FLUCTUATIONS OF THE LOCAL MAGNETIC FIELD IN FRUSTRATED MEAN-FIELD ISING MODELS

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Abstract. We consider fluctuations of the local magnetic field in frustrated mean-field Ising models. Frustration can come about due to randomness of the interaction as in the Sherrington-Kirkpatrick model, or through fixed interaction parameters but with varying signs. We consider central limit theorems for the fluctuation of the local magnetic field values w.r.t. the *a priori* spin distribution for both types of models. We show that, in the case of the Sherrington-Kirkpatrick model there is a central limit theorem for the local magnetic field, a.s. with respect to the randomness. On the other hand, we show that, in the case of non-random frustrated models, there is no central limit theorem for the distribution of the values of the local field, but that the probability distribution of this distribution does converge. We compute the moments of this probability distribution on the space of measures and show in particular that it is not Gaussian.

1. Frustrated Ising models and the local field distribution

The celebrated Sherrington-Kirkpatrick model of a spin glass is given by the Hamiltonian

$$H_{SK} = \frac{1}{\sqrt{N}} \sum_{i,j=1}^{N} J_{i,j} s_i s_j, \tag{1.1}$$

where the $s_i = \pm 1$ are Ising spins and the interaction parameters $J_{i,j}$ are i.i.d. random variables with Gaussian distribution. It was proposed and solved in [1, 2] by Sherrington and Kirkpatrick using the replica trick. However, their solution is flawed because it predicts negative entropy at low temperatures. An alternative solution scheme was proposed by Parisi [3, 4], which is generally regarded as being correct. However, it also involves the mathematically dubious replica trick and the mathematical status of the solution is therefore still unclear. Indeed, this model presents a considerable challenge to mathematicians [9]. Nevertheless, some progress has been made. Aizenman, Lebowitz and Ruelle [6] proved that, in the absence of an external field, the Sherrington-Kirkpatrick (SK) solution is correct in the high temperature domain. Pastur and Schcherbina [7] proved that the SK solution is correct unless the Edwards-Anderson order parameter is not self-averaging (which implies the latter). Guerra [8] derived a beautiful inequality which implies that the the SK solution is correct in the high-temperature domain,

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even in the presence of an external field, and recently, Talagrand [10, 11] extended Guerra's bounds in various ways.

A different class of models with frustrated interactions was introduced by Eisele and Ellis [14, 15, 16]. Their models have Hamiltonians of the form

$$H_{EE} = \frac{1}{N}\sum_{i,j=1}^{N} J(\frac{i}{N} - \frac{j}{N})s_i s_j, \tag{1.2}$$

where $J(x)$ is a bounded continuous function. If this function takes on both positive and negative values, for example $J(x) = \cos(x)$, then the model is frustrated. A detailed study of these models was made in [14, 15] using large deviation theory.

Here we consider the local magnetic field fluctuations for both types of models. Unsurprisingly, we find that the two models behave quite differently, but remarkably, the SK model behaves in a more regular way in that a central limit theorem holds with probability 1. More precisely, we define the occupation measure for the fluctuations by

$$\mu_N = \frac{1}{N}\sum_{i=1}^{N} \delta_{\frac{1}{\sqrt{N}}\sum_{j=1}^{N} J_{i,j}s_j} \tag{1.3}$$

and show that μ_N converges a.s. in the parameters $J_{i,j}$, in probability with respect to the *a priori* ('flat') distribution of the spins to a normal distribution as $N \to \infty$. This weak result (which does not seem to have been observed before) is a prerequisite for a large deviation property for these variables, which would be of interest in considering the thermodynamic limit of the model. The local fields play a pivotal role in mean-field models, see also the approach of Thouless, Anderson and Palmer [12] and Mézard, Parisi and Virasoro [13].

On the other hand, in the models introduced by Eisele and Ellis we show that μ_N does not converge, even in probability. In fact, we consider a slight variant of their models, which does not quite fit within their class, but is more closely related to the SK model. It is given by the interaction parameters

$$J_{i,j} = \begin{cases} -1, & \text{if } 0 < |i-j| \le M \text{ or } |i-j| \ge N-M; \\ 0, & \text{if } i = j; \\ +1, & \text{if } M < |i-j| < N-M = 3M+1. \end{cases} \tag{1.4}$$

Here $N = 4M + 1$. If the measure (1.3) were to converge to a stable law γ then we would have

$$\mathbb{E}\left[e^{i\langle f,\mu_N\rangle}\right] \to \delta_\gamma\left[e^{i\langle f,\mu\rangle}\right] = e^{i\langle f,\gamma\rangle} \tag{1.5}$$

for continuous functions f. We will show that the distribution of the measures μ_N does converge but (1.5) does not hold. Indeed, we compute all moments $\lim_{N\to\infty} \mathbb{E}[\langle f,\mu_N\rangle^k]$ and show that the series converges for bounded continuous functions f. We then prove that this implies the convergence of the probability distribution of the measures μ_N to a nontrivial measure on the set $\mathcal{M}^1(\mathbb{R})$ of probability measures on $\mathbb{R}$. (Notice that for fixed ferromagnetic couplings $J_{i,j} = J$, we have

$$\mathbb{E}\left(e^{i\langle \mu_N, f\rangle}\right) = \frac{1}{2^N}\sum_{\{s_i\}} e^{i\, f\left(\frac{J}{\sqrt{N}}\sum_{j=1}^N s_j\right)} = \int e^{if(Jx)}\gamma(dx) \tag{1.6}$$

which is also not the same as (1.5). In this case the limiting distribution is a Gaussian distribution of δ-measures.)

Probability measures on the space of states were introduced by Aizenman and Wehr [17] for general translation-invariant random spin systems and their importance in the case of short-range spin-glasses has been argued at length by Newman and Stein [18, 19, 20]. Our example shows that such measures can be remarkably complicated even in very simple situations.

2. The local field distribution in the SK model

We prove that $\mu_N \to \gamma$ in probability w.r.t. the spin configurations a.s. with respect to the randomness. This can also be formulated as follows:

Theorem 2.1. *The random measures μ_N defined by (1.3) where the parameters $J_{i,j}$ are i.i.d. random variables with standard normal distribution satisfy $\mathbb{E}[F(\mu_N)] \to F(\gamma)$ a.s. with respect to the distribution of the coupling parameters, for any continuous function F on the space of probability measures $\mathcal{M}^1(\mathbb{R})$ with the topology of weak convergence, where γ is the Gaussian measure on $\mathbb{R}$ with mean zero and variance 1, i.e. $\gamma(dx) = e^{-x^2/2}\frac{dx}{\sqrt{2\pi}}$.*

Proof. We first prove convergence for functions F of the form $F(\mu) = e^{i\langle\mu,f\rangle}$. This can be done by proving convergence of the moments $\mathbb{E}[\langle\mu_N, f\rangle^p]$. To prove that these converge to $\langle\gamma, f\rangle^p$ almost surely, we compute

$$\overline{(\mathbb{E}[\langle\mu_N, f\rangle^p] - \langle\gamma, f\rangle^p)^2}, \tag{2.7}$$

where the 'overline' denotes the average over the random couplings. (In case of long expressions, we also use the notation $[\cdots]^-$.) First consider the first moment ($p = 1$). We put $f(x) = e^{itx}$ so that $\langle\gamma, f\rangle = e^{-t^2/2} = c(t)$, and compute

$$\overline{(\mathbb{E}[\langle\mu_N, f\rangle] - \langle\gamma, f\rangle)^2} = \overline{\left(\frac{1}{N}\sum_{i=1}^N\prod_{j=1}^N \cos\left(\frac{t}{\sqrt{N}}J_{i,j}\right) - c(t)\right)^2}. \tag{2.8}$$

Now,

$$\overline{\prod_{j=1}^N \cos\left(\frac{t}{\sqrt{N}}J_{i,j}\right)} = \prod_{j=1}^N \frac{1}{2}\left(\overline{e^{i\frac{t}{\sqrt{N}}J_{i,j}}} + \overline{e^{-i\frac{t}{\sqrt{N}}J_{i,j}}}\right) = \prod_{j=1}^N e^{-\frac{1}{2N}t^2} = c(t) \tag{2.9}$$

and similarly

$$\overline{\prod_{j=1}^N \cos\left(\frac{t}{\sqrt{N}}J_{i_1,j}\right)\cos\left(\frac{t}{\sqrt{N}}J_{i_2,j}\right)} = c(t)^2 \tag{2.10}$$

if $i_1 \neq i_2$ and otherwise

$$\begin{aligned}\overline{\prod_{j=1}^{N} \cos\left(\frac{t}{\sqrt{N}} J_{i,j_1}\right)^2} &= \prod_{j=1}^{N} \overline{\frac{1}{4}\left(e^{i\frac{t}{\sqrt{N}}J_{i,j}} + e^{-i\frac{t}{\sqrt{N}}J_{i,j}}\right)^2} \\ &= \prod_{j=1}^{N} \overline{\frac{1}{2}\left(e^{\frac{2it}{\sqrt{N}}J_{i,j}} + 1\right)} = \left[\frac{1}{2}(e^{-\frac{2}{N}t^2} + 1)\right]^N. \qquad (2.11)\end{aligned}$$

Hence

$$\begin{aligned}&\overline{(\mathbb{E}[\langle \mu_N, f\rangle] - \langle \gamma, f\rangle)^2} \\ &= \frac{1}{N^2} \sum_{i_1,i_2=1}^{N} \overline{\prod_{j=1}^{N} \cos\left(\frac{t}{\sqrt{N}} J_{i_1,j}\right) \cos\left(\frac{t}{\sqrt{N}} J_{i_2,j}\right)} - c(t)^2 \\ &= \frac{1}{N}\left(\left[\frac{1}{2}(e^{-\frac{2}{N}t^2} + 1)\right]^N - c(t)^2\right) \qquad (2.12) \\ &\leq \frac{1}{N}\left(1 - \frac{t^2}{N} + \frac{t^4}{N^2}\right)^N - \frac{1}{N}e^{-t^2} \\ &\leq \frac{1}{N}e^{-t^2}\left(e^{t^4/N} - 1\right) = \mathcal{O}(N^{-2}). \qquad (2.13)\end{aligned}$$

By the Borel-Cantelli lemma we can now conclude that, almost surely,

$$\mathbb{E}\left(\int e^{itx}\mu_N(dx)\right) \to \int e^{itx}\gamma(dx) \qquad (2.14)$$

and by interchanging the order of integration and the fact that convergence of characteristic functions implies weak convergence of bounded measures (see [21]),

$$\mathbb{E}[\langle \mu_N, f\rangle] \to \langle \gamma, f\rangle \text{ a.s.} \qquad (2.15)$$

for all bounded continuous functions f.

The same strategy applies also to higher moments. In that case we consider

$$\overline{\left(\mathbb{E}\left[\prod_{\alpha=1}^{p} \langle \mu_N, f_\alpha\rangle\right] - \prod_{\alpha=1}^{p} \langle \gamma, f_\alpha\rangle\right)^2} \qquad (2.16)$$

with $f_\alpha(x) = e^{it_\alpha x}$. As above we get

$$\begin{aligned}
&\overline{\left(\mathbb{E}\left[\prod_{\alpha=1}^{p}\langle\mu_N, f_\alpha\rangle\right] - \prod_{\alpha=1}^{p}\langle\gamma, f_\alpha\rangle\right)^2} \\
&\quad = \overline{\left(\frac{1}{N^p}\sum_{i_1,\dots,i_p=1}^{N}\prod_{j=1}^{N}\cos\left(\frac{1}{\sqrt{N}}\sum_{\alpha=1}^{p}t_\alpha J_{i_\alpha,j}\right) - \prod_{\alpha=1}^{p}c(t_\alpha)\right)^2} \\
&\quad = \frac{1}{N^{2p}}\sum_{i_1,\dots,i_p}\sum_{i'_1,\dots,i'_p}\Bigg[\left(\prod_{j=1}^{p}\cos\left(\frac{1}{\sqrt{N}}(t_1J_{i_1,j}+\cdots+t_pJ_{i_p,j})\right) - \prod_{\alpha=1}^{p}c(t_\alpha)\right) \\
&\qquad\qquad \times\left(\cos\left(\frac{1}{\sqrt{N}}(t_1J_{i'_1,j}+\cdots+t_pJ_{i'_p,j})\right) - \prod_{j=1}^{p}c(t_\alpha)\right)\Bigg]^{-}. \qquad (2.17)
\end{aligned}$$

We distinguish three cases. The first case is when all i_α and all i'_β are different, i.e. $\#\{i_1,\dots,i_p,i'_1,\dots,i'_p\} = 2p$. These contribute nothing to the above sum because of (2.9). The second case is when one pair is equal, i.e. $\#\{i_1,\dots,i_p,i'_1,\dots,i'_p\} = 2p-1$. In this case there are two possibilities: either $\#\{i_1,\dots,i_p\} = \#\{i'_1,\dots,i'_p\} = p$ and there exists one pair (α,β) such that $i_\alpha = i'_\beta$ or $\{i_1,\dots,i_p\}\cap\{i'_1,\dots,i'_p\} = \emptyset$ but $\#\{i_1,\dots,i_p\} = p$ and $\#\{i'_1,\dots,i'_p\} = p-1$ or vice versa. The second possibility again gives no contribution because the two factors in the last expression of (2.17) separate and we can use (2.9) again in one of the factors. In case $\#\{i_1,\dots,i_p\}\cap\{i'_1,\dots,i'_p\} = 1$ we can assume $i_\alpha = i'_\beta$ and we have

$$\begin{aligned}
&\overline{\prod_{j=1}^{N}\cos\left(\frac{1}{\sqrt{N}}(t_1J_{i_1,j}+\cdots+t_pJ_{i_p,j})\right)\cos\left(\frac{1}{\sqrt{N}}(t_1J_{i'_1,j}+\cdots+t_pJ_{i'_p,j})\right)} \\
&\quad = \prod_{j=1}^{N}\frac{1}{2}\Bigg[\exp\left[\frac{i}{\sqrt{N}}\left((t_\alpha+t_\beta)J_{i_\alpha,j} + \sum_{\alpha'=1,\alpha'\neq\alpha}^{p}t_{\alpha'}J_{i_{\alpha'},j} + \sum_{\beta'=1,\beta'\neq\beta}^{p}t_{\beta'}J_{i'_{\beta'},j}\right)\right] \\
&\qquad + \exp\left[\frac{i}{\sqrt{N}}\left((t_\alpha-t_\beta)J_{i_\alpha,j} + \sum_{\alpha'=1,\alpha'\neq\alpha}^{p}t_{\alpha'}J_{i_{\alpha'},j} - \sum_{\beta'=1,\beta'\neq\beta}^{p}t_{\beta'}J_{i'_{\beta'},j}\right)\right]\Bigg]^{-} \\
&\quad = \left(\frac{e^{-\frac{1}{2N}(t_\alpha+t_\beta)^2} + e^{-\frac{1}{2N}(t_\alpha-t_\beta)^2}}{2}\right)^N e^{-(t_1^2+\cdots+t_p^2)-(t_\alpha^2+t_\beta^2)/2}. \qquad (2.18)
\end{aligned}$$

If $\#\{i_1,\dots,i_p,i'_1,\dots,i'_p\} \le 2p-2$ then we simply bound the two factors in the right-hand side of (2.17) by $2^2 = 4$. The total number of terms in the sum in this case is bounded by

$$\left[3\binom{2p}{4} + \binom{2p}{3}\right]N^{2p-2}, \qquad (2.19)$$

and the number of terms in the second case is less than p^2N^{2p-1}. We thus obtain

$$\begin{aligned}
&\overline{\left(\mathbb{E}\left[\prod_{\alpha=1}^{p}\langle\mu_N, f_\alpha\rangle\right] - \prod_{\alpha=1}^{p}\langle\gamma, f_\alpha\rangle\right)^2} \\
&\quad\le \frac{4}{N^2}\left[3\binom{2p}{4} + \binom{2p}{3}\right] \\
&\quad\quad + \frac{1}{N}\sum_{\alpha,\beta=1}^{p}\left[\left(\frac{e^{-(t_\alpha+t_\beta)^2/2N} + e^{-(t_\alpha-t_\beta)^2/2N}}{2}\right)^N - e^{-(t_\alpha^2+t_\beta^2)/2}\right] \\
&\quad = \mathcal{O}(N^{-2}).
\end{aligned} \tag{2.20}$$

As in the case of $p = 1$ above, we conclude that

$$\mathbb{E}[\int f(x_1, \dots, x_p)\mu_N(dx_1)\dots\mu_N(dx_p)] \to \int f(x_1, \dots, x_p)\gamma(dx_1)\dots\gamma(dx_p) \tag{2.21}$$

a.s. for all bounded continuous functions f. In particular,

$$\mathbb{E}[\langle\mu_N, f\rangle^p] \to \langle\gamma, f\rangle^p \tag{2.22}$$

a.s. for all bounded continuous functions f. By expanding the exponential, it then follows that

$$\mathbb{E}[e^{i\langle\mu_N, f\rangle}] \to e^{i\langle\gamma, f\rangle}. \tag{2.23}$$

Unlike the finite-dimensional situation, this is not sufficient for the convergence of the measures. By Prokhorov's theorem [22], we need to prove tightness of the sequence of probability distributions of the μ_N. This is done in the following lemma. □

Tightness of the sequence of probability measures on $\mathcal{M}^1(\mathbb{R})$ follows from:

Lemma 2.2. *For all $\epsilon > 0$ there exists a compact set $K_\epsilon \subset \mathcal{M}^1(\mathbb{R})$ such that for all $N \in \mathbb{N}$,*

$$\mathbb{P}(\mu_N \notin K_\epsilon) < \epsilon \tag{2.24}$$

a.s. with respect to the coupling parameters.

Proof. We define

$$K_n = \left\{\mu \in \mathcal{M}^1(\mathbb{R}) \middle| \mu[(-\infty, -a) \cup (a, +\infty)] \le n^4 e^{-a}\, \forall a \in \mathbb{N}\right\}. \tag{2.25}$$

Clearly, K_n is compact. By Chebyshev's inequality,

$$\mathbb{P}\left(\mu_N[(-\infty, -a) \cup (a, +\infty)] > n^4 e^{-a}\right) < n^{-4} e^a \mathbb{E}(\langle\mu_N, 1_{(-\infty,-a)\cup(a,+\infty)}\rangle). \tag{2.26}$$

We now bound $1_{(-\infty,-a)\cup(a,+\infty)}$ by $e^{2(x-a)} + e^{-2(x+a)}$ and compute

$$\overline{\mathbb{E}\left(\langle\mu_N, e^{\pm 2x}\rangle\right)} = \overline{\frac{1}{N}\sum_{i=1}^{N}\mathbb{E}\left(e^{\pm\frac{2}{\sqrt{N}}\sum_{j=1}^{N} J_{i,j}s_j}\right)} = e^2. \tag{2.27}$$

It follows that

$$\overline{\mathbb{P}\left(\mu_N[(-\infty) \cup (a, +\infty)] > n^4 e^{-a}\right)} \le 2n^{-4}e^{2-a}. \tag{2.28}$$

By the Borel-Cantelli lemma, we conclude that, for n large enough,

$$\mathbb{P}\left(\mu_N[(-\infty, -a) \cup (a, +\infty)] > n^4 e^{-a}\right) < 2n^{-2}e^{2-a} \tag{2.29}$$

a.s. for all $a \in \mathbb{N}$. The lemma now follows from

$$\begin{aligned}\mathbb{P}(\mu_N \in K_n^c) &\le \sum_{a=1}^{\infty} \mathbb{P}\left(\mu_N[(-\infty,-a)\cup(a,+\infty)] > n^4 e^{-a}\right) \\ &\le 2e^2 n^{-2} \sum_{a=1}^{\infty} e^{-a} \\ &= \frac{2e^2}{e-1} n^{-2} < \epsilon \end{aligned} \quad (2.30)$$

for n large enough. □

3. The local field distribution in the frustrated model

3.1. **The first and second moment.** For the first moment, the convergence is easy: Let us introduce the notation

$$L_N(\{s_x\}) = N\langle f, \mu_N\rangle = \sum_{x=1}^{N} f\left(\frac{1}{\sqrt{N}} \sum_{y=1}^{N} J_{x,y} s_y\right). \quad (3.1)$$

Then

$$\frac{1}{N}\sum_{\{s_x\}} L_N(\{s_x\}) = 2 \sum_{k=-2M}^{2M} \binom{4M}{2M-k} f\left(\frac{2k}{\sqrt{N}}\right). \quad (3.2)$$

(To see this, write $\sum_{y=1}^{N} J_{x,y} s_y = -\sum_{y=x-M}^{x-1} s_y - \sum_{y=x+1}^{x+M} s_y + \sum_{y=x+M+1}^{x+3M} s_y$ and sum over the possible values $2k$ of this variable with possible number of occurrences at fixed x. The sum over s_x gives an additional factor 2.) Hence

$$\lim_{N\to\infty} \mathbb{E}\left[\langle \mu_N, f\rangle\right] = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x) e^{-x^2/2} dx = \langle f, \gamma\rangle. \quad (3.3)$$

Therefore, if (1.5) were to hold the limiting measure γ would be a standard normal distribution. Computation of the higher moments is more complicated. We first consider the case $k=2$.

For the second moment, we need to compute the limit

$$\lim_{N\to\infty} \frac{1}{2^N N^2} \sum_{\{s_x\}} L_N(\{s_x\})^2. \quad (3.4)$$

Now

$$\sum_{\{s_x\}} L_N(\{s_x\})^2 = \sum_{x_1,x_2} \sum_{\{s_x\}} f\left(\frac{1}{\sqrt{N}} \sum_y J_{x_1,y} s_y\right) f\left(\frac{1}{\sqrt{N}} \sum_y J_{x_2,y} s_y\right). \quad (3.5)$$

To compute the limit of this expression we consider it as a quadratic form and insert $e^{it_1 z}$ and $e^{it_2 z}$ for f. We then need to compute

$$\begin{aligned}
&\lim_{N\to\infty}\frac{1}{2^N N^2}\sum_{x_1,x_2}\sum_{\{s_x\}}\exp\left[\frac{i}{\sqrt{N}}\sum_y (t_1 J_{x_1,y}+t_2 J_{x_2,y})s_y\right] \\
&= \lim_{N\to\infty}\frac{1}{2^N N^2}\sum_{x_1,x_2}\prod_y\sum_{s_y=\pm 1} e^{iN^{-1/2}(t_1 J_{x_1,y}+t_2 J_{x_2,y})s_y} \qquad (3.6)\\
&= \lim_{N\to\infty}\frac{1}{N^2}\sum_{x_1,x_2}\prod_y \cos\left(\frac{1}{\sqrt{N}}(t_1 J_{x_1,y}+t_2 J_{x_2,y})\right).
\end{aligned}$$

Now, if $|x_1 - x_2| \le 2M$, the number of y for which $J_{x_1,y} = J_{x_2,y}$ is $2(2M - |x_1 - x_2| - 1)$ and the number of y for which $J_{x_1,y} = -J_{x_2,y}$ is $2|x_1 - x_2| + 1$. (Except of course if $x_1 = x_2$, but this case is negligible as we are dividing by N^2. Similarly, the cases $y = x_1, x_2$ are irrelevant.) On the other hand, if $|x_1 - x_2| > 2M$ then the number of y with $J_{x_1,y} = J_{x_2,y}$ is $2(|x_1 - x_2| - 2M - 1)$ and the number of y with $J_{x_1,y} = -J_{x_2,y}$ is $2(N - |x_1 - x_2|) - 1$. Thus, the above limit equals

$$\begin{aligned}
&\lim_{N\to\infty}\frac{1}{N}\Bigg\{\sum_{k=1}^{2M}\left(\cos\left(N^{-1/2}(t_1+t_2)\right)\right)^{2(2M-k-1)}\left(\cos\left(N^{-1/2}(t_1-t_2)\right)\right)^{2k+1} \\
&+\sum_{k=2M+1}^{N-1}\left(\cos\left(N^{-1/2}(t_1+t_2)\right)\right)^{2(k-2M-1)}\left(\cos\left(N^{-1/2}(t_1-t_2)\right)\right)^{2(N-k)-1}\Bigg\} \\
&= \lim_{N\to\infty}\frac{1}{N}\Bigg\{\sum_{k=0}^{2M}\left(1-\frac{1}{2N}(t_1+t_2)^2\right)^{N-2k}\left(1-\frac{1}{2N}(t_1-t_2)^2\right)^{2k} \\
&\quad+\sum_{k=2M+1}^{N}\left(1-\frac{1}{2N}(t_1+t_2)^2\right)^{2k-N}\left(1-\frac{1}{2N}(t_1-t_2)^2\right)^{2(N-k)}\Bigg\} \\
&= \frac{1}{2}\Bigg\{\int_0^1 \exp\left[-\frac{1}{2}\{(1-s)(t_1+t_2)^2+s(t_1-t_2)^2\}\right]ds \\
&\quad+\int_1^2 \exp\left[-\frac{1}{2}\{s(t_1+t_2)^2+(2-s)(t_1-t_2)^2\}\right]ds\Bigg\}. \qquad (3.7)
\end{aligned}$$

This can be rewritten as

$$\frac{1}{2}e^{-(t_1^2+t_2^2)/2}\int_{-1}^{1} e^{st_1t_2}du. \qquad (3.8)$$

The limit (3.4) is therefore given by

$$\int_{-\infty}^{\infty}dx_1\int_{-\infty}^{\infty}dx_2\,\rho_2(x_1,x_2)f(x_1)f(x_2), \qquad (3.9)$$

where ρ_2 is the density of the measure with characteristic function given by (3.8), i.e.

$$\int_{-\infty}^{\infty} dx_1 \int_{-\infty}^{\infty} dx_2\, \rho_2(x_1, x_2) e^{i(t_1 x_1 + t_2 x_2)} = \frac{1}{2} e^{-(t_1^2+t_2^2)/2} \int_{-1}^{1} e^{s t_1 t_2} ds. \tag{3.10}$$

Diagonalising $t_1^2 - 2st_1t_2 + t_2^2$ we find

$$\rho_2(x_1, x_2) = \frac{1}{4\pi} \int_{-1}^{1} \frac{e^{-\frac{(x_1+x_2)^2}{4(1-s)} - \frac{(x_1-x_2)^2}{4(1+s)}}}{\sqrt{1-s^2}} ds. \tag{3.11}$$

Clearly, this is not equal to the density of $\gamma \otimes \gamma$, i.e. $\frac{1}{2\pi} e^{-(x_1^2+x_2^2)/2}$ which would be the result if (1.5) were to hold.

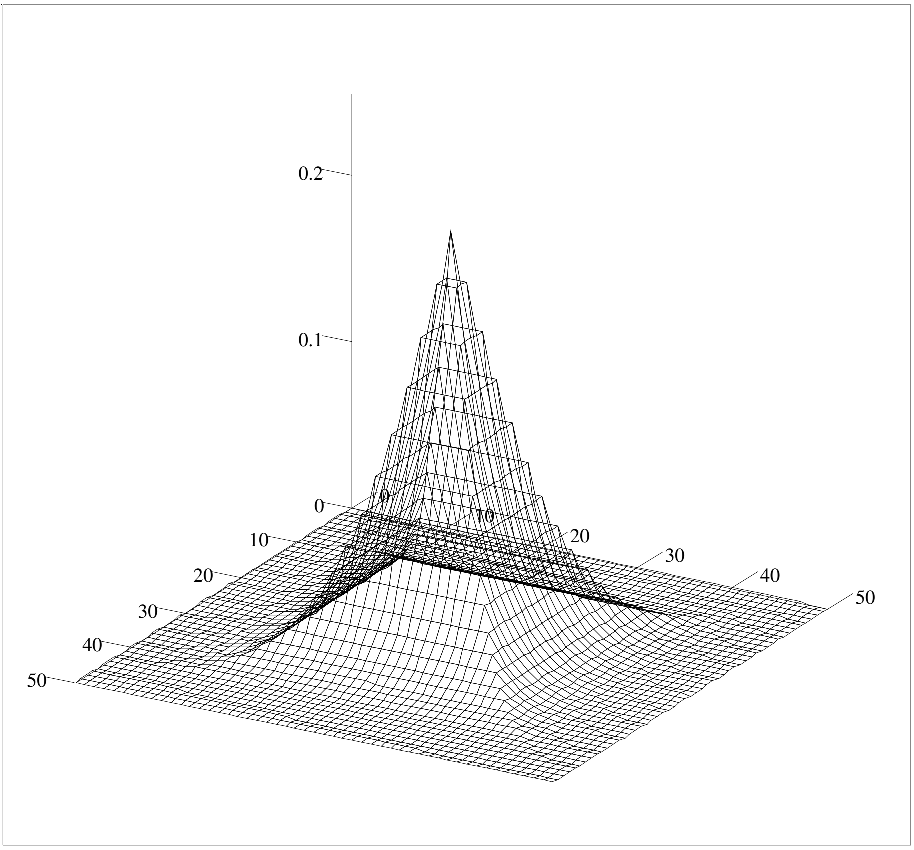

FIGURE 1. Plot of the function (3.11). It appears from this plot that ρ_2 is constant on squares, i.e. it only depends on $|x_1| \vee |x_2|$. This will be proved in the appendix.

3.2. **Higher moments.** The computation of the higher moments follows the same strategy but the result is more complicated and cannot be expressed in such a simple fashion. Analogous to (3.5) we have

$$\sum_{\{s_x\}} L_N(\{s_x\})^p = \sum_{x_1,\dots,x_p} \sum_{\{s_x\}} \prod_{i=1}^p f\left(\frac{1}{\sqrt{N}} \sum_y J_{x_i,y} s_y\right). \tag{3.12}$$

Inserting exponentials $e^{it_i z}$ we have to compute

$$\begin{aligned} &\lim_{N\to\infty} \frac{1}{2^N N^p} \sum_{x_1,\dots,x_p} \sum_{\{s_x\}} \exp\left[\frac{i}{\sqrt{N}} \sum_y s_y \sum_{i=1}^p t_i J_{x_i,y}\right] \\ &= \lim_{N\to\infty} \frac{1}{N^p} \sum_{x_1,\dots,x_p} \prod_y \cos\left(\frac{1}{\sqrt{N}} \sum_{i=1}^p t_i J_{x_i,y}\right). \end{aligned} \tag{3.13}$$

We now show that a calculation as in the case of the second moment yields the following:

Theorem 3.1. *For bounded continuous functions f, the limit*

$$\lim_{N\to\infty} \mathbb{E}\left[\langle f, \mu_N\rangle^p\right] = \lim_{N\to\infty} \frac{1}{N^p 2^N} \sum_{\{s_x\}} L_N\left(\{s_x\}\right)^p$$

exists and equals

$$\int_{-\infty}^{\infty} dx_1 \dots \int_{-\infty}^{\infty} dx_p \, \rho_p(x_1,\dots,x_p) f(x_1)\dots f(x_p),$$

where the probability density ρ_p is given by

$$\rho_p(x_1,\dots,x_p) = \frac{1}{(2\pi)^{p/2}} \int_0^1 \cdots \int_0^1 \frac{d\alpha_1 \dots d\alpha_p}{\sqrt{\det\ S(\alpha_1,\dots,\alpha_p)}} e^{-\frac{1}{2}\langle x,\, S(\alpha_1,\dots,\alpha_p)^{-1} x\rangle} \tag{3.14}$$

and the matrix $S(\alpha_1,\dots,\alpha_p)$ has matrix elements $s(\alpha_i-\alpha_j)$, $i,j=1,\dots,p$, where

$$s(\alpha) = \begin{cases} 1-4|\alpha| & \text{if } |\alpha| < \frac{1}{2}, \\ 4|\alpha|-3 & \text{if } |\alpha| \geq \frac{1}{2}. \end{cases} \tag{3.15}$$

.

Proof. First notice that for all pairs $i<j$,

$$\#\{y : J_{x_i,y} = J_{x_j,y}\} = \begin{cases} N - 2|x_j - x_i| & \text{if } |x_j-x_i| \leq 2M, \\ 2|x_j-x_i| - N & \text{if } |x_j - x_i| > 2M. \end{cases} \tag{3.16}$$

(N.B. The right-hand side is correct up to an error of at most 2, which is irrelevant in the limit $N \to \infty$.) We now rewrite the limit (3.13) as follows:

$$\begin{aligned}
&\lim_{N\to\infty} \frac{1}{N^p} \sum_{x_1,\dots,x_p} \prod_y \left\{ 1 - \frac{1}{2N}\left(\sum_{i=1}^p t_i J_{x_i,y}\right)^2 \right\} \\
&= \lim_{N\to\infty} \frac{1}{N^p} \sum_{x_1,\dots,x_p} \prod_y \exp\left[-\frac{1}{2N} \sum_{i,j=1}^p t_i t_j J_{x_i,y} J_{x_j,y} \right] \\
&= e^{-\frac{1}{2}(t_1^2+\dots+t_p^2)} \lim_{N\to\infty} \frac{1}{N^p} \sum_{x_1,\dots,x_p} \exp\left[-\frac{1}{N} \sum_{i<j} t_i t_j \sum_y J_{x_i,y} J_{x_j,y} \right] \\
&= e^{-\frac{1}{2}(t_1^2+\dots+t_p^2)} \lim_{N\to\infty} \frac{1}{N^p} \sum_{x_1,\dots,x_p} \exp\left[-\sum_{i<j} t_i t_j N s((x_j - x_i)/N) \right]
\end{aligned} \tag{3.17}$$

The last expression follows from the fact that

$$\begin{aligned}
\sum_y J_{x_i,y} J_{x_j,y} &= \#\{y : J_{x_i,y} J_{x_j,y} = 1\} - \#\{y : J_{x_i,y} J_{x_j,y} = -1\} \\
&= \begin{cases} N - 2|x_j - x_i| - 2|x_j - x_i| & \text{if } |x_i - x_j| \le 2M, \\ 2|x_j - x_i| - N - (2N - 2|x_j - x_i|) & \text{if } |x_j - x_i| > 2M. \end{cases}
\end{aligned} \tag{3.18}$$

Taking the limit $N \to \infty$ now yields

$$e^{-\frac{1}{2}(t_1^2+\dots+t_p^2)} \int_0^1 d\alpha_1 \dots \int_0^1 d\alpha_p \prod_{i<j} \exp\left[-t_i t_j s(\alpha_i - \alpha_j) \right]. \tag{3.19}$$

The result (3.14) then follows from the well-known Fourier transform formula for Gaussian functions. □

The formula (3.14) can be simplified by a transformation of variables. We subdivide the domain of integration into subdomains as follows. First let $u_j := \alpha_{j+1} - \alpha_1$ and define $s_j := s(u_j)$ for $j = 1, \dots, p-1$.

Lemma 3.2. *Let π be a permutation of $\{1, \dots, p-1\}$ and let $\sigma_1, \dots, \sigma_{p-1} \in \{\pm 1\}$. Define the region $\mathcal{R}(\pi, \sigma) \subset [0,1]^{p-1}$ by $(u_1, \dots, u_{p-1}) \in \mathcal{R}(\pi, \sigma)$ iff*

$$0 \le u_{\pi(1)} - \frac{1}{4}(\sigma_{\pi(1)} + 1) < \dots < u_{\pi(p-1)} - \frac{1}{4}(\sigma_{\pi(p-1)} + 1) \le \frac{1}{2}. \tag{3.20}$$

Then the region $\mathcal{R}(\pi, \sigma)$ is equivalent to

$$-1 < \sigma_{\pi(1)} s_{\pi(1)} < \dots < \sigma_{\pi(p-1)} s_{\pi(p-1)} < 1. \tag{3.21}$$

and the elements of the matrix S are given by

$$\begin{array}{rcrcll}
 & & S_{ii} & = & 1 & \text{for all } 1 \le i \le p \\
S_{1i} & = & S_{i1} & = & s_{i-1} & \text{for all } 1 < i \le p \\
S_{ij} & = & S_{ji} & = & \sigma_{j-1} s_{i-1} - \sigma_{i-1} s_{j-1} + \sigma_{i-1}\sigma_{j-1} & \text{if } \sigma_{i-1} s_{i-1} < \sigma_{j-1} s_{j-1}.
\end{array}$$

Moreover, if we denote $b_i := \sigma_{\pi(i)} s_{\pi(i)}$*, then*

$$\det S(\alpha_1, \ldots, \alpha_p) = 2^{p-2}(1 + b_1)(1 - b_{p-1}) \prod_{i=1}^{p-2} (b_{i+1} - b_i). \tag{3.22}$$

Proof. The equivalence of regions follows immediately from (3.15) which implies

$$s_k = \begin{cases} 1 - 4u_k, & \text{if } \sigma_k = -1 \\ 4u_k - 3, & \text{if } \sigma_k = +1 \end{cases}$$

which can be written as

$$s_k = 4\sigma_k u_k - 1 - 2\sigma_k \tag{3.23}$$

from which

$$u_k - \frac{1}{4}(\sigma_k + 1) = \frac{1}{4}(\sigma_k s_k + 1). \tag{3.24}$$

To determine the matrix elements of S, notice first that $S_{i,i} = s(0) = 1$ and $S_{i,1} = S_{1,i} = s(|u_{i-1}|) = s_{i-1}$ for $i > 1$. Moreover, if $1 < i < j$, $S_{i,j} = S_{j,i} = s(|u_{j-1} - u_{i-1}|)$, so we may assume $\sigma_i < \sigma_j s_j$. Now, by (3.23) we have

$$s(u_{i-1} - u_{j-1}) = 4\sigma|u_{i-1} - u_{j-1}| - 1 - 2\sigma, \tag{3.25}$$

where $\sigma = -1$ if $|u_{i-1} - u_{j-1}| \leq \frac{1}{2}$ and $\sigma = +1$ otherwise. By the above equivalence, $0 \leq u_{i-1} - \frac{1}{4}(\sigma_{i-1} + 1) < u_{j-1} - \frac{1}{4}(\sigma_{j-1} + 1) \leq \frac{1}{2}$ and hence

$$\frac{1}{4}(\sigma_{j-1} - \sigma_{i-1}) \leq u_{j-1} - u_{i-1} \leq \frac{1}{4}(\sigma_{j-1} - \sigma_{i-1}) + \frac{1}{2}. \tag{3.26}$$

¿From this it is easy to see that

$$|u_{i-1} - u_{j-1}| = \begin{cases} u_{j-1} - u_{i-1}, & \text{if } \sigma_{i-1} \leq \sigma_{j-1} \\ u_{i-1} - u_{j-1}, & \text{if } \sigma_{i-1} > \sigma_{j-1} \end{cases}$$

and

$$\sigma = \begin{cases} -1, & \text{if } \sigma_{i-1} \geq \sigma_{j-1}, \\ +1, & \text{if } \sigma_{i-1} < \sigma_{j-1}. \end{cases}$$

This implies

$$\sigma = -1 + \frac{1}{2}(1 - \sigma_{i-1})(1 + \sigma_{j-1}) \tag{3.27}$$

and

$$\frac{|u_{i-1} - u_{j-1}|}{u_{i-1} - u_{j-1}} = \sigma_{i-1}\sigma_{j-1}\sigma. \tag{3.28}$$

Inserting these identities into (3.25) we obtain

$$s(u_{i-1} - u_{j-1}) = 4\sigma_{i-1}\sigma_{j-1}(u_{i-1} - u_{j-1}) + \sigma_{i-1}\sigma_{j-1} - \sigma_{j-1} + \sigma_{i-1},$$

which is the stated result.

To evaluate $\det S$ we perform several elementary row and column operations and show the resulting matrix to be the matrix B, given in the appendix. For all $2 \le i \le p$ multiply each entry in row i by σ_i and each entry in column i by σ_i. Notice that if $1 < i < j \le p$ and $\sigma_{i-1}s_{i-1} < \sigma_{j-1}s_{j-1}$, then the resulting matrix $\tilde{S}$ satisfies

$$\begin{aligned}\tilde{S}_{ij} &= \sigma_{i-1}\sigma_{j-1}(\sigma_{j-1}s_{i-1} - \sigma_{i-1}s_{j-1} + \sigma_{i-1}\sigma_{j-1}) \\ &= \sigma_{i-1}s_{i-1} - \sigma_{j-1}s_{j-1} + 1 \\ &= b_{\pi^{-1}(i)} - b_{\pi^{-1}(j)} + 1.\end{aligned}$$

The entries in the row 1 now read $(1\ b_{\pi^{-1}(1)}\ \ldots\ b_{\pi^{-1}(p-1)})$. Reorder the rows and columns, according to the permutation π, so that the b indices are increasing in the first row and column. The resulting matrix is B. The total number of row and column operations is even, due to the symmetry of the matrix S, so the sign of the determinant is preserved. Thus $\det S = \det B$, which is evaluated in the Lemma A.2 in the appendix. □

The inverse of the matrix B can also be worked out: see Lemma A.2 in the Appendix. This leads to the following representation of the density ρ_p:

Corollary 3.3. *The density ρ_p of (3.14) can be written as*

$$\rho_p(x_0, x_1, \ldots, x_{p-1}) = \frac{2^{-p+1}}{(2\pi)^{p/2}} \sum_{\vec{\sigma},\pi} g_p(x_0, x_1, \ldots, x_{p-1}; \vec{\sigma}, \pi), \tag{3.29}$$

where

$$g(x_0, x_1, \ldots, x_{p-1}; \sigma, \pi) = \int_{\substack{v_1^2+\cdots+v_{p-2}^2 \le 4 \\ v_i \ge 0,\, \forall i}} dv_1 \ldots dv_{p-2} \int_{-\pi/2}^{\pi/2} d\alpha \tag{3.30}$$

$$\exp\Bigg\{ -\frac{1}{2} \Bigg(\frac{(x_0 + \sigma_{\pi(1)}x_{\pi(1)})^2}{v_1^2} + \frac{(x_0 - \sigma_{\pi(p-1)}x_{\pi(p-1)})^2}{\frac{1}{2}(4 - \sum_{i=1}^{p-2} v_i^2)(1 - \sin\alpha)} \tag{3.31}$$

$$+ \sum_{i=1}^{p-2} \frac{\left(\sigma_{\pi(i)}x_{\pi(i)} - \sigma_{\pi(i+1)}x_{\pi(i+1)}\right)^2}{v_{i+1}^2} \tag{3.32}$$

$$+ \frac{\left(\sigma_{\pi(p-2)}x_{\pi(p-2)} - \sigma_{\pi(p-1)}x_{\pi(p-1)}\right)^2}{\frac{1}{2}(4 - \sum_1^{p-2} v_i^2)(1 + \sin\alpha)} \Bigg) \Bigg\}. \tag{3.33}$$

In particular, $g_3(x_0, x_1, x_2; \vec{\sigma}, \pi)$ is given by

$$\begin{aligned} g_3(x_0, x_1, x_2; \vec{\sigma}, \pi) &= \int_0^2 dv_0 \int_{-\pi/2}^{\pi/2} d\alpha \\ &\exp\left\{ -\frac{1}{2} \left[\frac{(x_0 + \sigma_{\pi(1)}x_{\pi(1)})^2}{v_0^2} + \frac{(x_0 - \sigma_{\pi(2)}x_{\pi(2)})^2}{(2 - v_0^2/2)(1 - \sin\alpha)} \right.\right. \\ &\left.\left. + \frac{(\sigma_{\pi(1)}x_{\pi(1)} - \sigma_{\pi(2)}x_{\pi(2)})^2}{(2 - v_0^2/2)(1 + \sin\alpha)} \right] \right\}. \end{aligned} \tag{3.34}$$

Figure 3.2 shows a contour plot of the density ρ_3 at fixed x_2 from which it is apparent that the simple property of ρ_2 mentioned in the caption of Figure 3.1 does not generalise to higher p.

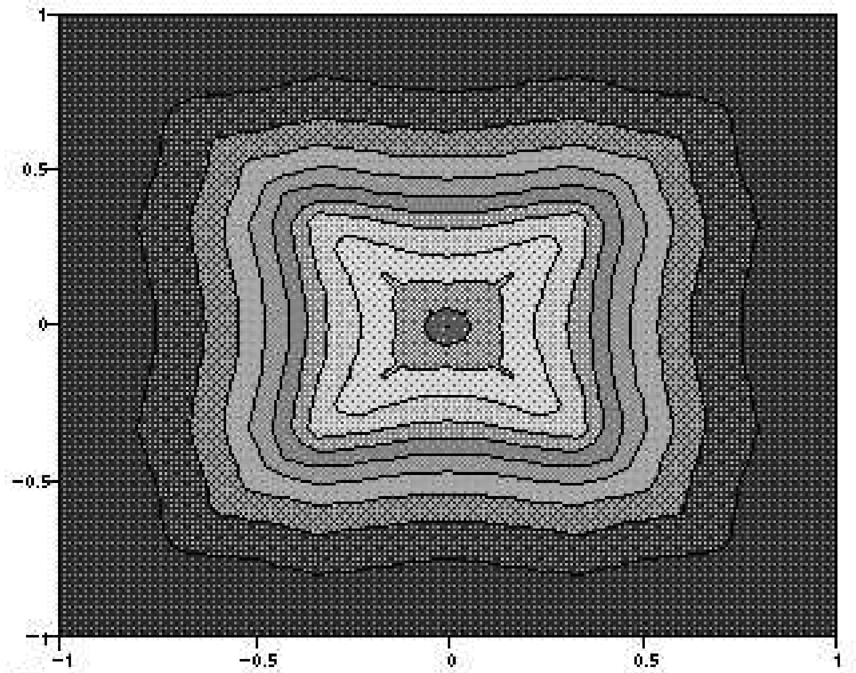

FIGURE 2. Contour plot of the function $\rho_3(x_0, x_1, 0.6)$ generated using the integrals in (3.34).

3.3. **Convergence of the probability distribution on the space of measures.** The convergence of the characteristic functions $\mathbb{E}\left[e^{i\langle f,\mu_N\rangle}\right]$ follows immediately from the theorem of Section 3, but as in Section 1 (Lemma 2.2), this does not yet imply that the corresponding distributions converge. This therefore requires a proof:

Theorem 3.4. *The sequence of probability distributions of the measures (1.5) converges weakly to the probability distribution on $\mathcal{M}^1(\mathbb{R})$ with characteristic function given by*

$$\mathbb{E}\left[e^{i\langle f,\cdot\rangle}\right] = \sum_{p=0}^{\infty} \frac{i^p}{p!} \int dx_1 \ldots \int dx_p \rho_p(x_1, \ldots, x_p) f(x_1) \ldots f(x_p). \quad (3.35)$$

This theorem follows from

Lemma 3.5. *For all $\epsilon > 0$, there is a compact $K_\epsilon \subset \mathcal{M}^1(\mathbb{R})$ and $N_0 \in \mathbb{N}$ such that*

$$\mathbb{P}(\mu_N \notin K_\epsilon) < \epsilon$$

for all $N = 4M + 1$ with $M \in \mathbb{N}$.

Proof. Let

$$K_\epsilon := \left\{ \mu \in \mathcal{M}^1(\mathbb{R}) \,\middle|\, \mu[(-\infty, -a) \cup (a, +\infty)] < \frac{e^{-a^2/4}}{\epsilon} \, \forall a \in \mathbb{N} \right\}.$$

Clearly this K_ϵ is compact, since for all $\delta > 0$ there exists $a > 1$ such that $\mu([-a, a]^c) \leq \delta$ for all $\mu \in K_\epsilon$, i.e. K_ϵ is tight. Using Chebychev's inequality we have

$$\mathbb{P}\left(\mu_N[(-\infty,-a)\cup(a,+\infty)] > \sqrt{2\pi}\frac{e^{-a^2/8}}{\epsilon}\right) \quad < \quad \frac{\epsilon\,\mathbb{E}(\langle\mu_N, 1_{(-\infty,-a)\cup(a,+\infty)}\rangle)}{e^{-a^2/8}} \tag{3.36}$$

where, as in (3.2) with $f(x) = 1_{(-\infty,-a)\cup(a,+\infty)}$

$$\begin{aligned}
\mathbb{E}(\langle\mu_N, 1_{(a,+\infty)}\rangle) &= \frac{1}{2^N}\sum_{\{s_x\}} L_N(\{s_x\}) \\
&= \frac{1}{2^{4M+1}}\sum_{k=-2M}^{2M}\binom{4M}{2M-k}1_{(-\infty,-a)\cup(a,+\infty)}\left(\frac{2k}{\sqrt{4M+1}}\right) \\
&= \frac{1}{2^{4M}}\sum_{k=[a\sqrt{M}]+1}^{2M}\binom{4M}{2M+k}.
\end{aligned}$$

We now use the bounds $\binom{4M}{2M} < \frac{2^{4M}}{\sqrt{2M}}$ for all $M \geq 1$ and $\frac{2M+1-i}{2M+i} < \exp\left(\frac{1-2i}{4M}\right)$ for all $1 \leq i \leq 2M$, which follows from $e^{-x} > 1-x$, to bound the coefficients $\binom{4M}{2M+k} < \frac{2^{4M}}{\sqrt{2M}}\exp\left(\frac{-k^2}{4M}\right)$. This gives the following bound

$$\begin{aligned}
\mathbb{E}(\langle\mu_N, 1_{(infty,-a)\cup(a,+\infty)}\rangle) &< \frac{1}{2^{4M}}\sum_{k=[a\sqrt{M}]+1}^{2M}\frac{2^{4M}}{\sqrt{2M}}\exp\left(\frac{-k^2}{4M}\right) \\
&\leq \frac{1}{\sqrt{2M}}\int_{a\sqrt{M}}^{+\infty} e^{-x^2/4M}dx \\
&= \int_{a/\sqrt{2}}^{+\infty} e^{-u^2/2}du \\
&< e^{-a^2/4}.
\end{aligned}$$

Applying this to (3.36) gives the result:

$$\mathbb{P}\left(\mu_N(a,\infty) \geq \sqrt{2\pi}\frac{e^{-a^2/8}}{\epsilon}\right) \quad < \quad \epsilon\frac{e^{-a^2/8}}{\sqrt{2\pi}}.$$

This yields as before

$$\begin{aligned}
\mathbb{P}(\mu_N \in K_\epsilon^c) &\leq \sum_{a=1}^{\infty}\mathbb{P}\left(\mu_N[(-\infty,-a)\cup(a,+\infty)] > \sqrt{2\pi}\frac{e^{-a^2/8}}{\epsilon}\right) \\
&\leq \frac{\epsilon}{\sqrt{2\pi}}\sum_{a=1}^{\infty} e^{-a^2/8} \\
&\leq \frac{\epsilon}{\sqrt{2\pi}}\int_0^{+\infty} e^{-u^2/8}du = \epsilon. \qquad (3.37)
\end{aligned}$$

□

Proof. (of Theorem 3.4.) By Prokhorov's theorem (see [22]) the lemma implies that the set of probability measures $\{\mu_N\}$ is relatively compact. This means that every subsequence has a convergent subsequence which must have the characteristic function given by (3.35) and is therefore uniquely determined. It follows by the usual subsequence argument that the sequence μ_N itself must converge to this measure. □

Appendix

Lemma A.1. *The value* $\rho_2(x,y)$ *given in (3.11) depends on* $|x| \vee |y|$ *only.*

Proof. Notice that it is clearly symmetric under interchange of x and y and also under sign change of x or y. We can therefore assume that $0 \leq x < y$. Differentiating w.r.t. x then yields

$$\frac{d}{dx}\rho_2(x,y) = \frac{1}{4\pi}\int_{-1}^{1}(x+sy)\frac{e^{-\frac{x^2+2sxy+y^2}{2(1-s^2)}}}{(1-s^2)^{3/2}}ds$$

Notice that the exponent can be rewritten as

$$\frac{(x+sy)^2}{2(1-s^2)} + \frac{1}{2}y^2.$$

Since $x < y$, $x + sy$ has a zero inside the integration interval $[-1, 1]$. We therefore divide this range into the intervals $[-1, -x/y]$ and $(-x/y, 1]$. On the second interval we change variables to s' in such a way that

$$s = 1 \implies s' = -1 \text{ and } s = -x/y \implies s' = -x/y$$

whereas

$$\frac{x+sy}{\sqrt{1-s^2}} = -\frac{x+s'y}{\sqrt{1-s'^2}}$$

and

$$\frac{ds}{1-s^2} = -\frac{ds'}{1-s'^2}$$

Solving the latter yields

$$s' = \frac{c(1-s)-(1+s)}{c(1-s)+(1+s)}$$

and inserting the boundary conditions then gives

$$c = \left(\frac{y-x}{y+x}\right)^2.$$

A simple calculation shows that the other identity also holds. The integral over the interval $(-x/y, 1]$ now transforms into minus the integral over $[-1, -x/y)$ so that the two contributions cancel and the derivative is zero. □

Lemma A.2. *Let* B *be the symmetric matrix with entries:*

$$\begin{array}{rcrcll} & & B_{ii} & := & 1 & \text{for } 1 \leq i \leq p \\ B_{1,i} & = & B_{i,1} & := & b_{i-1} & \text{for } 2 \leq i \leq p \\ B_{i,j} & = & B_{j,i} & := & b_{i-1} - b_{j-1} + 1, & \text{for } 1 \leq i < j \leq p. \end{array}$$

Then $\det\ B = 2^{p-2}(1+b_1)(b_2-b_1)\dots(b_{p-1}-b_{p-2})(1-b_{p-1})$. *Define* $b_0 := -1$, $b_p := 1$ *and* $w_i := (2(b_{i+1}-b_i))^{-1}$ *for all* $i = 0, \dots, p$. *Then the* *inverse B^{-1} is given by:*

$$
\begin{array}{rcrcll}
 & & B^{-1}_{1,1} & = & w_{p-1} + w_0 & \\
 & & B^{-1}_{i,i} & = & w_{i-2} + w_{i-1} & \textit{for } 2 \le i \le p \\
B^{-1}_{1,2} & = & B^{-1}_{2,1} & = & w_0 & \\
B^{-1}_{i,i+1} & = & B^{-1}_{i+1,i} & = & -w_{i-1} & \textit{for } 2 \le i \le p-1 \\
B^{-1}_{1,p} & = & B^{-1}_{p,1} & = & -w_{p-1}. &
\end{array}
$$

Proof. The determinant is obtained by elementary row and column operations: subtract row 1 from all other rows; then add column 2 to column 1; finally successively subtract column $i+1$ from column i for $i = 2, \ldots, p-1$. The resulting matrix is upper triangular and the product of diagonal elements is the said value.

To prove the statement about the inverse of B^{-1}, we multiply row x of B and column y of B^{-1} and consider various cases. If $x = y = 1$ we have $(BB^{-1})_{11} = b_{p-1}(-w_{p-1} + (w_0 + w_{p-1}) + b_1 w_0 = 1$. If $x = y = 2$ we have $(BB^{-1})_{22} = b_1 w_0 + (w_0 + w_1) + (b_1 - b_2 + 1)(-w_1) = 1$, and if $x = y > 1$, $(BB^{-1})_{yy} = (b_{y-2} - b_{y-1} + 1)(-w_{y-2}) + (w_{y-2} + w_{y-1}) + (b_{y-1} - b_y + 1)(-w_{y-1}) = 1$.

The (1,2)- and (2,1) elements are: $(BB^{-1})_{12} = w_0 + b_1(w_0 + w_1) + b_2(-w_1) = 0$ and $(BB^{-1})_{21} = (b_1 - b_{p-1} + 1)(-w_{p-1}) + b_1(w_0 + w_{p-1}) + w_0 = 0$. For $y > 2$ we get $(BB^{-1})_{1y} = b_{y-2}(-w_{y-2}) + b_{y-1}(w_{y-2} + w_{y-1}) + b_y(-w_{y-1}) = 0$ and $(BB^{-1})_{y1} = (b_{y-1} - b_{p-1} + 1)(-w_{p-1}) + b_{y-1}(w_{p-1} + w_0) + (b_1 - b_{y-1} + 1)w_0 = 0$.

For the cases $|x - y| \ge 2$, first assume, $1 < x < y < p$. Then $(BB^{-1})_{xy} = (b_{x-1} - b_{y-2} + 1)(-w_{y-2}) + (b_{x-1} - b_{y-1} + 1)(w_{y-2} + w_{y-1}) + (b_{x-1} - b_y + 1)(-w_{y-1}) = 0$. If $y = p$ we have $(BB^{-1})_{xp} = (b_{x-1} - b_{p-2} + 1)(-w_{p-2}) + (b_{x-1} - b_{p-1} + 1)(w_{p-2} + w_{p-1}) + b_{x-1}(-w_{p-1}) = 0$. If $1 < y < x < p$, $(BB^{-1})_{xy} = (b_{y-2} - b_{x-1} + 1)(-w_{y-2}) + (b_{y-1} - b_{x-1} + 1)(w_{y-2} + w_{y-1}) + (b_y - b_{x-1} + 1)(-w_{y-1}) = 0$. Finally, if $2 < y < p-1$, $(BB^{-1})_{py} = (b_{y-2} - b_{p-1} + 1)(-w_{y-2}) + (b_{y-1} - b_{p-1} + 1)(w_{y-2} + w_{y-2}) + (b_y - b_{p-1} + 1)(-w_{y-1}) = 0$ and $(BB^{-1})_{p2} = b_{p-1} w_0 + (b_1 - b_{p-1} + 1)(w_0 + w_1) + (b_2 - b_{p-1} + 1)(-w_1) = 0$. □

Lemma A.3. *The density $\rho_p(x_1, \ldots, x_p)$ may be expressed as*

$$
\rho_p(x_0, x_1, \ldots, x_{p-1}) \;=\; \frac{1}{2^{p-1}(2\pi)^{p/2}} \sum_{\substack{\sigma_1, \ldots, \sigma_{p-1} \in \pm 1 \\ \pi \in Perm[1, p-1]}} g(x_0, x_1, \ldots, x_{p-1}; \vec{\sigma}, \pi)
$$

where the sum is over all permutations π of $\{1,\dots,p-1\}$ and

$$g(x_0,x_1,\dots,x_{p-1};\sigma,\pi) = \int_{\substack{v_1^2+\dots+v_{p-2}^2\le 4 \\ v_i\ge 0,\,\forall i}} dv_1\dots dv_{p-2}\int_{-\pi/2}^{\pi/2} d\alpha$$

$$\exp\Bigg\{-\frac{1}{2}\Bigg(\frac{\big(x_0+\sigma_{\pi(1)}x_{\pi(1)}\big)^2}{v_1^2}+\frac{\big(x_0-\sigma_{\pi(p-1)}x_{\pi(p-1)}\big)^2}{\frac{1}{2}(4-\sum_{i=1}^{p-2}v_i^2)(1-\sin\alpha)}$$
$$+\sum_{i=1}^{p-2}\frac{\big(\sigma_{\pi(i)}x_{\pi(i)}-\sigma_{\pi(i+1)}x_{\pi(i+1)}\big)^2}{v_{i+1}^2}$$
$$+\frac{\big(\sigma_{\pi(p-2)}x_{\pi(p-2)}-\sigma_{\pi(p-1)}x_{\pi(p-1)}\big)^2}{\frac{1}{2}(4-\sum_1^{p-2}v_i^2)(1+\sin\alpha)}\Bigg)\Bigg\}.$$

Proof. Because of periodicity of the function $s(\alpha)$ the p-fold integral in (3.14) can be transformed into the following $p-1$-fold integral over $u_i=\alpha_{i+1}-\alpha_1$. Since s is an even function of u, we have that 2^{p-1} regions are similar and we obtain

$$\rho_p(x_1,\dots,x_p) = \frac{2^{p-1}}{(2\pi)^{p/2}}\int_{[0,1]^{p-1}}\frac{du_1\dots du_{p-1}}{\sqrt{\det\ S(u_1,\dots,u_{p-1})}} \times\exp\left\{-\frac{1}{2}\langle\vec{x},S(\vec{u})^{-1}\vec{x}\rangle\right\}$$

Next, we perform a change of variables to $\vec{s}$. By (3.23) we have $ds_i/du_i=4\sigma_i$, and the above integral may be written as

$$\rho_p(x_1,\dots,x_p) = \frac{2^{-p+1}}{(2\pi)^{p/2}}\sum_{\vec{\sigma},\pi}\int_{-1\le\sigma_{\pi(1)}s_{\pi(1)}\le\dots\le\sigma_{\pi(p-1)}s_{\pi(p-1)}\le 1}\frac{ds_1\dots ds_{p-1}}{\sqrt{\det\ S(\vec{s},\vec{\sigma},\pi)}} \times\exp\left\{-\frac{1}{2}\langle\vec{x},S(\vec{s},\vec{\sigma},\pi)^{-1}\vec{x}\rangle\right\}.$$

Now change variables to $\vec{b}$. This yields

$$\rho_p(x_1,\dots,x_p) = \frac{1}{2^{p-1}(2\pi)^{p/2}}\sum_{\vec{\sigma},\pi}\int_{-1\le b_1\le\dots\le b_{p-1}\le 1}\frac{db_1\dots db_{p-1}}{\sqrt{\det\ S(\vec{b},\vec{\sigma},\pi)}} \exp\left\{-\frac{1}{2}\langle\vec{x},S(\vec{b},\vec{\sigma},\pi)^{-1}\vec{x}\rangle\right\}$$

We now recall that $\det S(\vec{b},\vec{\sigma},\pi)=2^{p-2}(1+b_1)(b_2-b_1)\dots(b_{p-1}-b_{p-2})(1-b_{p-1})$. Moreover, the scalar product $\langle\vec{x},S(\vec{b},\vec{\sigma},\pi)^{-1}\vec{x}\rangle$ simplifies by reordering the vector $\vec{x}$ according to the permutation π, multiplying each entry by it's corresponding σ value and then using the matrix B rather than S. Thus we have

$$\langle(x_0,x_1,\dots,x_{p-1}),S(\vec{b},\vec{\sigma},\pi)^{-1}\,(x_0,x_1,\dots,x_{p-1})\rangle$$
$$= \langle(x_0,\sigma_{\pi(1)}x_{\pi(1)},\dots,\sigma_{\pi(p-1)}x_{\sigma(p-1)}),\ B\,(x_0,\sigma_{\pi(1)}x_{\pi(1)},\dots,\sigma_{\pi(p-1)}x_{\sigma(p-1)})\rangle$$

A brief calculation shows that this value is indeed

$$\frac{\big(x_0+\sigma_{\pi(1)}x_{\pi(1)}\big)^2}{2(1+b_1)}+\frac{\big(x_0-\sigma_{\pi(p-1)}x_{\pi(p-1)}\big)^2}{2(1-b_{p-1})}+\sum_{i=1}^{p-2}\frac{\big(\sigma_{\pi(i)}x_{\pi(i)}-\sigma_{\pi(i+1)}x_{\pi(i+1)}\big)^2}{2(b_{i+1}-b_i)}.$$

and so

$$g(x_0, x_1, \dots, x_{p-1}; \sigma, \pi) = \int_{-1 \le b_1 \le \dots \le b_{p-1} \le 1} db_1 \dots db_{p-1}$$

$$\frac{\exp\left\{ -\frac{1}{2} \left(\begin{array}{c} (x_0 + \sigma_{\pi(1)} x_{\pi(1)})^2/(2(1+b_1)) \\ +(x_0 - \sigma_{\pi(p-1)} x_{\pi(p-1)})^2/(2(1-b_{p-1})) \\ + \sum_{i=1}^{p-2} (\sigma_{\pi(i)} x_{\pi(i)} - \sigma_{\pi(i+1)} x_{\pi(i+1)})^2/(2(b_{i+1}-b_i)) \end{array} \right) \right\}}{\sqrt{2^{p-2}(1+b_1)(b_2-b_1)\dots(b_{p-1}-b_{p-2})(1-b_{p-1})}}.$$

Finally, we perform the following change of variables. For shorthand in this proof, let $V := v_1^2 + \ldots + v_{p-2}^2$. For $1 \le i \le p-2$, let $b_i = (v_1^2 + \cdots + v_i^2)/2 - 1$ and $b_{p-1} = \frac{1}{4}(V + (4-V)\sin\alpha)$. Then the Jacobian is $\left| \frac{\partial(b_1,\dots,b_{p-1})}{\partial(v_1,\dots,v_{p-2},\alpha)} \right| = \frac{1}{4} v_1 v_2 \dots v_{p-2} (4-V) \cos\alpha$. This is also the value of the denominator in the integral when expressed in the new variables. Thus the two cancel to leave only an exponential term. Under this change of variables, the region $\{\vec{b} \in \mathbb{R}^{p-1} \mid -1 \le b_1 \le \cdots \le b_{p-1} \le 1\}$ is equivalent to the region $\{\vec{v} \in \mathbb{R}_+^{p-2}, \alpha \mid \|\vec{v}\| \le 2, -\pi/2 \le \alpha \le \pi/2\}$. The resulting integral is just the one stated in the lemma, which is also (3.30).

□

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