# SIGNED INVOLUTIONS AVOIDING 2-LETTER SIGNED PATTERNS

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Abstract. Let $\mathcal{I}_n$ be the class of all signed involutions in the hyperoctahedral group $\mathfrak{B}_n$ and let $\mathcal{I}_n(T)$ be the set of involutions in $\mathcal{I}_n$ which avoid a set $T$ of signed patterns. In this paper, we complete a further case of the program initiated by Simion and Schmidt [6] by enumerating $\mathcal{I}_n(T)$ for all signed permutations $T \subseteq \mathfrak{B}_2$.

## 1. introduction

Let $\mathfrak{S}_n$ and $\mathfrak{B}_n$ be the symmetric and hyperoctahedral groups, respectively, on $n$ letters. We regard elements of the hyperoctahedral group $\mathfrak{B}_n$ as signed permutations written as $\pi = \pi_1\pi_2\dots\pi_n$ in which each of the symbols $1, 2, \dots, n$ appears, possibly signed. Clearly, the cardinality of $\mathfrak{B}_n$ is $2^n n!$. We define the signing operation as the one which changes the symbol $\pi_i$ to $-\pi_i$ and $-\pi_i$ to $\pi_i$, so it is an involution, and define the absolute value notation by $|\pi_i|$ if $\pi_i$ is $\pi_i$ positive and $-\pi_i$ otherwise.

**Definition 1.1.** *A signed permutation $\pi \in \mathfrak{B}_n$ is said to contain a pattern $\alpha \in \mathfrak{B}_k$ if there exists a sequence $1 \le i(1) < \cdots < i(k) \le n$ such that*

- $\{|\pi_{i(1)}|, \dots, |\pi_{i(k)}|\}$ *is an occurrence of the pattern* $\{|\alpha_1|, \dots, |\alpha_k|\}$*, and,*
- $\pi_{i(j)} > 0$ *if and only if* $\alpha_j > 0$ *for all* $1 \le j \le k$*.*

*A signed permutation $\pi$ which does not contain such a pattern $\alpha$ is said to* **avoid** *$\alpha$.*

Let $\pi$ be any signed permutations. Writing the permutation $|\pi|$ in disjoint cycle representation, and then replacing each entry $\pi_i$ with $-\pi_i$ if $-\pi_i$ is in the range of $\pi$, we obtain a cycle representation for $\pi$. For example, the cycle representation of $\pi$ = -3-41-25 is (-3, 1)(-4, -2)(5).

Let $I_n := \{\pi \in \mathfrak{S}_n : \pi^2 = \text{id}\}$ be the set of involutions in $\mathfrak{S}_n$ and we denote the cardinality of this set by $\text{inv}_n$. These numbers satisfy the well-known recursion

$$\text{inv}_n = \text{inv}_{n-1} + (n-1)\text{inv}_{n-2}, \quad \text{inv}_0 = \text{inv}_1 = 1. \tag{1.1}$$

Let $\mathcal{I}_n := \{\pi \in \mathfrak{B}_n : \pi^2 = \text{id}\}$ be the set of signed involutions on $n$ letters (see [2]). In other words, an involution on $n$ letters in $\mathfrak{B}_n$ is a signed permutation such that its cycle representation contains cycles of either two non-signed symbols or two signed symbols. Denote by $\mathcal{I}_n(T)$ the collection of signed involutions which avoid a set $T$ of signed permutations. For example, avoiding the pattern 21 or (resp. -2-1) in a signed involution means that all the positive (resp. negative) symbols are semi-fixed points (we say that $\pi$ has a semi-fixed point at $i$ if and only if $|\pi_i| = i$), and having a pattern -21 in a signed involution implies you must have a pattern 2-1.

We define three simple operations on signed permutations: the *reversal* (i.e., reading the permutation right-to-left: $\pi_1\pi_2\dots\pi_n \mapsto \pi_n\dots\pi_2\pi_1$), the *signing* (i.e., $\pi_1\pi_2\dots\pi_n \mapsto (-\pi_1)(-\pi_2)$

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$\cdots$(-$\pi_n$)), and the *complement* (i.e., $\pi_1\pi_2\ldots\pi_n \mapsto \sigma_1\sigma_2\ldots\sigma_n$ where $\sigma_i = n+1-\pi_i$ if $\pi_i > 0$ and $-(n+1)-\pi_i$ otherwise).

Let us denote by $G_b$ the group which is generated by the signing operation and the composition of the reversal and the complement operations.

**Proposition 1.2.** *Every element $g \in G_b$ provides a bijection which shows that if $T$ and $T'$ are both sets of signed patterns such that $T' = g(T) = \{g(\alpha) \mid \alpha \in T\}$, then $|\mathcal{I}_n(T)| = |\mathcal{I}_n(T')|$.*

In the symmetric group $\mathfrak{S}_n$, for every 2-letter pattern $\tau$ the number of $\tau$-avoiding permutations is 1, and for every pattern $\tau \in S_3$ the number of $\tau$-avoiding permutations is given by the Catalan numbers. Simion [5, Section 3] proved there are similar results for the hyperoctahedral group $\mathfrak{B}_n$ (generalized by Mansour [3]), for every 2-letter signed pattern $\tau$ the number of $\tau$-avoiding signed permutations is given by $\sum_{j=0}^{n}\binom{n}{j}^2 j!$. Mansour and West [4] enumerated the collections of signed permutations that avoid a signed pattern $T$, $\mathfrak{B}_n(T)$, for all possible $T \subseteq \mathfrak{B}_2$. In the present paper, we find the cardinalities of $\mathcal{I}_n(T)$ for all possible $T \subseteq \mathfrak{B}_2$. (The exhaustive treatment of cases was suggested by the influential paper of Simion and Schmidt [6], which followed a similar program for the cardinalities $|\mathfrak{S}_n(T)|$ where $T \subseteq S_3$).

## 2. The $|T| = 1$ cases for $T \subset \mathfrak{B}_2$

Taking advantage of Proposition 1.2, the question of determining the values $|\mathcal{I}_n(\tau)|$ for the 8 choices of a single 2-letter signed pattern, namely 12, 1-2, -12, -1-2, 21, 2-1, -21 and -2-1, can be reduced to 4 cases, which are $|\mathcal{I}_n(12)|$, $|\mathcal{I}_n(1\text{-}2)|$, $|\mathcal{I}_n(21)|$ and $|\mathcal{I}_n(2\text{-}1)|$. These numbers, for $n \le 9$ are given in Figure 1.

| $\lvert\mathcal{I}_n(\tau)\rvert_{n\ge 0}$ | $\tau$ |
|---|---|
| 1, 2, 5, 14, 43, 142, 499, 1850, 7193, 29186 | 12, 1-2, 21, -2-1, -1,-2, -1 2 |
| 1, 2, 6, 18, 58, 190, 642, 2206, 7746, 27662 | 2-1, -2 1 |

Figure 1. The values of $|\mathcal{I}_n(\tau)|$ for $n = 0, 1, \ldots, 9$ and $\tau \in \mathfrak{B}_2$.

**Theorem 2.1.** *For all $n \ge 0$,*

$$|\mathcal{I}_n(12)| = |\mathcal{I}_n(21)| = |\mathcal{I}_n(1\text{-}2)|.$$

*Proof.* We begin by proving $|\mathcal{I}_n(12)| = |\mathcal{I}_n(21)|$ for all $n \ge 0$. To do this we define a function $p : \mathcal{I}_n(12) \to \mathcal{I}_n(21)$ as follows: given $\pi = \pi_1\pi_2\ldots\pi_n \in \mathcal{I}_n(12)$ we define $p(\pi)$ by

$$p(\pi)_m = \begin{cases} \pi_m, & \text{if } \pi_m < 0, \\ n+1-\pi_m, & \text{otherwise.} \end{cases}$$

In other words, the map $p$ acts on $\pi$ by reversing the order of the unsigned symbols in $\pi$. For example, if $\pi = 6\text{-}543\text{-}21$ then $p(\pi) = 1\text{-}534\text{-}26$. From this definition it can be seen easily that $p$ is an involution, that is, $p^2$ is the identity function on $\mathcal{I}_n(12)$.

Now let us prove that $|\mathcal{I}_n(12)| = |\mathcal{I}_n(1\text{-}2)|$ for all $n \ge 0$. To do so we recursively define a function $f : \mathcal{I}_n(12) \to \mathcal{I}_n(1\text{-}2)$ as follows: given a signed permutation $\pi = \pi_1\pi_2\ldots\pi_n \in \mathcal{I}_n(12)$ we consider the four cases:

(1) if $\pi_1 = $ -1, then let $f(\pi) = $ -1$(f(\pi'-1)+1)$, where $\pi' = \pi_2\ldots\pi_n$ and for any sequence $\beta$ of signed numbers and positive number $a$ we define two operations

$$(\beta-a)_i = \begin{cases} \beta_i, & |\beta_i| \le a \\ \beta_i - 1, & \beta_i > a \\ \beta_i + 1, & \beta_i < -a \end{cases} \quad \text{and } (\beta+a)_i = \begin{cases} \beta_i + 1, & \beta_i \ge a \\ \beta_i - 1, & \beta_i < -a \end{cases}$$

(2) if $\pi_1 = \text{-}t$ with $t > 1$, then $f(\pi) = \alpha$, where $\alpha_1 = \text{-}t$, $\alpha_t = \text{-}1$, and

$$(\alpha_2, \ldots, \alpha_{t-1}, \alpha_{t+1}, \ldots, \alpha_n) = (f(((\pi_2, \ldots, \pi_{t-1}, \pi_{t+1}, \ldots, \pi_n) - t) - 1) + 1) + (t-1).$$

(3) if $\pi_1 = 1$, then $\pi_m$ is a signed number for all $m > 1$. So, $f(\pi)$ can be defined as the signed permutation $(1, -\pi_2, \ldots, -\pi_n)$.

(4) if $\pi_1 = t$ (so $\pi_t = 1$), then define $\alpha = f(\pi)$ as follows: $\alpha_m = \begin{cases} \pi_m, & \pi_m < 0, \\ -\pi_m, & \pi_m > 0, \end{cases}$ for all $m$ such that $|\pi_m| > t$ or $m > t$, and $\alpha_1 = t$, $\alpha_t = 1$. To define the remaining values of $\alpha$, let $i_1, i_2, \ldots, i_s$ be the positions such that $|\pi_{i_j}| < t$ and $1 < i_j < t$. Let $\pi'$ be the signed permutation of length $s$ which is order-isomorphic to $\pi_{i_1}, \ldots, \pi_{i_s}$. Then $(\alpha_{i_1}, \ldots, \alpha_{i_s}) = \beta$, where $\beta$ is the signed permutation that is order-isomorphic to $f(\pi')$ and $|f(\pi')|_j \in \{|\pi_{i_1}|, \ldots, |\pi_{i_s}|\}$ for all $j = 1, 2, \ldots, s$.

For example, if $\pi = \text{-}1\,\text{-}3\,\text{-}2\,6\,5\,4\,\text{-}7$ then using (1), (2) and (4) we obtain that

$$f(\pi) = \text{-}1(f(\text{-}2\,\text{-}1\,5\,4\,3\,\text{-}6) + 1) = \text{-}1\,\text{-}3\,\text{-}2(f(3\,2\,1\,\text{-}4) + 3) = \text{-}1\,\text{-}3\,\text{-}2\,\text{-}6\,\text{-}5\,\text{-}4\,7.$$

It is easy to see that $f^2 = id$. Hence $f$ is bijection between $\mathcal{I}_n(12)$ and $\mathcal{I}_n(1\,\text{-}2)$, as required. □

We now turn our attention to the sets $\mathcal{I}_n(21)$ and $\mathcal{I}_n(\text{-}2\,1)$.

**Proposition 2.2.** *The exponential generating function for the numbers $|\mathcal{I}_n(21)|_{n\geq 0}$ is*

$$\sum_{n\geq 0} |\mathcal{I}_n(21)| \frac{x^n}{n!} = \exp\left(2x + \frac{x^2}{2}\right).$$

*Proof.* Let $\pi \in \mathcal{I}_n$ be an involution which avoids 21 with exactly $j$ unsigned symbols. Since $\pi$ avoids 21, the unsigned symbols form an increasing subsequence, and since $\pi$ is an involution we have that if $\pi_m$ is positive, then $\pi_m = m$. Hence the number of involutions in $\mathcal{I}_n(21)$ is exactly $\sum_{j=0}^{n} \binom{n}{j} \text{inv}_j$. Thus the exponential generating function for the number involutions in $\mathfrak{B}_n$ that avoid 21 is $\exp(2x + x^2/2)$. □

To enumerate the second class of signed involutions, the set $\mathcal{I}_n(\text{-}2\,1)$, we require some further definitions. Indeed, these definitions may be used for the more general problem of enumerating $\mathcal{I}_n(T)$, for general subsets $T$ of signed patterns in $\mathfrak{B}_k$. Given $a_1, a_2, \ldots, a_d \in \mathbb{Z}$, we define

$$\mathcal{I}_{n;a_1,a_2,\ldots,a_d}(T) = \{\pi_1\pi_2\ldots\pi_n \in \mathcal{I}_n(T) \mid \pi_1\pi_2\ldots\pi_d = a_1a_2\ldots a_d\}.$$

As a direct consequence of the above definitions, we have

$$|\mathcal{I}_n(T)| = \sum_{j=1}^{n} |\mathcal{I}_{n;j}(T)| + \sum_{j=1}^{n} |\mathcal{I}_{n;-j}(T)|. \tag{2.1}$$

Also, we need the following lemma which holds immediately by induction and (1.1).

**Lemma 2.3.** *Let $d_n(t) = d_{n-1}(t-1) + (t-4)d_{n-2}(t-2)$, for all $4 \leq t \leq n-3$. Then*

$$d_n(t) = inv_{t-3} d_{n+3-t}(3),$$

*where $inv_{t-3}$ is the number of involutions in $I_{t-3}$.*

Using this decomposition and the above lemma, we may now enumerate the signed permutations in $\mathcal{I}_n(2\,\text{-}1)$.

**Proposition 2.4.** *Let $c_n := |\mathcal{I}_n(2\,\text{-}1)|$. The numbers satisfy $c_0 := 1$, $c_1 = 2$, $c_2 = 6$ and for $n > 2$,*

$$c_n \;=\; 2c_{n-1} + nc_{n-2} - \sum_{j=1}^{n-3} j \cdot inv_{n-2-j} c_j.$$

*Proof.* Define $c_n(t_1, \ldots, t_d) := |\mathcal{I}_{n;t_1,\ldots,t_d}(2\text{-}1)|$ for any $n, d$ and $a_n(t) := c_n(t) + c_n(\text{-}t)$. It is not difficult to see that $a_n(1) = 2c_{n-1}$, $a_n(2) = 2c_{n-2}$, and for $t \geq 3$,

$$\begin{aligned} c_n(t) &= c_n(t,2) + \textstyle\sum_{j=3}^{t-1} c_n(t,j) + \sum_{j=t+1}^{n} c_n(t,j) \\ &= c_{n-1}(t-1) + (t-3)c_{n-2}(t-2) + \textstyle\sum_{j=t-1}^{n-2} c_{n-2}(j), \end{aligned}$$

and

$$\begin{aligned} c_n(\text{-}t) &= c_n(\text{-}t,\text{-}2) + \textstyle\sum_{j=3}^{t-1} c_n(\text{-}t,\text{-}j) + \sum_{j=t+1}^{n} c_n(\text{-}t,\text{-}j) \\ &= c_{n-1}(\text{-}(t-1)) + (t-3)c_{n-2}(\text{-}(t-2)) + \textstyle\sum_{j=t-1}^{n-2} c_{n-2}(\text{-}j). \end{aligned}$$

Thus

$$a_n(t) = a_{n-1}(t-1) + (t-3)a_{n-2}(t-2) + \textstyle\sum_{j=t-1}^{n-2} a_{n-2}(j).$$

If $d_n(t) = a_n(t) - a_n(t-1)$, then from the above recurrence relation we obtain that

$$\begin{aligned} d_n(3) &= a_n(3) - a_n(2) \\ &= a_{n-1}(2) + \textstyle\sum_{j=2}^{n-2} a_{n-2}(j) - a_n(2) \\ &= 2c_{n-3} + c_{n-2} - a_{n-2}(1) - 2c_{n-2} \\ &= 2c_{n-3} + c_{n-2} - 2c_{n-3} - 2c_{n-2} = -c_{n-2}, \end{aligned}$$

and $d_n(t) = d_{n-1}(t-1) + (t-4)d_{n-2}(t-2)$, for $t \geq 4$. Thus Lemma 2.3 gives $d_n(t) = \text{inv}_{t-3} d_{n+3-t}(3)$. Since $c_n = \sum_{t=1}^{n} a_n(t)$, see (2.1), we have

$$c_n - 2c_{n-1} - 2c_{n-2} = (n-2)c_{n-2} + \sum_{t=4}^{n}\sum_{i=1}^{t-3} \text{inv}_i d_{n-i}(3),$$

which is equivalent to $c_n \;=\; 2c_{n-1} + nc_{n-2} - \sum_{j=1}^{n-3} j \cdot \text{inv}_{n-2-j} c_j$, as required. □

## 3. The $|T| = 2$ cases for $T \subseteq \mathfrak{B}_2$

By appealing to Proposition 1.2 again, the second question of determining the values $\mathcal{I}_n(\tau^1, \tau^2)$ for the 28 choices of two 2-letter signed patterns reduces to the following 12 cases:

$$\begin{array}{lclllcllcl} B_1 &=& \{1\,2,\ 1\,\text{-}2\}; & B_1^{(1)} &=& \{1\,2,\ 2\,1\}; & B_2 &=& \{1\,2,\ \text{-}1\,\text{-}2\}; \\ B_2^{(1)} &=& \{1\,2,\ \text{-}2\,\text{-}1\}; & B_2^{(2)} &=& \{2\,1,\ \text{-}2\,\text{-}1\}; & B_3 &=& \{1\,2,\ 2\,\text{-}1\}; \\ B_4 &=& \{1\,\text{-}2,\ \text{-}1\,2\}; & B_5 &=& \{1\,\text{-}2,\ 2\,1\}; & B_6 &=& \{1\,\text{-}2,\ 2\,\text{-}1\}; \\ B_6^{(1)} &=& \{\text{-}1\,2,\ 2\,\text{-}1\}; & B_7 &=& \{2\,1,\ 2\,\text{-}1\}; & B_8 &=& \{2\,\text{-}1\,\text{-}2\,1\}. \end{array}$$

Labels with equal subscripts in the collections above denote equicardinality of the numbers, i.e. $|\mathcal{I}_n(B_1)| = |\mathcal{I}_n(B_1^{(1)})|$ etc., which we now prove.

**Proposition 3.1.** *We have*

*(1) there exists a bijection between the set $\mathcal{I}_n(12, 1\,\text{-}2)$ and the set $I_{n+1}$ of involutions in $S_{n+1}$;*

*(2) there exists a bijection between the set $\mathcal{I}_n(12, 21)$ and the set $I_{n+1}$.*

*Proof.* To see (1) we define a function $p : \mathcal{I}_n(12, 1\,\text{-}2) \rightarrow I_{n+1}$ as follows: Given $\pi = \pi_1\pi_2 \ldots \pi_n \in \mathcal{I}_n(12, 1\,\text{-}2)$, either $\pi_1 < 0$ or $\pi_1 = n$. Therefore, we define $p(\pi)$ as follows:

- if $\pi_1 = -1$ then $p(\pi) = 1(p(\pi_2 \ldots \pi_n - 1) + 1)$,

- if $\pi_1 = n$ (so $\pi_n = 1$), then $p(\pi) = (n+1)(p(\pi_2 \ldots \pi_{n-1} - 1) + 1)1$,
- if $\pi_1 = -t$ with $t > 1$ (so $\pi_t = -1$), then $p(\pi) = \alpha$ where $\alpha_1 = t$, $\alpha_t = 1$, and $\alpha_2 \ldots \alpha_{t-1}\alpha_{t+1} \ldots \alpha_n = (p((\pi_2 \ldots \pi_{t-1}\pi_{t+1} \ldots \pi_n - t) - 1) + 1) + t$.

From the definition of $p$ and induction on $n$ with the initial conditions $p(1) = 21$ and $p(-1) = 12$, the fact that $p$ is a bijection is easily verified. For example, if $\pi = \text{-}1\,\text{-}4\,5\,\text{-}2\,3 \in \mathcal{I}_5(12, 1\,\text{-}2)$ then $p(\pi) = 1(p(\text{-}3\,4\,\text{-}1\,2) + 1) = 14x2y$ with $xy = (p(21) + 1) + 1 = 53$, thus $p(\pi) = 14523$.

To prove (2) let $\pi = \pi_1\pi_2 \ldots \pi_n \in \mathcal{I}_n(12, 21)$, so $\pi$ contains at most one unsigned element. More precisely, either there exists an unique $m$ such that $\pi_m = m > 0$ or for all $m$ we have $\pi_m < 0$. Now to prove our result we define a function $p : \mathcal{I}_n(12, 21) \to I_{n+1}$ as follows: for $\pi = \pi_1\pi_2 \ldots \pi_n \in \mathcal{I}_n(12, 21)$ with no unsigned elements we define $p(\pi) = (-\pi_1)(-\pi_2) \ldots (-\pi_n)(n+1)$, and for $\pi$ with exactly one unsigned element, say $\pi_m > 0$, we define $p(\pi) = \alpha$, where $\alpha_m = n+1$, $\alpha_{n+1} = m$, and $\alpha_1 \ldots \alpha_{m-1}\alpha_{m+1} \ldots \alpha_n = p(\pi_1 \ldots \pi_{m-1}\pi_{m+1} \ldots \pi_n - m) + m$. □

**Proposition 3.2.** *For all $n \geq 0$,*

$$|\mathcal{I}_n(21, \text{-}2\,\text{-}1)| = |\mathcal{I}_n(12, \text{-}1\,\text{-}2)| = |\mathcal{I}_n(21, \text{-}1\,\text{-}2)| = 2^n.$$

*Proof.* Using the same argument as Proposition 2.2, let $\pi \in \mathcal{I}_n$ be an involution which avoids the patterns 21 and -2-1. Suppose further that $\pi$ has $j$ unsigned symbols and $n - j$ signed symbols. If $\pi_m$ is positive, then $\pi_m = m$, and similarly if $\pi_m$ is negative then $\pi_m = -m$ (i.e. there can be no transpositions in the permutation since transpositions in signed involutions always lead to a 21 or -2-1 pattern.) Thus $|\mathcal{I}_n(21, \text{-}2\,\text{-}1)| = \sum_{j=0}^{n} \binom{n}{j} = 2^n$. Similarly, the other cases hold. □

**Proposition 3.3.** *We have* $\sum_{n\geq 0} |\mathcal{I}_n(21, 1\,\text{-}2)|\, x^n = \left(1 + \int_0^x e^{-t^2/2}dt\right) e^{x+x^2/2}$.

*Proof.* Let $\pi \in \mathcal{I}_n$ be an involution of length $n$ which avoids the patterns 21 and 1-2. It is easy to see the first letter of $\pi$, $\pi_1$ must be either $1$, $-1$, or $-t < -1$. This implies that for all $n \geq 2$, $|\mathcal{I}_n(21, 1\,\text{-}2)| = |\mathcal{I}_{n-1}(21, 1\,\text{-}2)| + (n-1)\,|\mathcal{I}_{n-2}(21, 1\,\text{-}2)| + 1$ with the initial condition $|\mathcal{I}_0(21, 1\,\text{-}2)| = 1$. The rest is easy to check. □

**Proposition 3.4.** *For all $n \geq 1$,* $|\mathcal{I}_n(12, 2\,\text{-}1)| \;=\; inv_n + \sum_{k=0}^{n-1} inv_k \sum_{m=0}^{n-1-k} inv_{n-1-k-m}$.

*Proof.* A simple argument shows that any permutation $\pi \in \mathcal{I}_n(12, 2\,\text{-}1)$ has all elements signed (thereby contributing $\text{inv}_n$ permutations), or there are elements which are unsigned. In this case, the permutation $\pi$ essentially consists of three blocks $\pi = \alpha\beta\gamma$ of sizes $k, m$ and $n - (k + m)$, respectively, where elements in $\alpha$ and $\gamma$ are signed and those in $\beta$ are unsigned. Since there is no occurrence of 2-1 (by the cycle representation of $\pi$, also there no occurrence of -2 1) we find the modulus of all elements in $\alpha$ are less than those in $\beta$, and in turn the modulus of those in $\beta$ are less than those in $\gamma$. Now since 12 is forbidden, the letters of $\beta$ must be strictly decreasing and since they are consecutive also, there is only one such configuration. There are, however, $\text{inv}_k\text{inv}_{n-k-m}$ choices for $\alpha$ and $\gamma$. Summing over $k$ and $m$ yields the result. □

**Proposition 3.5.** *Let $a_n := |\mathcal{I}_n(1\,\text{-}2, \text{-}12)|$. Then for all $n \geq 1$,* $a_n = 2inv_n + \sum_{j=1}^{\lfloor (n-1)/2 \rfloor} j!a_{n-2j}$ *with the initial condition $a_0 = 1$.*

*Proof.* The involutions in $\mathcal{I}_n(1\,\text{-}2, \text{-}12)$ may have all signs the same, in which case there are $2\text{inv}_n$ involutions of this type. Otherwise there are sign changes within the involution.

Let $\pi \in \mathcal{I}_n(1\,\text{-}2, \text{-}12)$ be such an involution and suppose the first sign-change occurs at position $j$, i.e. $sgn(\pi_1) = \ldots = sgn(\pi_j) \neq sgn(\pi_{j+1})$, for some $1 \leq j \leq \lfloor (n-1)/2 \rfloor$. Then

$$\pi \;=\; (\tau^{-1} + (n-j), \sigma + j, \tau)$$

where $\sigma \in \mathcal{I}_{n-2j}(1\text{-}2,\text{-}12)$ and $\tau$ is a permutation of $\{1,\ldots,j\}$ whose entries have sign $\neq sgn(\sigma)$.

It is easy to see why this latter condition must hold, for if $c \le j$ is the smallest entry which does not appear in $|\tau_1|,\ldots,|\tau_j|$, then there exists $c' > j$ which appears in $\{|\pi_{n-j}|,\ldots,|\pi_n|\}$, thereby forming the '2' in an occurrence of -1-2 (if $sgn(\pi_1 = +1)$ or 1-2 (if $sgn(\pi_1) = -1$).

There are $a_{n-2j}$ choices for $\sigma$, and $j!$ choices for $\tau$ (the sign of all entries in $\tau$ being different to $\sigma_1$). Summing over all $j$ we have

$$a_n = 2\mathrm{inv}_n + \sum_{j=1}^{\lfloor (n-1)/2 \rfloor} j! a_{n-2j}$$

with the initial condition $a_0 = 1$. □

**Proposition 3.6.** *For all $n \ge 0$, $|\mathcal{I}_n(1\text{-}2,2\text{-}1)| = |\mathcal{I}_n(\text{-}1\,2,2\text{-}1)| = \sum_{k=0}^{n} inv_k inv_{n-k}$.*

*Proof.* First, let us prove that there exists a bijection $p : \mathcal{I}_n(1\text{-}2,2\text{-}1) \to \mathcal{I}_n(\text{-}1\,2,2\text{-}1)$. Let $\pi = \pi_1\pi_2\ldots\pi_n \in \mathcal{I}_n(1\text{-}2,2\text{-}1)$. If $\pi_1 > 0$ then $\pi \in \mathfrak{S}_n$ and let $p(\pi)_i = \text{-}\pi_i$ for all $1 \le i \le n$, so that $p(\pi)$ avoids both -1 2 and 2-1. If $\pi_1 = -t < 0$ then let $\pi'$ be the signed involution on the elements $\pi_i$ of $\pi$ with $|\pi_i| > -\pi_1$ and $i > t$. It is easy to see that each element in $\pi$, but not in $\pi'$, is signed. Now define $p(\pi)$ by signing each element of $\pi$ which is not in $\pi'$ and then map the elements of $\pi'$ by $p(\pi')$. Hence, by induction on $n$ and the definition of $p$ we have that $p$ is a signing operation, thus $p$ is a bijection. For instance, if $\pi = (\text{-}3,\text{-}4,\text{-}1,\text{-}2,6,5) \in \mathcal{I}_6(1\text{-}2,2\text{-}1)$ then $p(\pi) = (3,4,1,2,\text{-}6,\text{-}5) \in \mathcal{I}_6(\text{-}1\,2,2\text{-}1)$ ($\pi' = 65$).

To count the number of elements of the set $\mathcal{I}_n(\text{-}1\,2,2\text{-}1)$ we require some new definitions. Let $c_n(t_1,\ldots,t_d) := |\mathcal{I}_{n;t_1,\ldots,t_d}(\text{-}1\,2,2\text{-}1)|$. It is not hard to see that $c_n(-1) = \mathrm{inv}_{n-1}$ and $c_n(-t) = \mathrm{inv}_{n-2}$ with $t = 2,3,\ldots,n$. Also, $c_n(1) = c_{n-1}$, $c_n(2) = c_{n-2}$, and for $t \ge 3$,

$$\begin{aligned} c_n(t) &= c_n(t,2) + \textstyle\sum_{j=3}^{t-1} c_n(t,j) + \sum_{j=t+1}^{n} c_n(t,j) \\ &= c_{n-1}(t-1) + (t-3)c_{n-2}(t-2) + \textstyle\sum_{j=t-1}^{n-2} c_{n-2}(j). \end{aligned}$$

If $d_n(t) = c_n(t) - c_n(t-1)$, then from the above recurrence relation we obtain that

$$\begin{aligned} d_n(3) &= c_n(3) - c_n(2) \\ &= c_{n-1}(2) + \textstyle\sum_{j=2}^{n-2} c_{n-2}(j) - c_n(2) \\ &= c_{n-3} + c_{n-2} - c_{n-2}(1) - \textstyle\sum_{j=1}^{n-2} c_{n-2}(-j) - c_{n-2} \\ &= c_{n-3} + c_{n-2} - c_{n-3} - \mathrm{inv}_{n-3} - (n-3)\mathrm{inv}_{n-4} - c_{n-2} \\ &= -\mathrm{inv}_{n-3} - (n-3)\mathrm{inv}_{n-4} = -\mathrm{inv}_{n-2}, \end{aligned}$$

and for $t \ge 4$,

$$d_n(t) = d_{n-1}(t-1) + (t-4)d_{n-2}(t-2).$$

Thus Lemma 2.3 gives $d_n(t) = \mathrm{inv}_{t-3} d_{n+3-t}(3)$. Since $c_n = \sum_{t=1}^{n}(c_n(t) + c_n(-t))$ (see Equation (2.1)), using Equation (1.1) we have

$$c_n = c_{n-1} + (n-1)c_{n-2} + \mathrm{inv}_{n-1} + \mathrm{inv}_{n-2} - \sum_{t=3}^{n}\sum_{j=4}^{t} \mathrm{inv}_{j-4}\mathrm{inv}_{n+1-j},$$

which is equivalent to $c_n = c_{n-1} + (n-1)c_{n-2} + \mathrm{inv}_{n-1} + \mathrm{inv}_{n-2} - \sum_{j=1}^{n-3} j \cdot \mathrm{inv}_{n-2-j}\mathrm{inv}_j$. Substituting $c_n = a_n + \sum_{k=0}^{n} \mathrm{inv}_k \mathrm{inv}_{n-k}$ and using (1.1) we arrive at $a_n = a_{n-1} + (n-2)a_{n-2}$ with $a_0 = a_1 = 0$. Hence $c_n = \sum_{k=0}^{n} \mathrm{inv}_k \mathrm{inv}_{n-k}$, as required. □

**Proposition 3.7.** *Let $c_n := |\mathcal{I}_n(21, 2\,\text{-}1)|$. The numbers satisfy $c_0 := 1$ and*

$$c_n \;=\; 2c_{n-1} + (n-1)c_{n-2} - \sum_{j=1}^{n-2} j \cdot inv_{n-2-j}c_{j-1}.$$

*Proof.* Let $c_n(t_1,\ldots,t_d) = |\mathcal{I}_{n;t_1,\ldots,t_d}(21, 2\,\text{-}1)|$. It is not hard to see that $c_n(1) = c_n(\text{-}1) = c_{n-1}$, $c_n(2) = 0$, $c_n(\text{-}2) = c_{n-2}$, $c_n(t) = 0$ and

$$\begin{array}{ll} c_n(-t) & = c_n(-t,-2) + \sum_{j=3}^{t-1} c_n(-t,-j) + \sum_{j=t+1}^{n} c_n(-t,-j) \\ & = c_{n-1}(-(t-1)) + (t-3)c_{n-2}(-(t-2)) + \sum_{j=t-1}^{n-2} c_{n-2}(-j). \end{array},$$

for all $t \geq 3$. If $d_n(t) = c_n(-t) - c_n(-(t-1))$, then from the above recurrence relation we obtain that

$$\begin{array}{ll} d_n(3) & = c_n(\text{-}3) - c_n(\text{-}2) \\ & = c_{n-1}(\text{-}2) + \sum_{j=2}^{n-2} c_{n-2}(\text{-}j) - c_n(\text{-}2) \\ & = c_{n-3} + (c_{n-2} - c_{n-2}(\text{-}1) - c_{n-2}(1)) - c_{n-2} \\ & = c_{n-3} + c_{n-2} - c_{n-3} - c_{n-3} - c_{n-2} \\ & = -c_{n-3}, \end{array}$$

and $d_n(t) = d_{n-1}(t-1) + (t-4)d_{n-2}(t-2)$ for $t \geq 4$. Thus Lemma 2.3 gives $d_n(t) = \text{inv}_{t-3}d_{n+3-t}(3)$. Since $c_n = \sum_{t=1}^{n}(c_n(t) + c_n(-t))$ we have

$$c_n = 2c_{n-1} + (n-1)c_{n-2} - \sum_{t=3}^{n}\sum_{j=3}^{t} \text{inv}_{j-3}c_{n-j},$$

which is equivalent to $c_n \;=\; 2c_{n-1} + (n-1)c_{n-2} - \sum_{j=1}^{n-2} j \cdot \text{inv}_{n-2-j}c_{j-1}$. □

**Proposition 3.8.** *For all $n \geq 0$,*

$$|\mathcal{I}_n(2\,\text{-}1, \text{-}2\,1)| \;=\; c_n,$$

*where $c_0 = 1$ and $c_n = 2c_{n-1} + nc_{n-2} - \sum_{j=1}^{n-3} j \cdot inv_{n-3-j}c_j$.*

*Proof.* To count the number of elements of the set $\mathcal{I}_n(2\,\text{-}1, \text{-}2\,1)$ let us define $c_n(t_1,\ldots,t_d)$ to be the number of involutions $\pi = \pi_1 \ldots \pi_n \in \mathcal{I}_n(2\,\text{-}1, \text{-}2\,1)$ such that $|\pi_j| = t_j$ for all $j = 1, 2, \ldots, d$. It is not hard to see that $c_n(1) = 2c_{n-1}$, $c_n(2) = 2c_{n-2}$, and for $t \geq 3$,

$$\begin{array}{ll} c_n(t) & = c_n(t,2) + \sum_{j=3}^{t-1} c_n(t,j) + \sum_{j=t+1}^{n} c_n(t,j) \\ & = c_{n-1}(t-1) + (t-3)c_{n-2}(t-2) + \sum_{j=t-1}^{n-2} c_{n-2}(j). \end{array}$$

If $d_n(t) = c_n(t) - c_n(t-1)$, then from the above recurrence relation we obtain that

$$\begin{array}{ll} d_n(3) & = c_n(3) - c_n(2) \\ & = c_{n-1}(2) + \sum_{j=2}^{n-2} c_{n-2}(j) - c_n(2) \\ & = c_{n-3} + (c_{n-2} - c_{n-2}(1)) - 2c_{n-2} \\ & = c_{n-3} + c_{n-2} - c_{n-3} - 2c_{n-2} \\ & = -c_{n-2}, \end{array}$$

and $d_n(t) = d_{n-1}(t-1) + (t-4)d_{n-2}(t-2)$, for $t \geq 4$. Thus Lemma 2.3 gives $d_n(t) = \text{inv}_{t-3}d_{n+3-t}(3)$. Since $c_n = \sum_{t=1}^{n} c_n(t)$ we have $c_n = 2c_{n-1} + nc_{n-2} - \sum_{t=4}^{n}\sum_{j=4}^{t} \text{inv}_{j-4}c_{n+1-j}$, which is equivalent to $c_n = 2c_{n-1} + nc_{n-2} - \sum_{j=1}^{n-3} j \cdot \text{inv}_{n-3-j}c_j$. □

## 4. The $|T| = 3$ case for $T \subset \mathfrak{B}_2$

Proposition 1.2 reduces determining the values $|\mathcal{I}_n(\tau^1, \tau^2, \tau^3)|$ for the 56 choices of two 2-letter signed patterns to the 20 cases listed below. The following proposition helps reduce repetitive arguments in what follows.

**Proposition 4.1.** *If $X \subset \mathfrak{B}_2$ contains 2-1 (resp. -21) but not -21 (resp. 2-1) and $Y := X \cup \{2\text{-}1, \text{-}21\}$, then $|\mathcal{I}_n(X)| = |\mathcal{I}_n(Y)|$. Denote this relation by $Y \sim X$.*

*Proof.* It is easy to see that an involution $\pi$ contains the pattern 2-1 if and only if it contains the pattern -21. □

Let
$$\begin{array}{llll}
C_1 = \{12, 1\text{-}2, \text{-}12\}; & C_2 = \{12, 1\text{-}2, \text{-}1\text{-}2\}; & C_3 = \{12, 1\text{-}2, 21\}; \\
C_4 = \{12, 1\text{-}2, 2\text{-}1\}; & C_4^{(1)} = \{12, 1\text{-}2, \text{-}21\}; & C_4^{(2)} = \{1\text{-}2, 21, 2\text{-}1\}; \\
C_4^{(3)} = \{1\text{-}2, 21, \text{-}21\}; & C_5 = \{12, 1\text{-}2, \text{-}2\text{-}1\}; & C_6 = \{12, \text{-}1\text{-}2, 21\}; \\
C_6^{(1)} = \{12, 21, \text{-}2\text{-}1\}; & C_6^{(2)} = \{1\text{-}2, 21, \text{-}2\text{-}1\}; & C_7 = \{12, \text{-}1\text{-}2, 2\text{-}1\}; \\
C_8 = \{12, 21, 2\text{-}1\}; & C_9 = \{12, 2\text{-}1, \text{-}21\}; & C_{10} = \{12, 2\text{-}1, \text{-}2\text{-}1\}; \\
C_{11} = \{1\text{-}2, \text{-}12, 21\}; & C_{12} = \{1\text{-}2, \text{-}12, 2\text{-}1\}; & C_{13} = \{1\text{-}2, 2\text{-}1, \text{-}21\}; \\
C_{14} = \{21, 2\text{-}1, \text{-}21\}; & C_{15} = \{21, 2\text{-}1, \text{-}2\text{-}1\}. &
\end{array}$$

We remind the reader that those classes listed above with the same subscripts will be shown to be equi-numerous.

**Proposition 4.2.** *The numbers $a_n = |\mathcal{I}_n(12,\ 1\text{-}2,\ \text{-}12)|$ are given by*
$$a_n = inv_n + ((n-1)/2)! + \sum_{k=0}^{(n-2)/2} k! a_{n-2-2k},$$
*where $a! := 0$ when $a$ is not a nonnegative integer.*

*Proof.* For case $C_1$: Let $\pi \in \mathcal{I}_n(12, 1\text{-}2, \text{-}12)$. Either $\pi$ is an involution of $\text{-}1, \text{-}2, \ldots, \text{-}n$, <u>or</u> $\pi = \beta((n+1)/2)\beta^{-1}$ with $\beta^{-1}$ a permutation of $\text{-}1, \text{-}2, \ldots, \text{-}((n-1)/2)$, <u>or</u> $\pi = \beta(n-m)\gamma(m+1)\beta^{-1}$ with $\beta$ a permutation of $\text{-}1, \text{-}2, \ldots, \text{-}m$ and $\gamma$ a signed involution of $\text{-}(m+2), \text{-}(m+2), \ldots, \text{-}(n-m-1)$ that avoid 12, 1-2, and -12. Hence the number of such signed involution $a_n$ satisfies the recurrence relation $a_n = \text{inv}_n + ((n-1)/2)! + \sum_{k=0}^{(n-2)/2} k! a_{n-2-2k}$, where $a! = 0$ when $a$ is not an integer with initial values $a_0 = 1$, $a_1 = 2$ and $a_2 = 3$. □

**Proposition 4.3.** *For all $n \geq 0$, $|\mathcal{I}_n(12, 1\text{-}2, \text{-}1\text{-}2)| = \binom{n+1}{\lceil n/2 \rceil}$.*

*Proof.* For case $C_2$: Let $a_n(t_1, \ldots, t_d) := |\mathcal{I}_{n;t_1,\ldots,t_d}(12, 1\text{-}2, \text{-}1\text{-}2)|$. It is not hard to see that $a_n(t) = 0$ for all $t = 1, 2, \ldots, n-1$, and $a_n(n) = a_n(\text{-}n) = a_{n-2}$ the number of such involutions of length $n-2$. Also, $a_n(\text{-}1) = 1$, and for all $t = 2, 3, \ldots, n-1$,
$$a_n(\text{-}t) = a_n(\text{-}t, n) + a_n(\text{-}t, \text{-}2) + \sum_{j=3}^{t-1} a_n(\text{-}t, \text{-}j) = a_{n-2}(\text{-}(t-1)) + a_{n-2}(\text{-}1) + \sum_{j=3}^{t-1} a_{n-2}(\text{-}(j-1)),$$
which is equivalent to
$$a_n(\text{-}t) = \sum_{j=1}^{t-1} a_{n-2}(\text{-}j). \tag{4.1}$$

Now, define $A_n(v)$ to the the polynomial $\sum_{t=1}^{n}(a_n(t)+a_n(\text{-}t))v^{t-1}$. Multiplying (4.1) by $v^{t-1}$ and summing over all $t=2,\dots,n-1$ we arrive at

$$A_n(v)=1+2v^{n-1}A_{n-2}(1)+\frac{v}{1-v}(A_{n-2}(v)-v^{n-2}A_{n-2}(1))-v^{n-2}A_{n-4}(1),$$

for all $n\geq 4$ with the initial conditions; $A_0(v)=1$, $A_1(v)=2$, $A_2(v)=1+2v$, and $A_3(v)=1+v+4v^2$. Multiplying the above recurrence relation by $x^n/v^n$ and summing over $n\geq 4$ we obtain the functional equation

$$\left(1-\frac{x^2}{v(1-v)}\right)A(x/v,v)=\frac{v^2(v+x)-x^2v(1+x)+x^4}{v^2(v-x)}+\frac{x^2(v-2v^2-x^2(1+v))}{v^2(1-v)}A(x,1).$$

This type of equation can be solved systematically using the *kernel method* [1]. Substitute $v=\frac{1+\sqrt{1-4x^2}}{2}$ in this equation to get $A(x,1)=\frac{4x^2-1+\sqrt{1-4x^2}}{x^2(1-2x)}=\sum_{n\geq 0}\binom{n+1}{\lceil n/2\rceil}x^n$. □

**Proposition 4.4.** *For all $n\geq 0$,*

$$\begin{aligned}|\mathcal{I}_n(12,1\text{-}2,21)| &= inv_n+\textstyle\sum_{k=0}^{(n-1)/2}k!\binom{n-k-1}{k}inv_{n-1-2k}\\ |\mathcal{I}_n(12,1\text{-}2,2\text{-}1)| &= \textstyle\sum_{j=0}^{n}inv_j\\ |\mathcal{I}_n(12,1\text{-}2,\text{-}2\text{-}1)| &= F_{n+2}.\end{aligned}$$

*where $F_n$ is the $n^{th}$ Fibonacci number given by; $F_0=0$, $F_1=1$ and $F_n=F_{n-1}+F_{n-2}$.*

*Proof.* For case $C_3$: Let $\pi\in\mathcal{I}_n(12,1\text{-}2,21)$, so either $\pi$ is an involution of -1, -2, …, -$n$, or $\pi$ contains exactly one positive element $\pi_{n-k}$ and $\pi_{n-k}=n-k$ is a fixed point. From this, it is easy to see that the number of such signed involutions is given by

$$\text{inv}_n+\sum_{k=0}^{(n-1)/2}k!\binom{n-k-1}{k}\text{inv}_{n-1-2k}.$$

For case $C_4$: If $\pi\in\mathcal{I}_n(12,1\text{-}2,2\text{-}1)$ (resp. $\pi\in\mathcal{I}_n(12,1\text{-}2,\text{-}21)$), then $\pi=\beta n(n-1)\dots(m+1)$, where $\beta$ is an involution of the numbers -1, -2, …, -$m$. Hence the number of signed involutions in $\mathcal{I}_n(12,1\text{-}2,2\text{-}1)$ (resp. $\pi\in\mathcal{I}_n(12,1\text{-}2,\text{-}21)$) is given by $\sum_{m=0}^{n}\text{inv}_m$. If $\pi\in\mathcal{I}_n(1\text{-}2,21,2\text{-}1)$ (or $\pi\in\mathcal{I}_n(1\text{-}2,21,\text{-}21)$), then $\pi=\beta(m+1)(m+2)\dots n$, where $\beta$ is an involution of the numbers -1, -2, …, -$m$. Thus the number of the signed involutions in $\mathcal{I}_n(1\text{-}2,21,2\text{-}1)$ (resp. $\mathcal{I}_n(1\text{-}2,21,\text{-}21)$) is given by $\sum_{m=0}^{n}\text{inv}_m$.

For case $C_5$: Let $\pi$ be any signed involution of length $n$ that avoids 12, 1-2 and -2-1. The first letter of $\pi$ must be either -1 or $n$, giving us $|\mathcal{I}_n(C_5)|=|\mathcal{I}_{n-1}(C_5)|+|\mathcal{I}_{n-2}(C_5)|$ with initial conditions $\mathcal{I}_0(C_5)=1$, $\mathcal{I}_1(C_5)=2$. Hence $\mathcal{I}_n(C_5)$ is $F_{n+2}$, the $(n+1)^{\text{th}}$ Fibonacci number. □

**Proposition 4.5.** *For all $n>2$,*

$$\begin{aligned}
|\mathcal{I}_n(12,\text{-}1\text{-}2,21)| &= n+1, & |\mathcal{I}_n(12,21,2\text{-}1)| &= inv_n+\textstyle\sum_{k=0}^{n-1}inv_k inv_{n-1-k},\\
|\mathcal{I}_n(12,\text{-}1\text{-}2,2\text{-}1)| &= 2n, & |\mathcal{I}_n(12,2\text{-}1,\text{-}21)| &= inv_n+\textstyle\sum_{k=0}^{n-1}inv_k\sum_{j=0}^{n-1-k}inv_j,\\
|\mathcal{I}_n(1\text{-}2,\text{-}12,2\text{-}1)| &= 2inv_n, & |\mathcal{I}_n(1\text{-}2,\text{-}12,21)| &= inv_n+\textstyle\sum_{k=1}^{n}((n-k)/2)!,\\
|\mathcal{I}_n(21,\text{-}2\text{-}1,2\text{-}1)| &= 2^n, & |\mathcal{I}_n(1\text{-}2,2\text{-}1,\text{-}21)| &= |\mathcal{I}_n(1\text{-}2,2\text{-}1)|,\\
|\mathcal{I}_n(21,2\text{-}1,\text{-}2\text{-}1)| &= 2^n, & |\mathcal{I}_n(21,2\text{-}1,\text{-}21)| &= |\mathcal{I}_n(21,2\text{-}1)|,
\end{aligned}$$

*where $a!:=0$ when $a$ is not a nonnegative integer.*

*Proof.* For case $C_6$: It is not hard to see that

$$\mathcal{I}_n(12,\text{-}1\,\text{-}2,21) = \{\text{-}n\,\text{-}(n-1)\dots\text{-}(n\text{-}m\text{-}2)m\,\text{-}(n\text{-}m+1)\dots\text{-}1) \mid m=0,\dots,n\} \quad (4.2)$$
$$\mathcal{I}_n(12,21,\text{-}2\,\text{-}1) = \{\text{-}1\,\text{-}2\dots\text{-}(m-1)m\,\text{-}(m+1)\dots\text{-}n) \mid m=0,\dots,n\} \quad (4.3)$$
$$\mathcal{I}_n(1\,\text{-}2,21,\text{-}2\,\text{-}1) = \{\text{-}1\,\text{-}2\dots\text{-}m(m+1)\dots n) \mid m=0,\dots,n\}, \quad (4.4)$$

giving $|\mathcal{I}_n(12,\text{-}1\,\text{-}2,21)| = |\mathcal{I}_n(12,21,\text{-}2\,\text{-}1)| = |\mathcal{I}_n(1\,\text{-}2,21,\text{-}2\,\text{-}1)| = n+1$.

For case $C_7$: One can easily notice that $\mathcal{I}_n(12,\text{-}1\,\text{-}2,2\,\text{-}1)$ is $\{m(m-1)\dots1\,\text{-}n\,\text{-}(n-1)\dots\text{-}(m+1) \mid m=0,\dots,n\} \cup \{\text{-}n\,\text{-}(n-1)\dots\text{-}(m+1)m\dots1 \mid m=1,...,n-1\}$, hence $|\mathcal{I}_n(12,\text{-}1\,\text{-}2,2\,\text{-}1)| = 2n$.

For case $C_8$: If $\pi\in\mathcal{I}_n(12,21,2\,\text{-}1)$, then either $\pi$ is an involution of -1,-2, $\dots$, $\text{-}n$ or $\pi=\beta(m+1)\gamma$, where $\beta$ is an involution of -1,-2, $\dots$, $\text{-}m$ and $\gamma$ is an involution of $\text{-}(m+2),\text{-}(m+3),\dots,\text{-}n$. The number of such signed involutions is $\text{inv}_n+\sum_{k=0}^{n-1}\text{inv}_k\text{inv}_{n-1-k}$.

For case $C_9$: Let $\pi\in\mathcal{I}_n(12,2\,\text{-}1,\text{-}2\,1)$. Either $\pi$ is an involution of -1,-2, $\dots$, $\text{-}n$ or $\pi=\beta(n-k)(n-k-1)\dots(m+1)\gamma$, where $\beta$ is an involution of -1,-2, $\dots$, $\text{-}m$ and $\gamma$ is an involution of $\text{-}(n-k+1),\text{-}(n-k+2),\dots,\text{-}n$. The number of such signed involutions is thus $\text{inv}_n+\sum_{k=0}^{n-1}\text{inv}_k\sum_{j=0}^{n-1-k}\text{inv}_j$.

For case $C_{10}$: Let $\pi\in\mathcal{I}_n(21,\text{-}2\,\text{-}1,2\,\text{-}1)$ and $a_n=|\mathcal{I}_n(21,\text{-}2\,\text{-}1,2\,\text{-}1)|$. Since $\pi$ avoids 21 and -2-1, $\pi$ cannot contain any transpositions. Thus $\pi$ contains only fixed points, each of which may be signed or unsigned. Hence $a_n=2^n$.

For case $C_{11}$: Let $\pi\in\mathcal{I}_n(1\,\text{-}2,\text{-}1\,2,21)$, so all the positive elements of $\pi$ are increasing giving us that all the positive elements of $\pi$ are fixed points. Thus $\pi$ has one of the following forms: either $\pi$ is an involution of -1,-2, $\dots$ $\text{-}n$ or $\pi=\beta(m+1)(m+2)\dots(n-m)\beta^{-1}$ where $\beta$ any permutation of $\text{-}(n-m+1),\dots,\text{-}n$ and $n\ge 2m+1$. Hence the number of such involutions is $\text{inv}_n+\sum_{k=1}^{n}((n-k)/2)!$.

For case $C_{12}$: Every signed involution avoiding 1-2, -12 and 2-1 must be an involution of either $1,2,\dots,n$ or -1,-2, $\dots$, $\text{-}n$. Hence $|\mathcal{I}_n(1\,\text{-}2,\text{-}12,2\,\text{-}1)| = 2\text{inv}_n$.

For cases $C_{13}$ and $C_{14}$: By Proposition 4.1, we have $C_{13}\sim B_6$ and $C_{14}\sim B_7$. For case $C_{15}$: This follows directly from Proposition 3.2 and we have $|\mathcal{I}_n(21,2\text{-}1,\text{-}2\text{-}1)|=2^n$. ☐

## 5. The $|T|\ge 4$ case for $T\subset\mathfrak{B}_2$

The 70 choices of four 2-letter signed patterns reduce to 26 in light of Proposition 1.2. Let

$D_1 = \{12,1\text{-}2,\text{-}12,\text{-}1\text{-}2\}$; $D_2 = \{12,1\text{-}2,\text{-}12,21\}$; $D_3 = \{12,1\text{-}2,\text{-}1\text{-}2,21\}$;
$D_3^{(1)} = \{1\text{-}2,\text{-}12,\text{-}1\text{-}2,21\}$; $D_4 = \{12,1\text{-}2,\text{-}12,2\text{-}1\}$; $D_4^{(1)} = \{1\text{-}2,\text{-}12,21,2\text{-}1\}$;
$D_5 = \{12,1\text{-}2,\text{-}1\text{-}2,2\text{-}1\}$; $D_5^{(1)} = \{12,\text{-}12,\text{-}1\text{-}2,2\text{-}1\}$; $D_5^{(2)} = \{1\text{-}2,\text{-}1\text{-}2,21,2\text{-}1\}$;
$D_5^{(3)} = \{\text{-}12,\text{-}1\text{-}2,21,2\text{-}1\}$; $D_5^{(4)} = \{12,21,2\text{-}1,\text{-}2\text{-}1\}$; $D_5^{(5)} = \{1\text{-}2,21,2\text{-}1,\text{-}2\text{-}1\}$;
$D_5^{(6)} = \{\text{-}12,21,2\text{-}1,\text{-}2\text{-}1\}$; $D_6 = \{12,1\text{-}2,\text{-}1\text{-}2,21,2\text{-}1\}$; $D_6^{(1)} = \{12,\text{-}12,\text{-}1\text{-}2,21,2\text{-}1\}$;
$D_7 = \{12,\text{-}1\text{-}2,21,2\text{-}1\}$; $D_8 = \{12,1\text{-}2,2\text{-}1,\text{-}21\}$; $D_8^{(1)} = \{1\text{-}2,21,2\text{-}1,\text{-}21\}$;
$D_9 = \{1\text{-}2,\text{-}12,2\text{-}1,\text{-}21\}$; $D_{10} = \{12,\text{-}1\text{-}2,2\text{-}1,\text{-}21\}$; $D_{11} = \{12,21,2\text{-}1,\text{-}21\}$;
$D_{12} = \{\text{-}1\text{-}2,21,2\text{-}1,\text{-}21\}$; $D_{13} = \{12,1\text{-}2,21,\text{-}2\text{-}1\}$; $D_{13}^{(1)} = \{1\text{-}2,\text{-}12,21,\text{-}2\text{-}1\}$;
$D_{14} = \{12,\text{-}1\text{-}2,21,\text{-}2\text{-}1\}$; $D_{15} = \{21,2\text{-}1,\text{-}21,\text{-}2\text{-}1\}$.

Due to the large number of restrictions, quite a few of the sequences which enumerate these classes are the same as $|T|=3$ cases via a simple application of Proposition 4.1. We list these here without proof and then concentrate on the new sequences which occur:

$$D_8 \sim C_4,\ D_8^{(1)} \sim C_4,\ D_9 \sim C_{12},\ D_{10} \sim C_7,\ D_{11} \sim C_8,\ D_{12} \sim C_{10},\ D_{15} \sim C_{15}.$$

**Proposition 5.1.** *For $n > 2$,*

$$\begin{array}{rclrcl} |\mathcal{I}_n(D_1)| &=& 2^{\lceil n/2 \rceil}, & |\mathcal{I}_n(D_2)| &=& inv_n + ((n-1)/2)!, \\ |\mathcal{I}_n(D_3)| &=& \lfloor (n+3)/2 \rfloor, & |\mathcal{I}_n(D_4)| &=& inv_n + 1, \\ |\mathcal{I}_n(D_5)| &=& n+1, & |\mathcal{I}_n(D_6)| &=& inv_n + inv_{n-1}, \\ |\mathcal{I}_n(D_7)| &=& 3, & |\mathcal{I}_n(D_{13})| &=& 2, \\ |\mathcal{I}_n(D_{14})| &=& 0, & & & \end{array}$$

*where $a! := 0$ when $a$ is not a nonnegative integer.*

*Proof.* For case $D_1$: If $\pi \in \mathcal{I}_n(12, 1\text{-}2, \text{-}1\,2, \text{-}1\,\text{-}2)$ then $|\pi_1| > |\pi_2| > \cdots > |\pi_n|$. Since each pair of elements $\pi_i$ and $\pi_{n+1-i}$, for $i = 1, 2, \ldots, \lceil n/2 \rceil$, can be signed or unsigned we find that $|\mathcal{I}_n(12, 1\text{-}2, \text{-}1\,2, \text{-}1\,\text{-}2)| = 2^{\lceil n/2 \rceil}$.

For case $D_2$: If we consider the number positive elements in a signed permutation $\pi \in \mathcal{I}_n(12, \text{-}1\,2, 1\text{-}2, 21)$, then either all the elements of $\pi$ are negative or all the elements of $\pi$ are negative except $\pi_{(n+1)/2} = (n+1)/2$. In the first case $\pi$ is an involution of -1, -2, ..., -$n$ and in the second case $\pi = \beta((n+1)/2)\beta^{-1}$ where $\beta$ is a permutation of the elements $n, n-1, \ldots, (n-1)/2$. Thus the number of signed involutions in $\mathcal{I}_n(12, \text{-}1\,2, 1\text{-}2, 21)$ is given by the formula $\mathrm{inv}_n + ((n-1)/2)!$.

For case $D_3$: This class is easily enumerated by ensuring involutions given in Equation (4.2) avoid the pattern 1-2. This corresponds to the cases $m = 0$ and $n \geq m > (n+1)/2$. The number of such $m$ is $1 + \lfloor (n+1)/2 \rfloor$.

For case $D_3^{(1)}$: An involution $\pi$ in this class must have either $\pi_1 = 1$ or $\pi_1 = \text{-}n$. If $\pi_1 = 1$ then all remaining entries of $\pi$ must be positive and therefore fixed points. However if $\pi_1 = \text{-}n$, then $\pi_n = \text{-}1$ and the remaining entries of $\pi$ must avoid the four patterns in this class. Thus we have $|\mathcal{I}_n(D_3^{(1)})| = 1 + |\mathcal{I}_{n-2}(D_3^{(1)})|$. Using the initial conditions $|\mathcal{I}_2(D_3^{(1)})| = 2$ and $|\mathcal{I}_3(D_3^{(1)})| = 3$, we have the size of this class to be $\lfloor (n+3)/2 \rfloor$, the same as $D_3$.

For case $D_4$: This follows by adding the restriction that involutions in $C_{12}$ cannot contain the pattern 12. Thus either involutions have all negative signs, or all positive signs. If the involution has all positive signs, it must avoid the pattern 12 and so there is only one such involutions, namely $(n, n-1, \ldots, 1)$. Thus $|\mathcal{I}_n(D_4)| = \mathrm{inv}_n + 1$.

For case $D_5$: This is straightforward by considering the exact form of the involutions in $\mathcal{I}_n(C_6)$ given in (4.2–4.4): $|\mathcal{I}_n(D_5)| = |\mathcal{I}_n(D_5^{(1)})| = \ldots = |\mathcal{I}_n(D_5^{(6)})| = n+1$.

For case $D_6$: By considering the involutions $\pi \in \mathcal{I}_n(D_6)$ avoiding 12, 1-2, 21 and 2-1, we must have that either $\pi_n = n$ (in which case the signs of all other elements are - and so there are $\mathrm{inv}_{n-1}$ of them) or $\pi_n = -i$ for some $i \leq n$ (in which case all elements of the involution are -, accounting for $\mathrm{inv}_n$ involutions). Thus $|\mathcal{I}_n(D_6)| = \mathrm{inv}_n + \mathrm{inv}_{n-1}$.

For case $D_7$: Only 3 involutions in $\mathcal{I}_n(C_7)$ avoid 21, hence $|\mathcal{I}_n(D_7)| = 3$. These involutions are;

$$(1\,\text{-}n\,\text{-}(n-1) \cdots \text{-}2),\ (\text{-}(n-1)\ \cdots\ \text{-}1\,n) \text{ and } (\text{-}n\,\text{-}(n-1) \cdots \text{-}1).$$

For case $D_{13}$: This is similar to case $C_5$ except the first letter of any such involution $\pi_1 \neq n$ (for otherwise it would contain a 21 pattern). Thus $\pi_1 = \text{-}1$ and we have $|\mathcal{I}_n(D_{13})| = |\mathcal{I}_{n-1}(D_{13})|$ with initial condition $|\mathcal{I}_1(D_{13})| = 2$.

For case $D_{14}$: Since $D_{14} \supset C_6$, from (4.2) for $n > 2$ it is impossible to avoid the pattern -2-1, hence $|\mathcal{I}_n(D_{14})| = 0$. □

The 56 choices for five 2-letter signed patterns reduce to 20 in light of Proposition 1.2. Many of these cases are simply degenerate cases of $|T| \le 4$. Let

$$\begin{array}{llllll}
E_1 &=& \{12,1\text{-}2,\text{-}12,\text{-}1\text{-}2,21\}; & E_2 &=& \{12,1\text{-}2,\text{-}12,\text{-}1\text{-}2,2\text{-}1\};\\
E_2^{(1)} &=& \{12,1\text{-}2,\text{-}1\text{-}2,21,2\text{-}1\}; & E_2^{(2)} &=& \{12,\text{-}12,\text{-}1\text{-}2,21,2\text{-}1\};\\
E_2^{(3)} &=& \{1\text{-}2,\text{-}12,\text{-}1\text{-}2,21,2\text{-}1\}; & E_2^{(4)} &=& \{12,1\text{-}2,21,2\text{-}1,\text{-}2\text{-}1\};\\
E_2^{(5)} &=& \{12,\text{-}12,21,2\text{-}1,\text{-}2\text{-}1\}; & E_2^{(6)} &=& \{1\text{-}2,\text{-}12,21,2\text{-}1,\text{-}2\text{-}1\};\\
E_3 &=& \{12,1\text{-}2,\text{-}12,21,2\text{-}1\}; & E_4 &=& \{12,1\text{-}2,\text{-}12,2\text{-}1,\text{-}21\};\\
E_4^{(1)} &=& \{1\text{-}2,\text{-}12,21,2\text{-}1,\text{-}21\}; & E_5 &=& \{12,1\text{-}2,\text{-}1\text{-}2,2\text{-}1,\text{-}21\};\\
E_5^{(1)} &=& \{1\text{-}2,\text{-}1\text{-}2,21,2\text{-}1,\text{-}21\}; & E_5^{(2)} &=& \{12,21,2\text{-}1,\text{-}21,\text{-}2\text{-}1\};\\
E_5^{(3)} &=& \{1\text{-}2,21,2\text{-}1,\text{-}21,\text{-}2\text{-}1\}; & E_6 &=& \{12,1\text{-}2,21,2\text{-}1,\text{-}21\};\\
E_7 &=& \{12,\text{-}1\text{-}2,21,2\text{-}1,\text{-}21\}; & E_8 &=& \{12,1\text{-}2,\text{-}12,21,\text{-}2\text{-}1\};\\
E_9 &=& \{12,1\text{-}2,\text{-}1\text{-}2,21,\text{-}2\text{-}1\}; & E_{10} &=& \{12,\text{-}1\text{-}2,21,2\text{-}1,\text{-}2\text{-}1\}.
\end{array}$$

The patterns equinumerous with the $D$ patterns are: $E_4 \sim D_4$, $E_4^{(1)} \sim D_4^{(1)}$, $E_5 \sim D_5$, $E_5^{(1)} \sim D_5^{(2)}$, $E_5^{(2)} \sim D_5^{(4)}$, $E_5^{(3)} \sim D_5^{(5)}$, $E_6 \sim D_6$, $E_7 \sim D_7$. Also trivial to see are $|\mathcal{I}_n(E_8)| = 1$ (since $E_8 \supset D_{13}$) and $|\mathcal{I}_n(E_9)| = |\mathcal{I}_n(E_{10})| = 0$ (since $E_9, E_{10} \supset D_{14}$). The non-trivial cases are

**Proposition 5.2.** $|\mathcal{I}_n(12,1\text{-}2,\text{-}12,\text{-}1\text{-}2,21)| = 1+(n \mod 2)$, $|\mathcal{I}_n(12,1\text{-}2,\text{-}12,\text{-}1\text{-}2,2\text{-}1)| = 2$ *and* $|\mathcal{I}_n(12,1\text{-}2,\text{-}12,21,2\text{-}1)| = inv_n$.

*Proof.* For case $E_1$: If $\pi \in \mathcal{I}_n(E_1)$, we must have $|\pi_i| = (n+1-i)$ for all $1 \le i \le n$. If $n$ is even, then $\pi_i < 0$ for all $i$ whereas if $n$ is odd we are allowed one unsigned letter $\pi_{(n+1)/2} = (n+1)/2$. Hence $|\mathcal{I}_n(E_1)| = 1 + (n \mod 2)$.

For cases $E_2$–$E_2^{(6)}$: We consider only $\mathcal{I}_n(E_2)$ since the other 6 cases follow by the same reasoning. If $\pi \in \mathcal{I}_n(E_2)$, then $\pi$ may only contain the patterns 2 1, -2 1, -2 -1. From this it is clear that $|\pi_i| = n+1-i$ for all $1 \le i \le n$, and since $\pi$ avoids 2-1 iff it also avoids -21, we must have either all letters of $\pi$ signed or all letters unsigned. Thus

$$|\mathcal{I}_n(E_2)| = |\mathcal{I}_n(E_2^{(1)}| = \ldots = |\mathcal{I}_n(E_2^{(6)}| = 2.$$

For case $E_3$: Permutations in $\mathcal{I}_n(E_3)$ may only contain the patterns -1-2 and -2-1 and -21. In this case the pattern -21 is irrelevant since 2-1 is forbidden. This class thus contains all signed involutions containing only -1-2, -2-1, of which there are $\text{inv}_n$. □

Proposition 1.2 reduces the 28 choices for six 2-letter signed patterns to 12.

Let
$$\begin{array}{lllllllll}
F_1 &=& \mathfrak{B}_2\backslash\{\text{-}21,\text{-}2\text{-}1\}, & F_1^{(1)} &=& \mathfrak{B}_2\backslash\{\text{-}1\text{-}2,\text{-}21\} & F_2 &=& \mathfrak{B}_2\backslash\{21,\text{-}2\text{-}1\},\\
F_2^{(1)} &=& \mathfrak{B}_2\backslash\{\text{-}12,\text{-}2\text{-}1\}, & F_2^{(2)} &=& \mathfrak{B}_2\backslash\{12,\text{-}2\text{-}1\}, & F_2^{(3)} &=& \mathfrak{B}_2\backslash\{\text{-}12,21\},\\
F_2^{(4)} &=& \mathfrak{B}_2\backslash\{12,\text{-}1\text{-}2\}, & F_3 &=& \mathfrak{B}_2\backslash\{\text{-}1\text{-}2,\text{-}2\text{-}1\}, & F_4 &=& \mathfrak{B}_2\backslash\{\text{-}21,2\text{-}1\},\\
F_5 &=& \mathfrak{B}_2\backslash\{\text{-}12,\text{-}21\}, & F_5^{(1)} &=& \mathfrak{B}_2\backslash\{1\text{-}2,\text{-}21\}, & F_6 &=& \mathfrak{B}_2\backslash\{\text{-}12,1\text{-}2\}.
\end{array}$$

**Proposition 5.3.** *For* $n > 2$, $|\mathcal{I}_n(F_1)| = 1$, $|\mathcal{I}_n(F_2)| = 2$, $|\mathcal{I}_n(F_3)| = inv_n$ *and* $|\mathcal{I}_n(F_i)| = 0$ *if* $i > 3$.

*Proof.* For case $F_1$: Permutations in $\mathcal{I}_n(F_1)$ may only contain the patterns {-21, -2-1}, but since an involution contains the pattern -21 iff it contains the pattern 2-1, the involutions in this class can only contain the pattern -2-1, of which there is only one. $F_1^{(1)}$ holds in this way also and we have $|\mathcal{I}_n(F_1)| = |\mathcal{I}_n(F_1^{(1)})| = 1$. For case $F_2$: Permutations in $\mathcal{I}_n(F_2)$ may only contain the patterns {21, -2-1} giving $\pi_i = n+1-i$ for all $1 \leq i \leq n$, or $\pi_i = -(n+1-i)$ for all $1 \leq i \leq n$, thereby giving 2 involutions. Cases $F_2^{(2)}$–$F_2^{(4)}$ are similar to this and so $|\mathcal{I}_n(F_2)| = 2$. For case $F_3$: Since $F_3 \sim E_3$, $|\mathcal{I}_n(F_3)| = \mathrm{inv}_n$.

For cases $F_4$–$F_6$: Notice that $F_4, F_5 \supset E_9$ and $F_5^{(1)}$ contains the sign change of $E_9$, so all three classes $\mathcal{I}_n(F_4), \mathcal{I}_n(F_5)$ and $\mathcal{I}_n(F_5^{(1)})$ are empty. $|\mathcal{I}_n(F_6)| = 0$ since $F_6 \supset E_{10}$ and $|\mathcal{I}_n(E_{10})| = 0$. □

These extreme avoidances for $|T| = 7$ are easy, with the $\mathcal{I}_n(\mathfrak{B}_2\backslash\{12\})$, $\mathcal{I}_n(\mathfrak{B}_2\backslash\{\text{-1-2}\})$, $\mathcal{I}_n(\mathfrak{B}_2\backslash\{21\})$ and $\mathcal{I}_n(\mathfrak{B}_2\backslash\{\text{-2-1}\})$ as the only non-empty sets containing exactly one involution each.

We mention that the techniques used in this paper may also be used to calculate the corresponding numbers for the number of signed involutions in $\mathfrak{B}_n$ with no fixed points, $\pi_i = i$, and no semi-fixed points, $\pi_i = \pm i$. We omit the details.

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