stones than there are holes to its left. The game is won if one can select the holes in such an order that we end up with all stones in hole 0.

Let us write $c = (c_1, c_2, \ldots)$ for a Tchoukaillon configuration whereby c_i is the number of stones in hole i and $n := c_1 + c_2 + \cdots$. It turns out that for every n there is a unique winning Tchoukaillon configuration, Tchouk_n, consisting of n stones. We list these configurations in Figure 4.

We find it important to mention that it is *not* the case that any order of selecting holes in Tchouk_n results in a win.

Example 3.1. Consider $\mathsf{Tchouk}_3 = (1, 2)$. If we select hole 1 first, then on performing our move the single stone is placed into hole 0. Next we select hole 2 that contains two stones, and on performing our move we drop one stone into hole 1 and the other into hole 0. Next we select hole 1 again, and drop the stone from there into hole 0. After this all three stones are in hole 0 and we have won the game.

However, had we selected hole 2 first, then one stone would have been placed into hole 1, and one into hole 0. There are then two stones in hole 1, and there is no way of winning so we will have lost the game.

n	Tchouk_n	Move vector
0	$()$	*none*
1	(1)	(1)
2	$(0,2)$	$(1,1)$
3	$(1,2)$	$(2,1)$
4	$(0,1,3)$	$(2,1,1)$
5	$(1,1,3)$	$(3,1,1)$
6	$(0,0,2,4)$	$(3,1,1,1)$
7	$(1,0,2,4)$	$(4,1,1,1)$
8	$(0,2,2,4)$	$(4,2,1,1)$
9	$(1,2,2,4)$	$(5,2,1,1)$
10	$(0,1,1,3,5)$	$(5,2,1,1,1)$
11	$(1,1,1,3,5)$	$(6,2,1,1,1)$
12	$(0,0,0,2,4,6)$	$(6,2,2,1,1)$
13	$(1,0,0,2,4,6)$	$(7,2,1,1,1,1)$

FIGURE 4. The first few unique winning configurations Tchouk_n of Tchoukaillon solitaire

One way to construct Tchouk_n is by recursion. Given Tchouk_{n-1}, suppose that position i is the leftmost position containing 0 stones. Then Tchouk_n is the configuration that results from Tchouk_{n-1} by adding i stones to hole i, and subsequently removing one stone from each of the holes $1, 2, \ldots, i-1$.

Recently, Jones, Taalman and Tongen [6] gave an explicit method to construct the winning configurations Tchouk_n. The configuration $\mathsf{Tchouk}_n = (c_1, c_2, \ldots)$ whereby

$$\begin{aligned} c_1 &= n \mod 2 \\ c_2 &= n - c_1 \mod 3 \\ c_3 &= n - (c_1 + c_2) \mod 4 \\ &\vdots \\ c_k &= n - (c_1 + c_2 + \cdots + c_{k-1}) \mod (k+1). \end{aligned}$$

Once the sum $c_1 + c_2 + \cdots + c_{k-1} = n$ one stops computing further entries.

For any winning configuration, there is some sequence of moves that will result in a win. As we saw in Example 3.1, it is not the case that any sequence of allowable moves on Tchouk_n will result in a win. The sequence of moves that is required to 'win' can be discovered by playing the game in reverse, and is essentially the same as the recursive rule for constructing Tchoukaillon configurations highlighted above (except, of course, executed in reverse since it starts from the empty board).

There is another interesting aspect to the winning configurations that instead looks at the *number* of times each hole was selected for a valid move during a 'win'. Given $c = (c_1, \ldots, c_k) = \mathsf{Tchouk}_n$, let $m = m(c) = (m_1, \ldots, m_k)$ be the sequence whereby m_i is the number of times that hole i was selected in playing the game. This sequence m is known in the literature as the *move vector*. For example, in Example 3.1 we considered $c = (1,2) = \mathsf{Tchouk}_3$. For that game hole 1 was selected twice (so $m_1 = 2$) and hole 1 was selected once (so $m_2 = 1$). The move vector for this c is $m(c) = (2,1)$.

In their paper [9], Taalman et al. gave an explicit expression for the entries of the move vector of Tchouk_n in terms of n:

Theorem 3.2. ([9, Thm.4]) The move vector for solving Tchouk_n is $m = (m_1, \ldots, m_\ell)$ where

$$\begin{aligned} m_1 &= \left\lceil \frac{n}{2} \right\rceil \\ m_2 &= \left\lceil \frac{n - m_1}{3} \right\rceil \\ m_3 &= \left\lceil \frac{n - (m_1 + m_2)}{4} \right\rceil \\ &\vdots \\ m_\ell &= \left\lceil \frac{n - (m_1 + m_2 + \cdots + m_{\ell-1})}{\ell + 1} \right\rceil . \end{aligned}$$

The link between Tchoukaillon solitaire and strange roots is now seen by comparing the theorem above to the expression in Proposition 2.6. The precise correspondence is given in the following proposition.

Proposition 3.3. Let $n \geq 2$. The sequence $\mathsf{Tchouk}_{n-1} = (b_1, \ldots, b_\ell)$ corresponds uniquely to $\mathsf{Alist}_n : \langle 1, y_1 = n \rangle \to \langle 2, y_2 \rangle \to \ldots \to \langle \mathsf{sr}(n), \mathsf{sr}(n) \rangle$ as follows:

(a) $\ell = \mathsf{sr}(n) - 1$.

(b) For $i = 1, 2, \ldots, \mathsf{sr}(n) - 1$,

$$b_i = 2i + 1 + iy_i - (i+1)y_{i+1}.$$

(c) For $i = 1, 2, \ldots, \mathsf{sr}(n)$,

$$y_i = i + \frac{1}{i}(b_i + b_{i-1} + \cdots + b_{\mathsf{sr}(n)-1}).$$

Proof. With the objects as stated in the proposition, the correspondences are established through the intermediate object of the move sequence $m = (m_1, m_2, \ldots, m_\ell)$ where $m_i := 1 + y_i - y_{i+1}$. The largest value of ℓ for which this is well defined is $\ell = \mathsf{sr}(n) - 1$, hence (a). That m is a valid move vector is verified by showing $0 \leq m_i < i$ for all i, and this is a consequence of Lemma 2.3.

The sum of the entries in a move sequence is the same as the number of stones in the Tchoukaillon game, and so $\sum_i m_i = (1 + y_1 - y_2) + (1 + y_2 - y_3) + \cdots = (\mathsf{sr}(n) - 1) + y_1 - y_{\mathsf{sr}(n)} = \mathsf{sr}(n) - 1 + n - \mathsf{sr}(n) = n - 1$. In other words the sequence Alist_n can be seen to correspond to a move sequence for a Tchoukaillon game with $n - 1$ stones. In order to be able to write the entries of the sequences $(b_1, \ldots, b_{\mathsf{sr}(n)-1})$ and $(y_1, \ldots, y_{\mathsf{sr}(n)})$ in terms of one another, we will make use of some identities.

(b) To describe the b_j's in terms of m_j's, we use the following identify from Taalman et al. [9, Theorem 2]:

$$b_i = im_i - \sum_{j=i+1}^{\mathsf{sr}(n)-1} m_j.$$

Substitute $m_j = 1 + y_j - y_{j+1}$ into this. We have

$$b_i = \begin{cases} i(1 + y_i - y_{i+1}) - \sum_{j=i+1}^{\mathsf{sr}(n)-1} (1 + y_j - y_{j+1}) & \text{if } i \leq \mathsf{sr}(n) - 2 \\ (\mathsf{sr}(n) - 1)(1 + y_{\mathsf{sr}(n)-1} - y_{\mathsf{sr}(n)}) & \text{if } i = \mathsf{sr}(n) - 1. \end{cases}$$

The expression in the top case simplifies to $2_i + 1 - \mathsf{sr}(n) + iy_i - (i+1)y_{i+1} + y_{\mathsf{sr}(n)} = 2i + 1 + iy_i - (i+1)y_{i+1}$. The expression in the bottom case simplifies, by using the fact that $y_{\mathsf{sr}(n)-1} = y_{\mathsf{sr}(n)}$ for $n \geq 2$, to $\mathsf{sr}(n) - 1$. In fact if we use $i = \mathsf{sr}(n) - 1$ in the top case then it reduces to this same expression $\mathsf{sr}(n) - 1$, and so $b_i = 2i + 1 + iy_i - (i+1)y_{i+1}$ for all $i = 1, 2, \ldots, \mathsf{sr}(n) - 1$.

(c) To describe the y_j's in terms of the b_j's. If $\mathsf{Tchouk}_{n-1} = (b_1, b_2, \ldots, b_{\mathsf{sr}(n)-1}) = b$ then the move vector corresponding to b is $m = (m_1, \ldots, m_{\mathsf{sr}(n)-1})$ where

$$m_i \; = \; \frac{1}{i} b_i + \frac{1}{i(i+1)} \sum_{k=i+1}^{\mathsf{sr}(n)-1} b_k, \tag{3.1}$$

for all $1 \leq i \leq \mathsf{sr}(n) - 1$ by using [9, Eqn. (2)]. As the m_i and y_i values are related via $m_i = 1 + y_i - y_{i+1}$ for all $1 \leq i \leq \mathsf{sr}(n) - 1$ and $y_1 = n$, we find that $y_i = n + (i-1) - \sum_{t=1}^{i-1} m_t$ and this holds for all $1 \leq i \leq \mathsf{sr}(n)$. (The ends of this sequence are well defined since $y_1 = n + 0 - 0 = n$ and $y_{\mathsf{sr}(n)} = n + (\mathsf{sr}(n) - 1) - (m_1 + \cdots + m_{\mathsf{sr}(n)-1}) = n + (\mathsf{sr}(n) - 1) - (n - 1) = \mathsf{sr}(n)$.)

Again by using equation 3.1,

$$m_i \; = \; \frac{(i+1)b_i + \sum_{k=i+1}^{\mathsf{sr}(n)-1} b_i}{i(i+1)},$$

we can express the partial sum

$$\sum_{t=1}^{i-1} m_t = (n-1) - \frac{1}{i} \sum_{k=i}^{\mathsf{sr}(n)-1} b_i.$$

Using this in the equation for y_i, and simplifying, we have that

$$y_i \; = \; i + \frac{1}{i} \sum_{k=i}^{\mathsf{sr}(n)-1} b_i,$$

for all $1 \leq i \leq \mathsf{sr}(n)$. Therefore the configuration $b = \mathsf{Tchouk}_{n-1}$ corresponds to $\mathsf{Alist}_n : \langle 1, y_1 \rangle \to \langle 2, y_2 \rangle \to \ldots \to \langle \mathsf{sr}(n), \mathsf{sr}(n) \rangle$, where the y_i's are as stated.

□

Notice that the end of a Tchoukaillon configuration is a fixed point in the following sense:

Lemma 3.4. Suppose $b = (b_1, \ldots, b_\ell) = \mathsf{Tchouk}_n$. Then $b_\ell = \ell$.

Proof. This is straightforward to see by using the recursive construction presented after Example 3.1 Since $\mathsf{Tchouk}_1 = (1)$, we have $b_\ell = 1 = \ell$ and it is true. Suppose it is true for $n = k$ so that $\mathsf{Tchouk}_k = (b_1, \ldots, b_\ell)$ with $\ell = b_\ell$. To construct Tchouk_{k+1} from Tchouk_k we must condition on the appearance of the first (i.e., lowest indexed) 0 in Tchouk_k.

(a) If $b_i = 0$ is the first zero of Tchouk_k and $i < \ell$ then only the entries in holes $\{1, \ldots, i\}$ are changed and the final entry of Tchouk_{k+1} will be the same as the final entry of Tchouk_k, hence $b_\ell = \ell$.

(b) If $b_{\ell+1}$ is the first zero of Tchouk_k, then Tchouk_{k+1} must have $b_{\ell+1} = \ell + 1$ and all entries to the left of this are decreased by one.

In both cases, the claim holds true and the result follows by induction. □

A natural corollary of Proposition 3.3 and Lemma 3.4 is the following, which allows us to interpret questions about the strange root of n in terms of winning Tchoukaillon configurations.

Corollary 3.5. For all $n \geq 2$, $\mathsf{sr}(n) = \text{length}(\mathsf{Tchouk}_{n-1}) = \text{final}(\mathsf{Tchouk}_{n-1})$, where length$(c)$ is the highest index i such that $c_i \neq 0$, and final(c) is the value of that c_i.

The correspondence established in this section allows us to gain some insight into the sr statistic through enumerative results on Tchoukaillon solitaire. The quantity that is most well-known in relation to Tchoukaillon solitaire is a statistic $t(k)$ that is defined as the smallest integer n for which k occurs for the first time in Tchouk_n. For example, if we look at Figure 4, we see that 4 first occurs in Tchouk_6, and so $t(4) = 6$.

Since the end of every winning Tchoukaillon sequence is a value equal to its index (by Lemma 3.4), $t(k)$ may be equivalently defined as the number of $n (\geq 0)$ for which there are no entries in holes $k, k+1, k+2, \ldots$. For example, for $k = 4$, if we look at Figure 4 then there are no stones in holes 4 or higher of the configurations Tchouk_0, Tchouk_1, …, Tchouk_5 and so $t(4) = 6$.

The sequence $t(1), t(2), \ldots$ is listed in the On-Line Encyclopedia of Integer Sequences [7, A002491] and begins $1, 2, 4, 6, 10, 12, 18, 22, 30, 34, 42, 48, \ldots$. It is known to have several curious properties. An extremely good exposition of these properties and further references can be found in the Jones et al. paper [6].

- $t(k)$ can be calculated by starting with k and successively rounding up to next multiple of $k-1$, $k-2$, …, 1. For example, $t(4)$ is calculated by starting with 4, round up to the next multiple of $k-1=3$ which is 6. Round up again to the next multiple of $k-2=2$ which is still 6, and rounding up to the next multiple of $k-3=1$ will not change the value at all. Thus $t(4) = 6$. (Brown [2]).
- It can be generated by a sieving process on the integers. This was described in Erdős & Jabotinsky [4] and David [3], and is very clearly explained in Sloane [8].
- $t(k) = \dfrac{k^2}{\pi} + O(n)$ (this result is due to Broline & Loeb [1] and improves on Erdős & Jabotinsky [4] result $t(k) = \dfrac{k^2}{\pi} + O(n^{4/3})$).

Brown's construction (in the first point above) bears a similarity to the construction that we are considering. It produces pairs of integers according to a rule similar to ours. However, it does not stop in the same manner that Fagan's construction or Definition 2.4 do, and so the notion of a 'root' seems to have been skipped over. In light of the correspondences we have established, we have the following:

Proposition 3.6.
(a) The number of integers n for which $\mathsf{sr}(n) = k$ is $t(k+1) - t(k)$. Equivalently, $t(k) = 1 + |\{n \geq 1 \ : \ \mathsf{sr}(n) < k\}|$.
(b) The number of integers whose strange root is less than k is approximately k^2/π for k large.

Part (b) helps justify the *strange root* terminology we have used as the number of non-negative integers whose *square root* is less than a natural number k is k^2. Although these connections give us some interesting information about the sr function, the known properties of $t(k)$ are not sufficient to aid us any further in considering Fagan's question. In the next section we will present a brief analysis of the Alist sequences with Fagan's question in mind.

4. Determining integers having a prescribed strange root

When we consider the sequence of pairs that arise from these constructions, is it possible to express those pairs that must precede some pair in a given sequence? Moreover, given an integer r, is it possible to determine the set $\{n \in \mathbb{N} \ : \ \mathsf{sr}(n) = r\}$?

Proposition 4.1. Let $n \in \mathbb{N}$ and consider

$$\mathsf{Alist}_n : \ \langle 1, n\rangle \to \cdots \to \langle i, u\rangle \to \langle i+1, v\rangle \to \cdots \to \langle r, r\rangle$$

where $r \geq 2$. Then u must be an integer that satisfies $(i+1)v > i(u+1) \geq (i+1)(v-1)$.

Proof. We may express u in terms of v as follows. As $\langle i+1, v\rangle$ comes from $\langle i, u\rangle$ we must have $u > i$. Since $v(\geq i+1)$ is the unique integer such that $(i+1)v > i(u+1) \geq (i+1)(v-1)$, we may rephrase this as: given $\langle i+1, v\rangle$ with $v \geq i+1$, $\langle i, u\rangle \to \langle i+1, v\rangle$ for all $u(> i)$ that satisfy $(i+1)v > i(u+1) \geq (i+1)(v-1)$. □

Example 4.2. For example, consider $\langle i+1, v\rangle = \langle 3, 5\rangle$. Then the set of pairs $\langle 2, u\rangle$ for which $\langle 2, u\rangle \to \langle 3, 5\rangle$ are those $u(> 2)$ such that $15 > 2(u+1) \geq 12$, i.e., for all $u > 2$ such that $6.5 > u \geq 5$. In other words, for $u = 5$ and $u = 6$.

In analysing the values that u can take, at a second glance it is more restricted than first appears. It transpires that there can be either one or two values of u that map to a given $\langle i+1, v\rangle$.

Proposition 4.3. Let $n \in \mathbb{N}$ and consider $\mathsf{Alist}_n : \langle 1, n\rangle \to \cdots \to \langle i, u\rangle \to \langle i+1, v\rangle \to \cdots \to \langle r, r\rangle$ where $r \geq 2$. Then

$$u \in \begin{cases} \{v-2+\lfloor \frac{v-1}{i} \rfloor,\ v-1+\lfloor \frac{v-1}{i} \rfloor\} & \text{if } i \mid v-1, \\ \{v-1+\lfloor \frac{v-1}{i} \rfloor\} & \text{if } i \not\mid v-1. \end{cases}$$

Proof. Suppose that $\langle i, u\rangle \to \langle i+1, v\rangle$ as stated in the proposition. Then by Proposition 4.1 u must satisfy $(i+1)v > i(u+1) \geq (i+1)(v-1)$. This inequality is equivalent to $v+(v/i)-1 = v-1+(v/i) > u \geq (v-1)+(v-1)/i-1 = v-2+(v-1)/i$, i.e.,

$$v-1+\left\lfloor \frac{v-1}{i} \right\rfloor \geq u \geq v-2+\left\lceil \frac{v-1}{i} \right\rceil.$$

Notice that if $\frac{v-1}{i}$ is an integer x, then this inequality is $v-1+x \geq u \geq v-2+x$, i.e., $u \in \{v-2+x, v-1+x\}$. However, if $\frac{v-1}{i}$ is not an integer but is $x+\epsilon$ for some integer $x \in \mathbb{N}$ and $\epsilon \in (0,1)$, then this inequality is $v-1+x \geq u \geq v-2+x+1 = v-1+x$, i.e., $u = v-1+x$. For example, consider $\langle i+1, v\rangle = \langle 4, 5\rangle$, we have that $v-1 = 4$ and $i = 3$. As $4/3$ is not an integer, the only u for which $\langle i, u\rangle \to \langle i+1, v\rangle$ is $u = v-1+\lfloor (v-1)/x \rfloor = 4+1 = 5$. The only pair $\langle 3, u\rangle$ that will produce $\langle 4, 5\rangle$ is $\langle 3, 5\rangle$. ☐

Proposition 4.3 allows us to give a description of those integers n whose strange root is some prescribed value by working backwards from the value of the root. Let us observe that in Proposition 4.3, when $i = 1$, the value i will always divide $v-1$, and there will be two possible values for u such that $\langle 1, u\rangle \to \langle 2, v\rangle$ for all $v \geq 2$. Thus given a pair $\langle 2, v\rangle$, both $\langle 1, 2v-3\rangle \to \langle 2, v\rangle$ and $\langle 1, 2v-2\rangle \to \langle 2, v\rangle$.

The following proposition provides a characterisation of the r that are the roots of at most two integers.

Proposition 4.4. Suppose that $n \geq 5$. Let $x_r = r$ and for every $i = r-1, \ldots, 1$ define

$$x_i \ := \ x_{i+1} - 1 + \left\lfloor \frac{x_{i+1}-1}{i} \right\rfloor \ = \ \left\lfloor \frac{(i+1)(x_{i+1}-1)}{i} \right\rfloor.$$

Then there are only two integers (x_1 and $x_1 - 1$) that have r as its strange root if and only if $x_{i+1} - 1 \not\equiv 0 \mod i$ for all $i \in \{2, \ldots, r-2\}$.

Example 4.5. Consider $r = 14$. Then we have $x_{14} = 14$, and we apply the rule to derive the second row of the following table:

i	14	13	12	11	10	9	8	7	6	5	4	3	2	1
x_i	14	14	14	14	14	14	14	14	15	16	18	22	31	60
$\frac{x_{i+1}-1}{i}$			$\frac{13}{12}$	$\frac{13}{11}$	$\frac{13}{10}$	$\frac{13}{9}$	$\frac{13}{8}$	$\frac{13}{7}$	$\frac{13}{6}$	$\frac{14}{5}$	$\frac{15}{4}$	$\frac{17}{3}$	$\frac{21}{2}$	

Using the top two rows we can compute the values in the bottom row. None of the quotients in the bottom row are integers hence, by the above proposition, there are only two integers ($x_1 = 60$ and $x_1 - 1 = 59$) that have 14 its strange root.

Proposition 4.4 classifies those r that are the strange root of only two integers. There are precisely $r-3$ (non-)divisibility conditions to be satisfied in order for r to be a unique strange root. Thus as r grows it would appear less and less likely to find an r such that the sequence $(x_r, \ldots, x_1)$ satisfies the stated condition. There is nothing suggesting that there is a maximal such value of r after which no more unique strange roots may exist. Based on the form of the condition in Prop 4.4 we present the following conjecture.

Conjecture 4.6. There is in infinite number of integers $r \in \mathbb{N}$ for which $|\{n \in \mathbb{N} \ : \ \mathsf{sr}(n) = r\}| = 2$.

m	$\mathsf{cf}(m)$	$\mathsf{sr}(n)$	n
1	2	2	2
2	3	3	$3, 4$
3	4	4	$5, 6$
6	6	6	$11, 12$
30	14	14	$59, 60$
493080	1760	1760	$986159, 986160$
242650650	39046	39046	$485301299, 485301300$

FIGURE 5. Note that $\mathsf{cf}(m) = x$ is equivalent to $\mathsf{sr}(2m) = x$. This is easily seen as the first entry of the sequence $\mathsf{CF}(m)$ is the second entry of the sequence Blist_{2m}.

Fagan's Question translates into the question that we have considered, since an integer r is the strange root of only two integers iff $\{m \in \mathbb{N} : \mathsf{cf}(m) = r\}$ is a singleton set. In Figure 5 we record the first few values of both cf and sr to summarise how they are related.

Since the numbers in Figure 5 seem to be growing so fast, it is not easy to get a clearer picture on the next value (if it exists). It would be interesting to see if some of the theory regarding the game of Tchoukaillon solitaire could be utilized to give insights into strange roots that are the strange roots of at most two integers.

Acknowledgments

I am very grateful to Colm Fagan for sharing his interesting problem with me, and to Neil Sloane for providing a reference for the the construction due to K.S. Brown.

References

[1] D.M. Broline and D.E. Loeb. The combinatorics of Mancala-Type games: Ayo, Tchoukaillon and 1/Pi. *J. Undergrad. Math. Applic.* **16** (1995), pp. 21–36.
[2] K.S. Brown. Rounding up to π. https://www.mathpages.com/home/kmath001/kmath001.htm
[3] Y. David. On a sequence generated by a sieving process. Riveon Lematematika **11** (1957), 26–31.
[4] P. Erdős and E. Jabotinsky. On a sequence of integers generated by a sieving process. *Indagationes Math.* **20** (1958), 115–128.
[5] C. Fagan. Personal Communication, June 2020.
[6] B. Jones, L. Taalman; A. Tongen. Solitaire Mancala games and the Chinese remainder theorem. *Amer. Math. Monthly* **120** (2013), no. 8, 706–724.
[7] OEIS Foundation Inc. (2020), The On-Line Encyclopedia of Integer Sequences, http://oeis.org.
[8] N.J.A. Sloane. My favorite integer sequences. *Sequences and their Applications (Proceedings of SETA '98)*, C. Ding, T. Helleseth and H. Niederreiter (editors), Springer-Verlag, London, 1999, pp. 103-130. arXiv:math/0207175
[9] L. Taalman, A. Tongen, B. Warren, F. Wyrick-Flax, I. Yoon. Mancala Matrices. *College Math Journal* **44**, No. 4, September 2013, 273–283.

SCHOOL OF MATHEMATICS AND STATISTICS, UNIVERSITY COLLEGE DUBLIN, DUBLIN 4, IRELAND.
E-mail address: `mark.dukes@ccc.oxon.org`
URL: `https://maths.ucd.ie/~dukes`