A SHORT NOTE ON THE LACKING POLYNOMIAL OF THE COMPLETE BIPARTITE GRAPH

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ABSTRACT. We classify the stochastically recurrent states of the stochastic sandpile model on the complete bipartite graphs $K_{2,n}$ and $K_{m,2}$. We use these characterizations to give formulae for the lacking polynomials on these graphs. Log-concavity of the sequence of coefficients of these two lacking polynomials is proven.

The stochastic sandpile model (SSM) is a variant of the Abelian sandpile model [2] that was introduced by Chan, Marckert and Selig [3] in 2013. In the SSM the classical sandpile toppling rule is replaced with an alternative rule whereby, on toppling an unstable vertex, a grain may (but does not have to) be sent to each neighbouring vertex. One consequence of this is that the toppling of a vertex does not necessarily result in it becoming stable. As part of their paper the authors introduced a notion of stochastically recurrent states as the analog of recurrent states for the classical model. They provided an expression for such states in terms of *orientations compatible with configurations*. Further research into this model can be found in the papers [6, 7].

Let G be a simple graph with a sink s. A *stable configuration* on G is an assignment of non-negative integers to each non-sink vertex such that the number of grains at a given vertex is less than its degree. First, we will recall a definition from Chan et al. [3] which explains what it means for an orientation to be compatible with a configuration.

Definition 1 (Chan et al. [3])**.** Let c be the configuration on G. An orientation $\mathcal{O}$ of G is an assignment of a direction to each of the edges of G. We say that configuration c *is compatible with* $\mathcal{O}$ (and likewise $\mathcal{O}$ *is compatible with* c) if for all non-sink vertices v_i in G,

$$in_{\mathcal{O}}(v_i) \;\geq\; d(v_i) - c_i,$$

where $in_{\mathcal{O}}(v_i)$ is the number of incoming edges to v_i w.r.t. $\mathcal{O}$.

We denote the set of stable configurations on G that are compatible with $\mathcal{O}$ as $comp(\mathcal{O})$. Note that, in comparison to the paper [3], the inequality in Definition 1 is missing one on the right hand side. This is because of a subtle change to the model. In [3] they considered a vertex unstable if the number of grains at a vertex exceeds its degree, whereas we consider a vertex unstable if the number of grains at vertex is not less than the degree which is in line with the definition of the ASM in [4]. The two notions are equivalent.

Theorem 2 (Chan et al. [3])**.** A stable configuration c on G is *stochastically recurrent* if and only if there exists an orientation $\mathcal{O}$ on G such that $c \in comp(\mathcal{O})$. In other words,

$$\mathsf{Sto}(K_{m,n}) = \bigcup_{\substack{\text{orientations } \mathcal{O} \\ \text{of } K_{m,n}}} comp(\mathcal{O})$$

where the union is taken over all orientations on G.

Key words and phrases. Stochastic sandpile model; classification problem; lacking polynomial; complete bipartite graph; log-concavity.

Chan et al. [3] also introduced the lacking polynomial of a graph to be the generating function counting the stochastically recurrent configurations according to the number of grains by which a given configuration differs from the maximally stable configuration.

Definition 3 (Chan et al. [3])**.** The lacking polynomial $L_G(x)$ is

$$L_G(x) := \sum_{c \in \mathsf{Sto}(G)} x^{\ell(c)},$$

where

$$\ell(c) := \sum_{v \in V(G)\setminus\{s\}} l(v)$$

and $l(v) := d(v) - c(v) - 1$ is the *lacking number* of vertex v.

In this note we consider the SSM on the complete bipartite graph $K_{m,n}$ with vertex set $\{v_0, v_1, \dots, v_{m+n-1}\}$. We will treat v_0 as the sink and this graph has edges connecting vertices in the sets $\{v_0, v_1, v_2, \dots, v_{m-1}\}$ and $\{v_m, v_{m+1}, \dots, v_{m+n-1}\}$.

We characterise stochastically recurrent states on the graphs $K_{2,n}$ and $K_{m,2}$ and use these characterisations to give expressions for the lacking polynomials $L_{2,n}(x)$ and $L_{m,2}(x)$ on those graphs. We also prove that the sequence of coefficients of both $L_{2,n}(x)$ and $L_{m,2}(x)$ are log-concave. This note is motivated by Alofi and Dukes [1] that considers rectangular tableaux representations of recurrent states of the Abelian sandpile model on the complete bipartite graph, and transformations upon them.

Theorem 2 allows us to write an expression for stochastically recurrent states on the complete bipartite graph$K_{m,n}$:

Proposition 4. The set of stochastically recurrent states of the SSM on $K_{m,n}$ is

$$\mathsf{Sto}(K_{m,n}) = \bigcup_{\substack{\text{orientations } \mathcal{O} \\ \text{of } K_{m,n}}} \{(c_1, \cdots, c_{m+n-1}) \ : \ out_{\mathcal{O}}(v_i) \le c_i < d(v_i), \ \forall 1 \le i \le m+n-1\}.$$

Proof. Let $c = (c_1, c_2, \cdots, c_{m+n-1})$ be a stable configuration on $K_{m,n}$. Suppose it is compatible with an orientation $\mathcal{O}$ where $c_i < d(v_i)$ for all $1 \le i \le m+n-1$. According to the Definition 1 we must have:

$$in_{\mathcal{O}}(v_i) \ge d(v_i) - c_i.$$

This means $c_i \ge d(v_i) - in_{\mathcal{O}}(v_i) = out_{\mathcal{O}}(v_i)$. When we combine the application of Definition 1 with Theorem 2 we find

$$\mathsf{Sto}(K_{m,n}) = \bigcup_{\substack{\text{orientations } \mathcal{O} \\ \text{of } K_{m,n}}} \{(c_1, \cdots, c_{m+n-1}) \ : \ out_{\mathcal{O}}(v_i) \le c_i < d(v_i), \ \forall 1 \le i \le m+n-1\}.$$

□

Example 5. Consider the graph $K_{2,2}$. To determine the stochastically recurrent states for each orientation $\mathcal{O}$ of graph $K_{2,2}$ first we find $out_{\mathcal{O}}(v_i)$ for all $1 \le i \le 3$, and then when we apply Prop. 4. See Table 1 for the derivations for each orientation. It follows that

$$\mathsf{Sto}(K_{2,2}) = \{(0,1,1), (1,0,1), (1,1,0), (1,1,1)\}.$$

We can provide crude lower and upper bounds on the number of stochastically recurrent configurations by making use of the fact [3, Prop. 2.3] that $\mathsf{Sto}(K_{m,n})$ properly contains the set of classically recurrent states, and such states are in 1-1 correspondence with the number of spanning tree of the underlying graph. Fieldler and Sedlacek [5] showed the number of spanning trees of the complete bipartite graph $K_{m,n}$ is $n^{m-1}m^{n-1}$. Thus

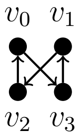
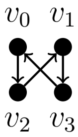
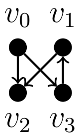
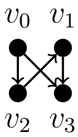
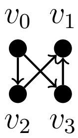
	$out_{\mathcal{O}}(v_1) = 1, out_{\mathcal{O}}(v_2) = 1, out_{\mathcal{O}}(v_3) = 1$. $\implies 1 \leq c_1 < 2$, $1 \leq c_2 < 2$, and $1 \leq c_3 < 2$. The contribution to $\mathsf{Sto}(K_{2,2})$ is $\{(1,1,1)\}$.
	$out_{\mathcal{O}}(v_1) = 1, out_{\mathcal{O}}(v_2) = 1, out_{\mathcal{O}}(v_3) = 1$. $\implies 1 \leq c_1 < 2$, $1 \leq c_2 < 2$, and $1 \leq c_3 < 2$. The contribution to $\mathsf{Sto}(K_{2,2})$ is $\{(1,1,1)\}$.
	$out_{\mathcal{O}}(v_1) = 1, out_{\mathcal{O}}(v_2) = 0, out_{\mathcal{O}}(v_3) = 1$. $\implies 1 \leq c_1 < 2$, $0 \leq c_2 < 2$, and $1 \leq c_3 < 2$. The contribution to $\mathsf{Sto}(K_{2,2})$ is $\{(1,0,1),(1,1,1)\}$.
	$out_{\mathcal{O}}(v_1) = 1, out_{\mathcal{O}}(v_2) = 1, out_{\mathcal{O}}(v_3) = 0$. $\implies 1 \leq c_1 < 2$, $1 \leq c_2 < 2$, and $0 \leq c_3 < 2$. The contribution to $\mathsf{Sto}(K_{2,2})$ is $\{(1,1,0),(1,1,1)\}$.
	$out_{\mathcal{O}}(v_1) = 0, out_{\mathcal{O}}(v_2) = 1, out_{\mathcal{O}}(v_3) = 1$. $\implies 0 \leq c_1 < 2$, $1 \leq c_2 < 2$, and $1 \leq c_3 < 2$. The contribution to $\mathsf{Sto}(K_{2,2})$ is $\{(0,1,1),(1,1,1)\}$.

TABLE 1. Checking all orientations of $K_{2,2}$.

the number of stochastically recurrent configurations on $K_{m,n}$ is at least $n^{m-1}m^{n-1}$. A trivial upper bound is achieved by noting the stochastically recurrent states are stable configurations, of which there are $n^{m-1}m^n$ many. Thus

$$n^{m-1}m^{n-1} \ \leq \ |\mathsf{Sto}(K_{m,n})| \ \leq \ n^{m-1}m^n. \tag{1}$$

Note that the upper bound differs from the lower bound only by a factor of m.

Question 6. Can it be determined whether or not the number of stochastically recurrent states dominates the set of stable states? I.e. can it be decided

$$|\mathsf{Sto}(K_{m,n})| \lesseqgtr \frac{n^{m-1}m^n}{2}?$$

The set of stable configurations on $K_{2,n}$ is

$$\{(c_1, c_2, \cdots, c_{n+1}) \ : \ 0 \leq c_1 < n \text{ and } c_i \in \{0,1\}, \ \forall \ 2 \leq i \leq n+1\}.$$

Theorem 7. The lacking polynomial of the graph $K_{2,n}$ is

$$L_{2,n}(x) = \sum_{k=0}^{n-1} \sum_{i=0}^{k} \binom{n}{i} x^k.$$

Proof. Definition 3 gives

$$L_{2,n}(x) = \sum_{c \in \mathsf{Sto}(K_{2,n})} x^{\ell(c)}$$

where

$$\ell(c) := \sum_{v_i \in V(K_{2,n}) \setminus \{v_0\}} l(v_i) \qquad \text{and} \qquad l(v_i) := d(v_i) - c(v_i) - 1.$$

In order to calculate this sum we need to have an explicit characterization of the set $\mathsf{Sto}(K_{2,n})$. Using Proposition 4 we know

$$\mathsf{Sto}(K_{2,n}) = \{c = (c_1, \cdots, c_{n+1}) \ : \ \exists \text{ an orientation } \mathcal{O} \text{ of } K_{2,n} \text{ compatible with } c\}.$$

Let us suppose $c = (c_1, \cdots, c_{n+1})$ is a member of $\mathsf{Sto}(K_{2,n})$. Then we must have that there exists an orientation $\mathcal{O}$ of $K_{2,n}$ that is compatible with c. This means

$$n - in_{\mathcal{O}}(v_1) = out_{\mathcal{O}}(v_1) \leq c_1 < n$$

and

$$2 - in_{\mathcal{O}}(v_j) = out_{\mathcal{O}}(v_j) \leq c_j < 2 \text{ for all } 2 \leq j \leq n+1.$$

Let us now see under what conditions one can construct an orientation $\mathcal{O}$ on $K_{2,n}$ that is compatible with a given c. Let X be the set of indices j in $[2, n+1]$ for which $c_j = 0$, and where we use the notation $[a, b] := \{a, a+1, \ldots, b\}$. For any j in X the number of outgoing edges at vertex v_j is zero ($out_{\mathcal{O}}(v_j) = 0$), because we know that $out_{\mathcal{O}}(v_j) \leq c_j < 2$, so if $c_j = 0$ then $out_{\mathcal{O}}(v_j) = 0$. Therefore, for all j in X there is one outgoing edge from v_1 to v_j, and hence vertex v_j has one incoming edge from v_1. So the number of outgoing edges from v_1 is greater than or equal to the number of elements in X.

For any j in $[2, n+1] \backslash X$, we know that there is at most one outgoing edge from v_j as $out_{\mathcal{O}}(v_j) \leq c_j < 2$. So if $c_j = 1$ then $out_{\mathcal{O}}(v_j) \leq 1$. With these considerations in mind, we can construct an orientation $\mathcal{O}$ on $K_{2,n}$ that is compatible with a stable configuration c:

(i) If $c_j = 1$ with $2 \leq j \leq n+1$ then there is at most one outgoing edge from v_j.
(ii) If $c_j = 0$ and $2 \leq j \leq n+1$ then there are no outgoing edges from v_j.
(iii) The outgoing edges from vertex v_1 is greater than or equal the the number of vertices v_j when $c_j = 0$ for all $2 \leq j \leq n+1$. We know that the $out_{\mathcal{O}}(v_1)$ is less than or equal to c_1 and greater than or equal to the number of vertices v_j when $c_j = 0$ for all $2 \leq j \leq n+1$. Therefore c_1 is greater than or equal to the number of vertices v_j when $c_j = 0$ for all $2 \leq j \leq n+1$.

There are no other restrictions that forbid us from constructing such an orientation $\mathcal{O}$. Therefore we can write down the following self-contained expression for the set $\mathsf{Sto}(K_{2,n})$ that does not depend on an orientation $\mathcal{O}$:

$$\mathsf{Sto}(K_{2,n}) = \{(c_1, c_2, \cdots, c_{n+1}) : c_2, \cdots, c_{n+1} \in \{0, 1\} \text{ and } c_1 \geq |j \in [2, n+1] : c_j = 0|\}.$$

Now that we have an explicit expression for the set $\mathsf{Sto}(K_{2,n})$, we can use this expression to calculate the lacking polynomial.

Let c be in $\mathsf{Sto}(K_{2,n})$. Let $l_1 = n - c_1 - 1$ be the lacking number at vertex v_1, and let $l_j = 1 - c_j$ be the lacking number at vertices $v_2, \cdots, v_{n+1}$ for all $j \in [2, n+1]$, so $l_j = 0$ when $c_j = 1$ and $l_j = 1$ when $c_j = 0$. Let i be the number of vertices v_j with $j \in [2, n+1]$ and $c_j = 0$ for which $l_j = 1$. The remaining vertices have $l_j = 0$. Then $x^{\ell(c)}$ factors as $x^{\ell(c)} = x^i x^{l_1}$. Since $c_1 \geq |j \in [2, n+1] : c_j = 0|$, we conclude that $c_1 \geq i$, therefore the lacking number at v_1 will be between 0 and $n - 1 - i$. Since there are $\binom{n}{i}$ combinations of i vertices with lacking number 1 among $\{v_2, \cdots, v_{n+1}\}$, we obtain

$$L_{2,n}(x) = \sum_{i=0}^{n} \binom{n}{i} x^i \sum_{l_1=0}^{n-1-i} x^{l_1}.$$

Now setting $k = i + l_1$ we have

$$L_{2,n}(x) \ = \ \sum_{k=0}^{n-1} \sum_{i=0}^{k} \binom{n}{i} x^k.$$

□

A configuration $c = (c_1, c_2, \cdots, c_{m+1})$ on $K_{m,2}$ is stable precisely when $c_i \in \{0, 1\}$ for all $i \in \{1, \cdots, m-1\}$ and $0 \leq c_j < m$ for all $j \in \{m, m+1\}$.

Theorem 8. The lacking polynomial for the graph $K_{m,2}$ is

$$L_{m,2}(x) = \sum_{k=0}^{m-1} S(m-1, k)x^k$$

where

$$S(m-1, k) = \sum_{q=0}^{k} \sum_{r=0}^{q} \binom{m-1}{r} \text{ for all } \ 0 \leq k \leq m-1.$$

Proof. The lacking polynomial $L_{m,2}(x)$, given in Definition 3, is

$$L_{m,2}(x) = \sum_{c \in \mathsf{Sto}(K_{m,2})} x^{\ell(c)}$$

where

$$\ell(c) = \sum_{v_i \in V(K_{m,2}) \setminus \{v_0\}} l(v_i), \qquad \text{and} \qquad l(v_i) = d(v_i) - c(v_i) - 1.$$

To calculate this we require an explicit characterization of the set $\mathsf{Sto}(K_{m,2})$. From Proposition 4 we have

$$\mathsf{Sto}(K_{m,2}) = \{c = (c_1, \cdots, c_{m+1}) \ : \ \exists \text{ orientation } \mathcal{O} \text{ of } K_{m,2} \text{ compatible with } c\}.$$

Suppose $c = (c_1, \cdots, c_{m+1})$ is a member of $\mathsf{Sto}(K_{m,2})$. Then we must have that there exists an orientation $\mathcal{O}$ of $K_{m,2}$ that is compatible with c. This means that

$$n - in_{\mathcal{O}}(v_i) = out_{\mathcal{O}}(v_i) \leq c_i < 2 \text{ for all } i \in \{1, \cdots, m-1\}$$

and

$$2 - in_{\mathcal{O}}(v_j) = out_{\mathcal{O}}(v_j) \leq c_j < m \text{ for all } j \in \{m, m+1\}.$$

Let us see under what conditions one can construct an orientation $\mathcal{O}$ on $K_{m,2}$ that is compatible with a given c. Let X be the set of indices i in $[1, m-1]$ for which $c_i = 0$. For any $i \in X$ the number of outgoing edges at vertex v_i is zero ($out_{\mathcal{O}}(v_i) = 0$) since $out_{\mathcal{O}}(v_i) \leq c_i < 2$. So if $c_i = 0$ then $out_{\mathcal{O}}(v_i) = 0$.

Therefore, for all i in X there are is one outgoing edge from each of v_m and v_{m+1} to v_i. Moreover, vertex v_i has one incoming edge from each of v_m and $v_m + 1$. So the number of outgoing edges from v_m is greater than or equal to the number of elements in X. Also, the number of outgoing edges from v_{m+1} is greater than or equal to the number of elements in X. Therefore $c_m \geq |X|$ and $c_{m+1} \geq |X|$.

For any $i \in [1, m-1] \setminus X$, we know that there is at most one outgoing edge from v_i since $out_{\mathcal{O}}(v_j) \leq c_j < 2$, so if $c_j = 1$ then $out_{\mathcal{O}}(v_j) \leq 1$. So the total number of outgoing edges from v_m and v_{m+1} to v_i must at least equal $m - 1 - |X|$. Therefore we can then construct such an orientation $\mathcal{O}$ of $K_{m,2}$ that is compatible with a stable configuration c precisely when $c_m + c_{m+1} \geq 2|X| + m - 1 - |X| = m - 1 + |X|$. There are no other restrictions that forbid us from constructing such an orientation $\mathcal{O}$. Therefore we can write down the following self-contained expression for the set $\mathsf{Sto}(K_{m,2})$ that does not depend on an orientation $\mathcal{O}$:

$$\mathsf{Sto}(K_{m,2}) = \{(c_1, c_2, \cdots, c_{m+1}) \ : \ c_1, \cdots, c_{m-1} \in \{0, 1\} \text{ and } c_m + c_{m+1} \geq m - 1 + |X|\}.$$

Now that we have an explicit expression for the set $\mathsf{Sto}(K_{m,2})$, we can use this expression to calculate the lacking polynomial. Let c be in $\mathsf{Sto}(K_{m,2})$. Let $l_i = 1 - c_i$ be the lacking number of vertex v_i for all $i \in [1, m-1]$, so that $l_i = 0$ when $c_i = 1$ and $l_i = 1$ when $c_i = 0$. For the vertices v_m and v_m the lacking numbers are $l_m = m - 1 - c_m$ and $l_{m+1} = m - 1 - c_{m+1}$, respectively. We factor $x^{\ell(c)} = x^{|X|}x^{l_m + l_{m+1}}$ and can now write $c_m + c_{m+1} \geq m - 1 + |X|$ in terms of lacking numbers as:

$$m - 1 - l_m + m - 1 - l_{m+1} \geq m - 1 + |X|$$

which equivalent to

$$m - 1 - |X| \geq l_m + l_{m+1}.$$

Now suppose that $r = l_m + l_{m+1}$, then r ranges in between 0 and $m - 1 - |X|$. For each choice of r we have exactly $r + 1$ choices for $(l_m, l_m + 1)$. Now let $j = |X|$ be the number of vertices v_i with $i \in [1, m-1]$ and $c_i = 0$ for which then $l_i = 1$. The remaining vertices will have $l_i = 0$. So there are $\binom{m-1}{j}$ combinations of j vertices with lacking number 1 among $\{v_1, \cdots, v_{m-1}\}$, and we obtain

$$L_{m,2}(x) = \sum_{j=0}^{m-1} \binom{m-1}{j} x^j \sum_{r=0}^{m-1-j} (r+1)x^r = \sum_{j=0}^{m-1} \sum_{r=0}^{m-1-j} \binom{m-1}{j} (r+1)x^{r+j}.$$

Set $k = j + r$. Then k runs from 0 to $m - 1$ so that j can run from 0 to k and, with $r = k - j$, we obtain the sum

$$L_{m,2}(x) = \sum_{k=0}^{m-1} \sum_{j=0}^{k} \tbinom{m-1}{j} (k - j + 1)x^k.$$

This is equal to

$$L_{m,2}(x) = \sum_{k=0}^{m-1} S(m-1, k)x^k$$

where

$$S(m-1, k) = \sum_{q=0}^{k} \sum_{r=0}^{q} \binom{m-1}{r} \text{ for all } 0 \leq k \leq m-1. \quad \square$$

A sequence $a_0, a_1, \ldots, a_n$ of non-negative real numbers is said to be *logarithmically concave*, or *log-concave*, if for all $0 < k < n$, $a_k^2 - a_{k-1}a_{k+1} \geq 0$.

Lemma 9. Suppose $(x_0, x_1, \ldots, x_n)$ is a log-concave sequence. Then the sequence $(z_0, \ldots, z_n)$ of partial sums defined by

$$z_k = \sum_{i=0}^{k} x_i$$

is also log-concave.

Proof. Wang and Yeh [8] proved that if sequences (x_k) and (y_k) are log-concave, then the sequence (z_k) where z_k is the ordinary convolution

$$z_k = \sum_{i=0}^{k} x_i y_{n-i}$$

is log-concave. The sequence $(y_0, \ldots, y_k) = (1, \ldots, 1)$ is trivially log-concave. The statement of the lemma follows since it is the special case with all y_j's replaced with 1s. $\square$

Theorem 10.
(i) The sequence of coefficients of the lacking polynomial $L_{2,n}(x)$ is log-concave.
(ii) The sequence of coefficients of the lacking polynomial $L_{m,2}(x)$ is log-concave.

Proof. (i) From Theorem 7 we have

$$L_{2,n}(x) = \sum_{k=0}^{n-1} T(n,k)x^k \quad \text{where} \quad T(n,k) := \sum_{i=0}^{k} \binom{n}{i}.$$

It is well-known that a sequence of binomial coefficients $\left(\binom{n}{k}\right)_{k=0,1,2,\ldots,n}$ is log-concave. By Lemma 9, the sequence $(T(n,k))_{k=0,\ldots,n}$ is log-concave. That is, for any $0 < k \leq n-1$

$$(T(n,k))^2 - T(n,k-1)T(n,k+1) \geq 0$$

Therefore the sequence of coefficients of the lacking polynomial $L_{2,n}(x)$ is log-concave.
(ii) From Theorem 8 we have

$$L_{m,2}(x) = \sum_{k=0}^{m-1} S(m-1,k)x^k$$

where

$$S(m-1,k) := \sum_{q=0}^{k} \sum_{r=0}^{q} \binom{m-1}{r} \text{ for all } 0 < k \leq m-1.$$

By Lemma 9 the sequence $R(m-1,q) := \left(\sum_{r=0}^{q} \binom{m-1}{r}\right)_{q=0,\ldots,m-1}$ is log-concave. Then, again by an application of Lemma 9, the sequence $(S(m-1,k) = \sum_{q=0}^{k} R(m-1,q))_{k=0}^{m-1}$ is also log-concave. Therefore the sequence of coefficients of the lacking polynomial $L_{m,2}(x)$ is log-concave. □

The two results in Theorem 10 suggest that log-concavity might be a property of these lacking polynomials for the general complete bipartite graph.

Conjecture 11. The sequence of coefficients of the lacking polynomial $L_{m,n}(x)$ for the complete bipartite graph $K_{m,n}$ is log-concave for all $m, n \geq 2$.

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