NEW EQUIVALENCES FOR PATTERN AVOIDING INVOLUTIONS

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Abstract. We complete the Wilf classification of signed patterns of length 5 for both signed permutations and signed involutions. New general equivalences of patterns are given which prove Jaggard's conjectures concerning involutions in the symmetric group avoiding certain patterns of length 5 and 6. In this way, we also complete the Wilf classification of S_5, S_6, and S_7 for involutions.

1. Introduction

Pattern avoidance has proved to be a useful concept in a variety of seemingly unrelated problems, including Kazhdan-Lusztig polynomials [2], singularities of Schubert varieties [3, 4, 5, 6, 7, 15], Chebyshev polynomials [18], rook polynomials for a rectangular board [17] and various sorting algorithms, sorting stacks and sortable permutations [8, 9, 10, 19, 20, 21].

In this paper, we deal with pattern avoidance in the symmetric group S_n and the hyperoctahedral group B_n. The group B_n, which is isomorphic to the automorphism group of the n-dimensional hypercube, can be represented as the group of all bijections ω of the set $X = \{-n, \ldots, -1, 1, \ldots, n\}$ onto itself such that $\omega(-i) = -\omega(i)$ for all $i \in X$, with composition as the group operation. However, for our purposes it is more convenient to represent the elements of S_n as permutation matrices, and the elements of B_n as signed permutation matrices, where a signed permutation matrix is a $0, 1, -1$-matrix with exactly one nonzero entry in every row and every column. We may also write the elements of B_n as words $\pi = \pi_1\pi_2 \ldots \pi_n$ in which each of the letters $1, 2, \ldots, n$ appears, possibly barred to signify negative letters; a matrix p corresponds to the word π such that $p_{ij} = 1$ if $\pi_i = j$, $p_{ij} = -1$ if $\pi_i = -j$, and $p_{ij} = 0$ otherwise. In our paper, we will make no explicit distinction between these two representations of a signed permutation. Let I_n and SI_n be the set of involutions in S_n and B_n, respectively. Note that involutions correspond precisely to symmetric matrices.

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A signed permutation $\pi \in B_n$ is said to *contain the pattern* $\tau \in B_k$ if there exists a sequence $1 \leq i_1 < i_2 < \ldots < i_k \leq n$ such that $|\pi_{i_a}| < |\pi_{i_b}|$ if and only if $|\tau_a| < |\tau_b|$ and $\pi_{i_a} > 0$ if and only if $\tau_a > 0$ for all $1 \leq a, b \leq k$. Otherwise, π is called a τ*-avoiding* permutation. Note that π contains τ if and only if the matrix representing π contains the matrix representing τ as a submatrix. By $M(\tau)$ we denote the set of all elements of M which avoid the pattern τ.

Two signed patterns σ and τ are called *Wilf equivalent*, in symbols $\sigma \sim \tau$, if they are avoided by the same number of signed n-permutations, i.e., if $|B_n(\sigma)| = |B_n(\tau)|$ for each $n \geq 1$. Similarly, σ and τ are called *I-Wilf equivalent*, denoted by $\sigma \overset{I}{\sim} \tau$, if $|SI_n(\sigma)| = |SI_n(\tau)|$ for each n. Note that two unsigned permutations $\sigma, \tau \in S_k$ are Wilf-equivalent if and only if they satisfy the identity $|S_n(\sigma)| = |S_n(\tau)|$ for each n, and they are I-Wilf equivalent if and only if they satisfy $|I_n(\sigma)| = |I_n(\tau)|$ for each n. The classification given by the Wilf equivalence is slightly coarser than that which is based on the symmetries of permutations, that is, the mappings generated by the reversal, transpose, and barring operation. The same is true for the I-Wilf equivalence, where the available symmetries are generated by the two diagonal reflections and the barring operation.

The question of whether two patterns are Wilf equivalent or not is difficult to answer in many cases. By the few generic equivalences known so far, it has been possible to completely determine the Wilf classes of S_n up to level $n = 7$. The decomposition of S_n into I-Wilf classes has been completely determined for $n = 4$ and almost solved for $n = 5$ as well. Jaggard [13] conjectured the last case of a possible equivalence for patterns of length 5: 12345 (or equivalently, 54321) and 45312 are equally restrictive for I_n up to $n = 11$.

Continuing the I-Wilf classification of signed patterns that began in [12], we will first prove a general equivalence result which confirms Jaggard's conjecture mentioned above, as well as another conjecture he made about the equivalence of certain patterns of length 6. The correspondence behind this result is based on a bijection between pattern avoiding transversals of Young diagrams given by Backelin, West and Xin [1]. In this way, we complete the classification of S_5 with respect to $\overset{I}{\sim}$, which is fundamental for the analogous classification of B_5. The result even covers all missing I-Wilf equivalences in S_6 and S_7.

Furthermore, we will show that barring some blocks of a signed block diagonal pattern preserves the Wilf class of the pattern, and it also (under some additional assumptions) preserves the I-Wilf class. These results not only allow us to determine the Wilf as well as the I-Wilf classes in B_5 but they also have consequences for longer signed patterns.

2. Jaggard's conjectures

In 2003, Jaggard [13] proved the equivalences $12\tau \overset{I}{\sim} 21\tau$ and $123\tau \overset{I}{\sim} 321\tau$, and completed the classification of S_4 according to pattern avoidance by involutions in this way. Furthermore, he conjectured that

(1) $12\dots k\tau \overset{I}{\sim} k(k-1)\dots 1\tau$ for any $k \geq 1$,
(2) $12345 \overset{I}{\sim} 45312$ (or equivalently, $54321 \overset{I}{\sim} 45312$),
(3) $123456 \overset{I}{\sim} 456123 \overset{I}{\sim} 564312$ (or equivalently, $654321 \overset{I}{\sim} 456123$).

In [1], Backelin, West and Xin defined a transformation to prove $12\dots k\tau \sim k(k-1)\dots 1\tau$. (As already mentioned in [12], their proof also works for a signed pattern τ.) This map acts not only on permutation matrices, but more generally, on transversals of Young diagrams. Bousquet-Mélou and Steingrímsson [11] showed that this map commutes with the diagonal reflection of the diagram, which proves the first of the three conjectures above. From this result, it follows that

$$\begin{pmatrix} \alpha_k & 0 & 0 \\ 0 & \chi & 0 \\ 0 & 0 & \alpha_l \end{pmatrix} \overset{I}{\sim} \begin{pmatrix} \beta_k & 0 & 0 \\ 0 & \chi & 0 \\ 0 & 0 & \beta_l \end{pmatrix}$$

for every signed permutation matrix χ and any $k, l \geq 0$, where α_n and β_n denote the $n \times n$ diagonal and antidiagonal permutation matrices corresponding to $12\dots n$ and $n(n-1)\dots 1$, respectively. In this section, we will show that

$$\begin{pmatrix} 0 & 0 & 0 & \alpha_k \\ 0 & 0 & \chi & 0 \\ 0 & \chi^t & 0 & 0 \\ \alpha_k & 0 & 0 & 0 \end{pmatrix} \overset{I}{\sim} \begin{pmatrix} 0 & 0 & 0 & \beta_k \\ 0 & 0 & \chi & 0 \\ 0 & \chi^t & 0 & 0 \\ \beta_k & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 0 & 0 & 0 & \alpha_k \\ 0 & 0 & 0 & \chi & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & \chi^t & 0 & 0 & 0 \\ \alpha_k & 0 & 0 & 0 & 0 \end{pmatrix} \overset{I}{\sim} \begin{pmatrix} 0 & 0 & 0 & 0 & \beta_k \\ 0 & 0 & 0 & \chi & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & \chi^t & 0 & 0 & 0 \\ \beta_k & 0 & 0 & 0 & 0 \end{pmatrix},$$

where χ^t denotes the transpose of χ. Note that, different to the general case, the reverse operation is not a symmetry for involutions, so these equivalences are really new.

Our proof will also use the Backelin, West and Xin bijection [1]. Therefore, let us first recall the extended notion of pattern avoidance they have used. A *Young diagram* (or Young shape) is a top-justified and left-justified array of cells, i.e., an array whose rows have non-increasing lengths from top to bottom, and its columns have non-increasing lengths from left to right. A cell of a Young shape is called a *corner* if the array obtained by removing the cell is still a Young shape. Occasionally, it will be convenient to use top-right justified diagrams instead of the top-left justified diagrams defined above. We will refer to the top-right justified shapes as *NE-shapes* to avoid confusion with the ordinary Young shapes.

A (*signed*) *transversal* of a Young diagram λ is an assignment of 0's and 1's (of 0's, 1's and -1's) to the cells of λ, such that each row and column contains exactly one nonzero entry. A *sparse filling* of λ is an arrangement of 0's, 1's and -1's which has at most one nonzero entry in every row and column.

For a $k \times k$ permutation matrix τ, we say that a filling L of a shape λ *contains* τ if there exists a $k \times k$ subshape within λ whose induced filling is equal to τ. The set of all transversals (or signed transversals) of a shape λ which do not contain τ is denoted by $S_\lambda(\tau)$ (or $B_\lambda(\tau)$, respectively). Two signed permutation matrices σ and τ are called *shape Wilf equivalent* if $|B_\lambda(\sigma)| = |B_\lambda(\tau)|$ for all Young shapes λ. Shape Wilf equivalence clearly implies Wilf equivalence. We will also say that σ and τ are *NE-shape Wilf equivalent* if $|B_\lambda(\sigma)| = |B_\lambda(\tau)|$ for each NE-shape λ. Observe that if σ and τ are permutation matrices, then they are shape Wilf equivalent if and only if $|S_\lambda(\sigma)| = |S_\lambda(\tau)|$ for each Young diagram λ.

By [1, Proposition 2.2], α_k and β_k are shape Wilf equivalent for all k. The following proposition, which is also largely based on [1], will allow us to extend this equivalence to more general patterns.

Proposition 2.1. *Let λ be a Young shape, and let χ, χ_1, χ_2 be signed permutations, such that χ_1 and χ_2 are shape Wilf equivalent. We set*

$$\theta = \begin{pmatrix} \chi_1 & 0 \\ 0 & \chi \end{pmatrix} \quad \text{and} \quad \omega = \begin{pmatrix} \chi_2 & 0 \\ 0 & \chi \end{pmatrix}.$$

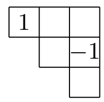
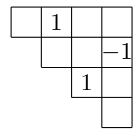
There is a bijection between θ-avoiding and ω-avoiding sparse fillings of λ. This bijection preserves the number of nonzero entries in each row and column; in particular, θ and ω are shape Wilf equivalent. Furthermore, if χ is nonempty, the bijection preserves the values of the filling in the corners of λ.

Proof. The proof is essentially the same as the proof given in [1, Proposition 2.3]. We briefly sketch the argument here. By assumption, there is a bijection φ between the χ_1-avoiding and χ_2-avoiding signed transversals of an arbitrary Young shape. Let L be an arbitrary θ-avoiding sparse filling of λ. Let us colour a cell of λ if there is no occurrence of χ to the south-east of this cell. Also, if λ has a row or column where all the uncoloured cells contain zeros, then we colour each cell of this row or column. Note that if χ is nonempty, then all the corners of λ are coloured. The uncoloured cells induce a χ_1-avoiding signed transversal of a Young subdiagram of λ. We apply the bijection φ to the subdiagram of uncoloured cells, and preserve the filling of all the coloured cells. This transforms the original filling of λ into a ω-avoiding sparse filling. This transformation is a bijection which has all the claimed properties. □

Note that Proposition 2.1 yields some information even when χ is the empty matrix. In such situation, the proposition shows that a bijection between pattern avoiding signed transversals can be extended to a bijection between pattern-avoiding sparse fillings, by simply ignoring the rows and columns with no nonzero entries.

We will now show how the results on shape Wilf equivalence may be applied to obtain new classes of I-Wilf equivalent patterns. Let us first give the necessary definitions. For an $n \times n$

matrix π let π^+ denote the subfilling of π formed by the cells of π which are strictly above the main diagonal, and let π_0^+ denote the subfilling formed by the cells on the main diagonal and above it. For example, for $\pi = 2\bar{4}31$ we have

$\pi^+ =$  and $\pi_0^+ =$  .

The coordinates of the entries in π are used for the cells of π^+ as well. Thus, for instance, the cell $(1, 2)$ is the top-left corner of π^+. Analogously, we define π^- to be the filled shape corresponding to the entries strictly below the main diagonal of π. Clearly, a symmetric matrix π is completely determined by π_0^+. Observe that a symmetric $0, 1, -1$-matrix π is a signed involution if and only if, for every $i = 1, \ldots, n$, the filling π_0^+ has exactly one nonzero entry in the union of all cells of the i-th row and i-th column.

Note that i is a fixed point of a signed involution π, that is $|\pi_i| = i$, if and only if the i-th row and the i-th column of π^+ have all entries equal to zero. In general, a signed involution π need not be completely determined by the filling π^+; however, if we have two signed involutions π, ρ with $\pi^+ = \rho^+$, then π and ρ only differ by the signs of their fixed points. If π is a signed involution, then, for each $i = 1, \ldots, n$, the filling π^+ has at most one nonzero entry in the union of the i-th row and i-th column; conversely, any filling π^+ of appropriate shape with these properties can be extended into a signed involution π, which is determined uniquely up to the sign of its fixed points.

For a signed permutation σ, let σ' denote the involution $\left(\begin{smallmatrix} 0 & \sigma \\ \sigma^t & 0 \end{smallmatrix}\right)$, where σ^t is the transpose of σ. We are now ready to state our first result on I-Wilf equivalence.

Theorem 2.2. *If σ and τ are two NE-shape Wilf equivalent signed permutation matrices, then $\sigma' \overset{I}{\sim} \tau'$. Moreover, the bijection between $SI_n(\sigma')$ and $SI_n(\tau')$ preserves fixed points.*

Proof. Let $\pi \in SI_n$ be an involution. We claim that π avoids σ' if and only if π^+ avoids σ. To see this, notice that any occurrence of σ' in π can be restricted either to an occurrence of σ in π^+ or an occurrence of σ^t in π^-; however, since π^+ is the transpose of π^-, we know that π^- contains σ^t if and only if π^+ contains σ. The converse is even easier to see.

Let us choose $\pi \in SI_n(\sigma')$. Since π^+ is a sparse σ-avoiding filling, we may apply the bijection from Proposition 2.1 (adapted for NE-shapes) to π^+, to obtain a τ-avoiding sparse filling of the same shape, which has a nonzero entry in a row i (or column i) whenever π^+ has a nonzero entry in the same row (or column, respectively). Hence this filling also corresponds to an involution, more exactly, to ρ^+ for an involution $\rho \in SI_n$, and furthermore, the fixed points of ρ are in the same position as the fixed points of π, because the position of the fixed points is determined by the zero rows and columns, which are preserved by the bijection from Proposition 2.1. By defining the signs of the fixed points of ρ to be the same as the signs of the fixed points of π, the involution ρ is determined uniquely. Clearly, since ρ^+ avoids τ, we know that ρ avoids τ'. Each step of this construction can be inverted which proves the bijectivity. Furthermore, the bijection preserves fixed points by construction. $\square$

By a similar reasoning, we obtain an analogous result for patterns of odd size. For a signed permutation σ, let σ'' denote the involution matrix

$$\left(\begin{smallmatrix} 0 & 0 & \sigma \\ 0 & 1 & 0 \\ \sigma^t & 0 & 0 \end{smallmatrix}\right),$$

and let σ^* denote the signed permutation $\left(\begin{smallmatrix} 0 & \sigma \\ 1 & 0 \end{smallmatrix}\right)$.

Theorem 2.3. *If σ and τ are NE-shape Wilf equivalent, then $\sigma'' \overset{I}{\sim} \tau''$. Moreover, the bijection between $SI_n(\sigma'')$ and $SI_n(\tau'')$ preserves fixed points.*

Proof. By an argument analogous to the proof of Theorem 2.2, we may observe that an involution π avoids σ'' if and only if π_0^+ avoids the pattern σ^*. By Proposition 2.1 (adapted for NE-shapes), the two patterns σ^* and τ^* are NE-shape Wilf equivalent and furthermore, the bijection realizing this equivalence preserves the corners of the shape. Note that in our situation, the corners correspond exactly to the diagonal cells of the original signed permutation matrix.

Now we consider π_0^+ for an involution $\pi \in SI_n(\sigma'')$. By Proposition 2.1, π_0^+ is in bijection with a τ^*-avoiding filling ρ_0^+. Since the bijection preserves the number of nonzero entries in each row and each column of π_0^+, and it also preserves the entries on the intersection of i-th row and i-th column (these are precisely the corners), we know that the bijection preserves, for each i, the number of nonzero entries in the union of the i-th row and i-th column. In particular, ρ_0^+ has exactly one nonzero entry in the union of i-th row and i-th column, which guarantees that ρ_0^+ can be (uniquely) extended into an involution ρ.

Because the bijection preserves the entries in the diagonal cells (i,i), $i = 1, \ldots, n$, the permutations π and ρ have the same fixed points. This provides the required bijection. $\square$

Let us apply these two theorems to some special cases of shape Wilf equivalent patterns. For an integer $k \geq 0$ and a signed permutation χ, let us define

$$\theta = \left(\begin{smallmatrix} 0 & \alpha_k \\ \chi & 0 \end{smallmatrix}\right) \quad \text{and} \quad \omega = \left(\begin{smallmatrix} 0 & \beta_k \\ \chi & 0 \end{smallmatrix}\right).$$

As we know, the two patterns θ and ω are NE-shape Wilf equivalent. From our results, we then obtain the following classes of I-Wilf equivalent patterns.

Corollary 2.4. *We have*

$$\begin{pmatrix} 0 & 0 & 0 & \alpha_k \\ 0 & 0 & \chi & 0 \\ 0 & \chi^t & 0 & 0 \\ \alpha_k & 0 & 0 & 0 \end{pmatrix} \overset{I}{\sim} \begin{pmatrix} 0 & 0 & 0 & \beta_k \\ 0 & 0 & \chi & 0 \\ 0 & \chi^t & 0 & 0 \\ \beta_k & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 0 & 0 & 0 & \alpha_k \\ 0 & 0 & 0 & \chi & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & \chi^t & 0 & 0 & 0 \\ \alpha_k & 0 & 0 & 0 & 0 \end{pmatrix} \overset{I}{\sim} \begin{pmatrix} 0 & 0 & 0 & 0 & \beta_k \\ 0 & 0 & 0 & \chi & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & \chi^t & 0 & 0 & 0 \\ \beta_k & 0 & 0 & 0 & 0 \end{pmatrix}.$$

The special cases $\chi = \emptyset$ and $\chi = (1)$ show both of Jaggard's conjectures to be correct.

Corollary 2.5. *We have* $54321 \overset{I}{\sim} 45312$ *and* $654321 \overset{I}{\sim} 456123 \overset{I}{\sim} 564312$.

3. Barring of blocks

In [12] it was shown that the barring of τ in $12\dots k\tau$ and $k(k-1)\dots 1\tau$ preserves both the Wilf class and the I-Wilf class. Furthermore it was proved that

$$\begin{pmatrix} \alpha_k & 0 & 0 \\ 0 & \chi & 0 \\ 0 & 0 & \alpha_k \end{pmatrix} \overset{I}{\sim} \begin{pmatrix} \alpha_k & 0 & 0 \\ 0 & -\chi & 0 \\ 0 & 0 & \alpha_k \end{pmatrix}$$

for every signed permutation matrix χ and $k \geq 0$. Basically, the assertion follows from $123 \overset{I}{\sim} 1\bar{2}3$. By a similar reasoning, we can show the I-Wilf equivalence of the reversed patterns because $321 \overset{I}{\sim} 3\bar{2}1$ as well. Now we turn our attention to the general block pattern

$$\begin{pmatrix} \chi_1 & 0 & 0 \\ 0 & \chi_2 & 0 \\ 0 & 0 & \chi_3 \end{pmatrix}$$

where the χ_i are signed permutation matrices. First we prove the following crucial statement.

Theorem 3.1. *Let χ_1 and χ_2 be signed permutation matrices and set*

$$\theta = \begin{pmatrix} \chi_1 & 0 \\ 0 & \chi_2 \end{pmatrix} \quad \text{and} \quad \omega = \begin{pmatrix} \chi_1 & 0 \\ 0 & -\chi_2 \end{pmatrix}.$$

For any Young shape λ, there is a bijection between θ-avoiding and ω-avoiding sparse fillings of λ. The bijection preserves the position of all nonzero entries, i.e., it transforms the filling only by changing the signs of some of the entries. In particular, the patterns θ and ω are shape Wilf equivalent. Moreover, if λ is self-conjugate and at least one of the matrices χ_1 and χ_2 is symmetric, then the bijection maps symmetric fillings to symmetric fillings.

Proof. Given a θ-avoiding sparse filling of λ, we construct the corresponding ω-avoiding filling as follows: Colour each cell of λ for which there is an occurrence of χ_1 to the north-west of the cell. Note that the cells left uncoloured then form a Young subdiagram of λ. By assumption, the coloured part does not contain χ_2. Switching the signs of all entries of this part consequently yields a signed transversal of λ which avoids ω. Note that even after the transformation has been performed, it is still true that the coloured cells are precisely those cells that have an occurrence of χ_1 to their north-west. The transformation may have created new copies of χ_1 in the diagram, but it may be easily seen that these copies do not alter the colouring of the cells. This shows that the transformation is indeed a bijection.

Let λ now be self-conjugate with a symmetric θ-avoiding filling. Obviously, if χ_1 is symmetric, then a cell is coloured if and only if its reflection (along the main diagonal) is coloured. Hence the signs of both entries must have been changed, so the resulting filling is symmetric again. If χ_2 is symmetric but χ_1 is not, then we slightly modify the definition of the bijection. Colour a cell if there is an occurrence of χ_2 to the south-east. The restriction to these cells is a symmetric filling of a self-conjugate subshape which avoids χ_1. Now change the signs of all nonzeros in uncoloured cells. The resulting filling avoids ω and is still symmetric. It is again easy to see that this provides the required symmetry-preserving bijection. □

An immediate consequence of the previous theorem is the following:

Corollary 3.2. *For any signed permutation matrices χ_1, χ_2, χ_3, we have*

$$\begin{pmatrix} \chi_1 & 0 & 0 \\ 0 & \chi_2 & 0 \\ 0 & 0 & \chi_3 \end{pmatrix} \sim \begin{pmatrix} \chi_1 & 0 & 0 \\ 0 & -\chi_2 & 0 \\ 0 & 0 & \chi_3 \end{pmatrix}.$$

Because of the symmetry property of the bijection we can prove an analogous result for pattern avoiding involutions.

Corollary 3.3. *Let χ_1, χ_2, χ_3 be signed permutation matrices, at least two of which are symmetric. Then we have*

$$\begin{pmatrix} \chi_1 & 0 & 0 \\ 0 & \chi_2 & 0 \\ 0 & 0 & \chi_3 \end{pmatrix} \overset{I}{\sim} \begin{pmatrix} \chi_1 & 0 & 0 \\ 0 & -\chi_2 & 0 \\ 0 & 0 & \chi_3 \end{pmatrix}.$$

Proof. By Theorem 3.1, the signed pattern $\mathrm{diag}(\chi_1, \chi_2, \chi_3)$ is I-Wilf equivalent with the signed pattern $\mathrm{diag}(\chi_1, \chi_2, -\chi_3)$ (note that at least one of the two matrices $\mathrm{diag}(\chi_1, \chi_2)$ and χ_3 is symmetric). By the same argument, the pattern $\mathrm{diag}(\chi_1, \chi_2, \chi_3)$ is I-Wilf equivalent with $\mathrm{diag}(\chi_1, -\chi_2, -\chi_3)$. Combining these facts with the observation that changing the signs of all the three blocks clearly preserves the I-Wilf class, we may even conclude that any matrix obtained by changing the signs of any of the three blocks is I-Wilf equivalent with the original matrix. □

Combining Theorem 3.1 with Theorems 2.2 and 2.3, we obtain more classes of I-Wilf equivalent patterns. The following corollary gives an example.

Corollary 3.4. *Let χ_1 and χ_2 be signed permutation matrices. Then we have*

$$\begin{pmatrix} 0 & 0 & 0 & 0 & \chi_1 \\ 0 & 0 & 0 & \chi_2 & 0 \\ 0 & 0 & \varepsilon & 0 & 0 \\ 0 & \chi_2^t & 0 & 0 & 0 \\ \chi_1^t & 0 & 0 & 0 & 0 \end{pmatrix} \overset{I}{\sim} \begin{pmatrix} 0 & 0 & 0 & 0 & \chi_1 \\ 0 & 0 & 0 & -\chi_2 & 0 \\ 0 & 0 & \varepsilon & 0 & 0 \\ 0 & -\chi_2^t & 0 & 0 & 0 \\ \chi_1^t & 0 & 0 & 0 & 0 \end{pmatrix}.$$

where ε is empty or $\varepsilon = (1)$.

4. Classification

The proof of Jaggard's conjecture provides the complete classification of the I-Wilf equivalences among the patterns from S_5. It turns out that there are 36 different classes (in comparison with 45 symmetry classes). By the results of [12], it has been known that B_5 has at most 405 I-Wilf equivalence classes. Applying the new equivalences, we obtain 402 classes which are definitively different. (By the symmetries of an involutive permutation, the patterns are divided into 566 classes.) Table 1 shows representatives of all classes, each with the number of involutions in $SI_9, \dots, SI_{12}$ avoiding the patterns of this class. The enumeration is done for $n = 9$ in any case; higher levels are only computed up to the final distinction. Classes containing patterns of S_5 are in bold; hence the classification of S_5 according to the I-Wilf equivalence can be read from the table as well.

The classification of the patterns of B_5 by Wilf equivalence becomes complete by Corollary 3.2. The relations given in [12] did not cover seven pairs of patterns whose Wilf equivalence was indicated by numerical results. All these cases are proved now by the corollary. Consequently, B_5 falls into 130 Wilf classes (in comparison with 284 symmetry classes). See [12, Table 7] for the complete list.

The bijections of Theorem 2.2 and Theorem 2.3 also provide the complete classification of S_6 and S_7 with respect to the I-Wilf equivalence. Table 2 lists all classes of S_6 obtained by all equivalences, already known (see [12] and the references therein) or proven here. As the enumeration of involutions in I_{12} avoiding the patterns shows, they are different. In a similar way, we obtain 1291 Wilf classes for S_7 whose table is available from [16].

It is very possible that the results given here and in [12] suffice to solve the I-Wilf classification of signed patterns up to length 7. However, the numerical proof that two classes are really different for a rapidly increasing number of classes is the challenge we (and computers) have to master.

Remark 4.1. After publishing this paper in arXiv, Aaron Jaggard mentioned that he and Joseph Marincel had shown that the patterns $(k-1)k(k-2)\dots 312$ and $k(k-1)\dots 21$ are I-Wilf equivalent for any $k \geq 5$ by using generating tree techniques [14].

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$3\bar{5}1\bar{4}\bar{2}$	160482	**35142**	160519	$14\bar{5}2\bar{3}$	160623	$3\bar{5}14\bar{2}$	160627
$4\bar{5}31\bar{2}$	160647	$351\bar{4}2$	160662	**14325**	160668	**12435**	160670
$52\bar{4}\bar{3}1$	160682	**12345**	160684 856400 4724160	**52431**	160684 856400 4724162	$523\bar{4}1$	160684 856396
$52\bar{3}41$	160686	**52341**	160702	$15\bar{3}42$	160817	**14523**	160819
$153\bar{4}2$	160831	**15342**	160834	$125\bar{4}3$	160843	$15\bar{4}\bar{3}2$	160845
$15\bar{3}\bar{4}2$	160861	$14\bar{3}25$	160944	$124\bar{3}5$	164848	$13\bar{4}25$	165194
$1325\bar{4}$	165198	$13\bar{5}\bar{4}\bar{2}$	165227	$1235\bar{4}$	165230	**13542**	165269
$524\bar{3}1$	165304	**13425**	165310	$124\bar{5}\bar{3}$	165365	$143\bar{5}2$	165389
14352	165416	$154\bar{3}2$	165484	$124\bar{5}3$	165525	$2\bar{5}\bar{4}\bar{3}\bar{1}$	165557
25431	165560	$13\bar{5}2\bar{4}$	165585	$2\bar{5}1\bar{4}\bar{3}$	165588	**45231**	165596
12453	165598	$15\bar{4}3\bar{2}$	165600	$2\bar{1}5\bar{4}3$	165604	**25143**	165627
$4\bar{5}2\bar{3}1$	165734	$53\bar{4}21$	165777	**13524**	165788	**53421**	165990
$143\bar{2}5$	166106	$134\bar{2}5$	166279	$1254\bar{3}$	166337	$13\bar{5}4\bar{2}$	166363
$13\bar{4}5\bar{2}$	166398	$13\bar{4}\bar{5}\bar{2}$	166404 896272	$135\bar{4}2$	166404 896308	$13\bar{4}52$	166418
13452	166429	$14\bar{5}3\bar{2}$	166451	$2\bar{5}14\bar{3}$	166467	**14532**	166479
$3\bar{5}2\bar{4}\bar{1}$	166488	$1245\bar{3}$	166498	$251\bar{4}3$	166505	$14\bar{3}\bar{5}2$	166527 897293
35241	166527 897923	$14\bar{3}52$	166538	$143\bar{5}\bar{2}$	166544	$1543\bar{2}$	166550
$2\bar{5}3\bar{4}\bar{1}$	166567	$25\bar{3}41$	166569	$13\bar{5}\bar{4}2$	166572	**32541**	166575
$2\bar{5}\bar{3}4\bar{1}$	166581	$253\bar{4}1$	166583	$24\bar{5}1\bar{3}$	166586	$2\bar{5}\bar{3}4\bar{1}$	166587
$134\bar{5}\bar{2}$	166591 898088	**25341**	166591 898195	$14\bar{5}\bar{3}\bar{2}$	166607	$13\bar{4}52$	166615
$13\bar{4}\bar{5}2$	166619	$14\bar{5}\bar{3}2$	166627	**24513**	166628 898700	$543\bar{2}1$	166628 898668
$145\bar{3}2$	166655	$3\bar{5}24\bar{1}$	166658	$352\bar{4}1$	166662	$135\bar{2}4$	166701
$2\bar{5}43\bar{1}$	166720	$25\bar{4}\bar{3}1$	166723	$1354\bar{2}$	166725 899209	$1435\bar{2}$	166725 899210
$135\bar{2}\bar{4}$	166727	$2\bar{5}34\bar{1}$	166737	$25\bar{3}41$	166739	$2\bar{4}5\bar{1}3$	166741
$32\bar{5}\bar{4}1$	166742	$25\bar{1}43$	166754	$14\bar{5}32$	166755	$2\bar{5}\bar{1}4\bar{3}$	166756
$2\bar{4}\bar{3}5\bar{1}$	166757	$243\bar{5}1$	166758	$2\bar{3}\bar{5}4\bar{1}$	166759 899733	**24351**	166759 899753
23541	166760	$2\bar{4}5\bar{1}\bar{3}$	166761	$2\bar{3}5\bar{1}4$	166762	$2351\bar{4}$	166769
$1345\bar{2}$	166773 899813	$2\bar{5}4\bar{3}\bar{1}$	166773 899906	$25\bar{4}31$	166775 899951	$534\bar{2}1$	166775 900042
$23\bar{5}\bar{4}\bar{1}$	166776	$2\bar{3}541$	166777	$2\bar{3}5\bar{1}\bar{4}$	166780	**45321**	166788
$5432\bar{1}$	166790	**23514**	166791	$4\bar{5}32\bar{1}$	166800	$3\bar{5}41\bar{2}$	166805
35412	166809	$2\bar{5}\bar{4}1\bar{3}$	166816	$35\bar{2}41$	166818	**25413**	166822
$3\bar{5}\bar{2}4\bar{1}$	166834	$25\bar{4}13$	166861	$13\bar{5}24$	166863	$1352\bar{4}$	166875
$2\bar{5}41\bar{3}$	166876	$2\bar{3}54\bar{1}$	166933	$23\bar{5}\bar{4}1$	166934 901415	$2\bar{5}\bar{4}\bar{3}1$	166934 901421
$2\bar{3}\bar{4}5\bar{1}$	166938	$234\bar{5}1$	166939	**23451**	166941	$3\bar{5}41\bar{2}$	166942
$35\bar{4}12$	166943	$452\bar{3}1$	166945	$2543\bar{1}$	166950	$3254\bar{1}$	166951
$23\bar{4}5\bar{1}$	166955	$23\bar{4}\bar{5}\bar{1}$	166956 901718	$2\bar{3}\bar{4}51$	166956 901724	$23\bar{4}51$	166957
$2\bar{3}51\bar{4}$	166959	$2\bar{3}\bar{5}41$	166969	$2\bar{5}\bar{4}1\bar{3}$	166974	$2\bar{3}514$	166978
$3\bar{5}\bar{4}2\bar{1}$	166980	$2435\bar{1}$	166982	$2\bar{4}\bar{3}51$	166983	$2354\bar{1}$	166985 921184
35421	166985 902215	$2345\bar{1}$	166991	$2\bar{3}\bar{4}51$	166992 902202	$3\bar{5}\bar{2}4\bar{1}$	166992 902120
$4\bar{5}321$	166992 902206	$35\bar{2}\bar{4}1$	166997	$2435\bar{1}$	166998 902230	$25\bar{1}\bar{4}3$	166998 902155
$2\bar{5}\bar{1}4\bar{3}$	167001	$254\bar{1}3$	167004	$5\bar{4}32\bar{1}$	167006	$2345\bar{1}$	167008
$25\bar{4}\bar{3}\bar{1}$	167009	$45\bar{3}\bar{2}1$	167010	$2\bar{5}4\bar{3}1$	167011	$45\bar{2}\bar{3}1$	167014
$25\bar{4}\bar{1}3$	167031	$2\bar{5}\bar{4}1\bar{3}$	167034	$2\bar{4}153$	167068	$241\bar{5}\bar{3}$	167091
$4\bar{5}23\bar{1}$	167106	$2\bar{5}143$	167110	$251\bar{4}\bar{3}$	167111	$53\bar{4}2\bar{1}$	167122
$35\bar{4}1\bar{2}$	167131	**24153**	167133	$4\bar{5}231$	167139	$3\bar{4}512$	167141
$235\bar{1}\bar{4}$	167143 903551	$34\bar{5}12$	167143 903656	$23\bar{5}\bar{1}\bar{4}$	167144	$4\bar{5}\bar{2}31$	167158
$3\bar{4}51\bar{2}$	167161	$34\bar{5}1\bar{2}$	167163	$5\bar{3}\bar{4}2\bar{1}$	167188	**34512**	167202
$4523\bar{1}$	167277	$5342\bar{1}$	167300	$3\bar{5}412$	167321	$35\bar{4}\bar{2}1$	167330
$3\bar{5}42\bar{1}$	167332	$13\bar{2}5\bar{4}$	167408	$153\bar{4}\bar{2}$	167560	$2\bar{1}45\bar{3}$	167561 905557 5067054
$2\bar{1}\bar{4}53$	167561 905557 5067055	$145\bar{2}3$	167601	$2\bar{1}453$	167602 906143 5073953 29335370	$2\bar{1}\bar{4}5\bar{3}$	167602 906143 5073953 29335426
$1452\bar{3}$	167646	$35\bar{1}42$	167670	$54\bar{3}2\bar{1}$	167744	$2\bar{1}4\bar{5}3$	167748 907383 5083238 29397202
$2\bar{1}\bar{4}5\bar{3}$	167748 907383 5083238 29397203	$1534\bar{2}$	167749 907398	$325\bar{4}1$	167749 907418	$52\bar{4}\bar{3}\bar{1}$	167815
$2\bar{1}35\bar{4}$	167818 907708 5083642 29380782	$2\bar{1}3\bar{5}4$	167818 907708 5083642 29380784	$2\bar{4}\bar{5}\bar{3}\bar{1}$	167826	**24531**	167828
$4\bar{5}3\bar{2}1$	167832	$5243\bar{1}$	167833	$45\bar{3}21$	167835	$13\bar{5}24$	167844
$2\bar{4}5\bar{3}\bar{1}$	167848	$24\bar{5}31$	167850	$1354\bar{2}$	167855 908182	$14\bar{3}5\bar{2}$	167855 908181

continued

$14\bar{3}\bar{5}\bar{2}$	167863	$351\bar{4}\bar{2}$	167869	$13\bar{5}42$	167877	$3514\bar{2}$	167886
$32\bar{5}4\bar{1}$	167923	$32\bar{5}41$	167940	$2\bar{3}\bar{5}4\bar{1}$	167942 909327	$2\bar{5}\bar{4}3\bar{1}$	167942 909336
$235\bar{4}1$	167943	$254\bar{3}1$	167944	$2\bar{4}15\bar{3}$	167951	$23\bar{5}4\bar{1}$	167959
$1453\bar{2}$	167960 909582	$2\bar{3}5\bar{4}1$	167960 909568	$2\bar{3}5\bar{1}\bar{4}$	167961	$2\bar{3}\bar{5}\bar{1}4$	167962
$2\bar{4}53\bar{1}$	167963	$24\bar{5}\bar{3}1$	167965	$23\bar{5}14$	167967	$2351\bar{4}$	167968 909719
$25\bar{3}14$	167968 909740	$2\bar{4}513$	167974	$2\bar{4}\bar{5}1\bar{3}$	167977	$523\bar{4}\bar{1}$	167981 909851
$52\bar{3}4\bar{1}$	167981 909855	$2\bar{5}\bar{3}\bar{1}4$	167988	$35\bar{1}\bar{4}2$	167990	$241\bar{5}3$	167991
$2\bar{5}1\bar{4}3$	167993	$14\bar{5}\bar{2}3$	167998 910090	$2531\bar{4}$	167998 910112	$5234\bar{1}$	167998 910078
$45\bar{3}1\bar{2}$	168007	**25314**	168008 910322	$3\bar{5}\bar{4}\bar{2}1$	168008 910269	$4531\bar{2}$	168008 910276
$2514\bar{3}$	168011 910256	$453\bar{2}1$	168011 910347	$135\bar{2}\bar{4}$	168012	$4\bar{5}3\bar{2}\bar{1}$	168024
$24\bar{5}3\bar{1}$	168027	$2\bar{4}5\bar{3}1$	168029 910494	$3\bar{5}2\bar{4}1$	168029 910481	$2\bar{1}54\bar{3}$	168039 909957 5104177 29555753
$2\bar{1}\bar{5}43$	168039 909957 5104177 29555755	$2\bar{4}35\bar{1}$	168054	$24\bar{3}\bar{5}1$	168055	$24\bar{3}51$	168056
$45\bar{3}2\bar{1}$	168084	$2\bar{1}45\bar{3}$	168088 910579 5110667 29617694	$2\bar{1}\bar{4}\bar{5}3$	168088 910579 5110667 29617699	$25\bar{3}4\bar{1}$	168108
$2\bar{5}3\bar{4}1$	168109	$25\bar{4}\bar{3}\bar{1}$	168116	$2\bar{5}431$	168118	$24\bar{1}5\bar{3}$	168123
$325\bar{4}\bar{1}$	168133	$2\bar{3}5\bar{4}\bar{1}$	168134	$23\bar{5}41$	168135	$25\bar{3}1\bar{4}$	168136
$3\bar{5}\bar{4}1\bar{2}$	168137	$253\bar{4}\bar{1}$	168140	$2\bar{5}\bar{3}41$	168141	$2415\bar{3}$	168146
$24\bar{5}\bar{1}\bar{3}$	168147 911472	$2\bar{5}\bar{3}\bar{4}1$	168147 911476	$354\bar{1}2$	168152	$35\bar{4}21$	168155
$2\bar{3}\bar{5}14$	168159	$2\bar{3}51\bar{4}$	168160 911630	$25\bar{4}1\bar{3}$	168160 911639	$2534\bar{1}$	168163 911669
$4532\bar{1}$	168163 911687	$2\bar{4}5\bar{3}\bar{1}$	168166	$245\bar{1}3$	168167	$245\bar{3}1$	168168 911687
$25\bar{3}\bar{1}4$	168168 911692	$235\bar{4}\bar{1}$	168169	$2\bar{3}\bar{5}41$	168170 911718	$3\bar{5}\bar{4}2\bar{1}$	168170 911823
$24\bar{5}\bar{3}\bar{1}$	168174	$2\bar{4}531$	168176	$2\bar{5}31\bar{4}$	168177	$3\bar{5}\bar{4}\bar{2}\bar{1}$	168184
$2\bar{4}\bar{1}5\bar{3}$	168200	$154\bar{3}\bar{2}$	168202	$354\bar{2}1$	168203	$2541\bar{3}$	168207
$24\bar{5}13$	168211	$2\bar{4}\bar{5}\bar{3}1$	168212	$3\bar{5}4\bar{2}1$	168215	$35\bar{4}2\bar{1}$	168216
$3542\bar{1}$	168217	$3\bar{5}241$	168219	$2453\bar{1}$	168228	$3524\bar{1}$	168255
$2\bar{4}51\bar{3}$	168265	$145\bar{3}\bar{2}$	168266	$3\bar{2}5\bar{4}1$	168268	$24\bar{3}5\bar{1}$	168279
$24\bar{3}\bar{5}\bar{1}$	168280	$2\bar{4}351$	168281	$2\bar{5}\bar{1}43$	168292	$25\bar{3}4\bar{1}$	168296
$2\bar{5}341$	168297	$3\bar{4}5\bar{2}1$	168300	$2\bar{5}3\bar{1}4$	168304 912844	**34521**	168304 913052
$25\bar{1}\bar{4}\bar{3}$	168308 912905	$52\bar{4}3\bar{1}$	168308 912922	$35\bar{4}\bar{1}2$	168312	$34\bar{5}21$	168317 913171
$34\bar{5}2\bar{1}$	168317 913172	$23\bar{5}\bar{1}4$	168328 913181	$3\bar{5}\bar{4}\bar{1}\bar{2}$	168328 913277	$145\bar{2}\bar{3}$	168330 913130
$3\bar{4}5\bar{2}\bar{1}$	168330 913304	$2\bar{5}\bar{4}\bar{1}3$	168333	$3541\bar{2}$	168343	$235\bar{1}\bar{4}$	168344
$3\bar{4}52\bar{1}$	168353	$253\bar{1}\bar{4}$	168354	$24\bar{1}\bar{5}3$	168355	$3\bar{5}\bar{2}4\bar{1}$	168361
$253\bar{1}4$	168363 913662	$2\bar{5}\bar{4}\bar{1}3$	168363 913651	$2451\bar{3}$	168366	$2\bar{4}5\bar{1}\bar{3}$	168367
$345\bar{2}1$	168369	$2\bar{5}413$	168386	$3\bar{4}521$	168389	$352\bar{4}\bar{1}$	168394
$45\bar{3}12$	168396	$2\bar{5}4\bar{1}3$	168397	$345\bar{2}1$	168402	$3\bar{5}421$	168423
$35\bar{4}\bar{1}2$	168431	$2\bar{4}513$	168435 914602	$345\bar{2}\bar{1}$	168435 914677	$245\bar{1}3$	168438
$3\bar{2}541$	168460	$53\bar{4}\bar{2}\bar{1}$	168475	$53\bar{4}\bar{2}\bar{1}$	168486	$3451\bar{2}$	168493
$3\bar{4}5\bar{1}\bar{2}$	168509	$3\bar{5}\bar{4}12$	168515	$3\bar{5}\bar{2}41$	168521	$2\bar{4}5\bar{1}3$	168522
$3452\bar{1}$	168525	$25\bar{1}4\bar{3}$	168526	$24\bar{1}5\bar{3}$	168527 915136	$2\bar{5}\bar{1}43$	168527 915161
$34\bar{5}\bar{1}\bar{2}$	168527 915307	$3\bar{5}4\bar{1}2$	168537	$25\bar{4}1\bar{3}$	168542	$254\bar{3}\bar{1}$	168546
$2\bar{5}\bar{4}31$	168547	$3\bar{5}421$	168554	$34\bar{5}\bar{1}2$	168563	$35\bar{2}4\bar{1}$	168567
$35\bar{4}\bar{2}\bar{1}$	168583	$245\bar{3}\bar{1}$	168584	$2\bar{4}\bar{5}31$	168585	$254\bar{1}\bar{3}$	168587
$543\bar{2}\bar{1}$	168588	$45\bar{3}\bar{2}\bar{1}$	168597	$3\bar{5}\bar{1}42$	168621	$245\bar{1}\bar{3}$	168625
$35\bar{1}4\bar{2}$	168636	$452\bar{3}\bar{1}$	168648	$354\bar{2}\bar{1}$	168661	$3\bar{2}5\bar{4}\bar{1}$	168670
$354\bar{1}\bar{2}$	168670	$345\bar{2}\bar{1}$	168673	$345\bar{1}\bar{2}$	168682	$524\bar{3}\bar{1}$	168691
$35\bar{4}\bar{1}\bar{2}$	168745	$534\bar{2}\bar{1}$	168757	$35\bar{2}4\bar{1}$	168760	$45\bar{2}3\bar{1}$	168766
$453\bar{2}\bar{1}$	168820	$453\bar{1}\bar{2}$	168829				

TABLE 1. I-Wilf classes of B_5 and the numbers $|SI_n(\tau)|$ for $n = 9, 10, 11, 12$. To determine the class to which the pattern $\bar{1}\bar{4}\bar{5}23$ belongs, calculate $|SI_9(\bar{1}\bar{4}\bar{5}23)| = 168330$. This number corresponds to both the patterns $145\bar{2}\bar{3}$ and $3\bar{4}5\bar{2}\bar{1}$ above. To decide which of these is the correct one, it is necessary to calculate $|SI_{10}(\bar{1}\bar{4}\bar{5}23)| = 913130$. Thus $\bar{1}\bar{4}\bar{5}23$ belongs to the class represented by $145\bar{2}\bar{3}$.

361542	97405	465132	97511	361452	98805	351624	99133	426153	99287	146253	99321
132546	99432	125436	99521	154326	99585	153624	99650	124356	99653	123546	99729
624351	99857	625431	99885	123456	99991	623541	100021	645231	100088	632541	100156
563412	100293	623451	100615	163542	100879	463152	100992	164352	101197	125634	101405
156423	101451	145236	101662	126453	101754	163452	101918	153426	102109	135426	104236
136542	105312	124653	105971	124536	106788	154362	106857	156342	107185	125463	107578
326154	107772	134526	108083	136254	108336	265431	108967	143625	108969	145326	109293
261543	109404	143652	109443	462513	109514	132564	109674	135246	109943	136452	110137
123564	110264	134652	110707	124563	110872	135462	110964	146352	111024	143562	111229
635421	111594	264351	111647	135624	111648	263541	111733	153462	111836	124635	111871
362541	111963	125643	112058	624531	112186	462531	112231	156432	112493	261453	112598
153642	112738	253614	112805	145263	112830	246153	112962	134625	113031	326541	113101
134562	113121	463251	113154	236154	113168	263451	113331	362451	113424	164532	113439
154623	113690	136524	113837	426513	113909	136245	114046	351642	114060	236541	114071
254361	114129	462351	114245	146325	114470	256341	114598	326514	114730	146523	114833
146532	115050	364152	115051	562431	115131	251634	115165	463512	115289	564321	115297
261354	115305	243615	115357	264513	115506	365142	115532	324651	115600	635241	115605
256413	115714	243651	115741	264153	115762	634521	116018	564231	116084	154632	116098
264531	116206	365421	116214	265413	116546	241653	116580	234651	116603	135642	116656
145362	116665	562341	116676	236514	116688	235461	116747	251364	117002	645321	117190
465312	117342	234615	117530	135264	117649	234561	117661	325614	117792	256314	118369
265143	118372	231564	118450	231645	118517	346152	118533	563421	118646	326451	118724
145623	118881	465321	119049	264315	119084	246513	119204	136425	119269	251643	119284
236145	119306	261534	119411	256431	119481	426531	119592	256134	119745	236451	119864
456312	120024	356412	120049	356142	120195	364251	120269	235614	120277	254613	120434
265341	120451	362514	120655	253461	120790	246351	120922	254631	121026	365412	121073
246315	121125	465231	121289	263154	121348	145632	121395	263514	121571	251463	121692
254163	121697	235164	121719	253641	121786	263415	121892	325641	121936	246135	121959
246531	122125	356241	122422	245163	122425	426351	122452	256143	122484	436512	122608
241635	122668	364521	122725	352641	122840	235641	122894	245613	122957	245361	123195
346251	123251	463521	123375	465213	123413	456132	123474	364512	123518	456231	123756
236415	123833	356214	123835	354621	123935	365241	124192	346512	124405	356124	124936
265134	125054	265314	125541	245631	125665	365214	125736	356421	126250	345612	126268
436521	126552	346521	126743	354612	127013	456321	127598	345621	128803		

TABLE 2. I-Wilf classes of S_6 and the numbers $|I_{12}(\tau)|$