Three different aspects of Knuth's big-chooser matchbox process: tableaux, a Markov chooser process, and a selfish-chooser generalization

Mark Dukes and Andrew Mullins

Abstract. In this paper we consider three variants of Knuth's generalization of Banach's matchbox problem. That paper introduced the notion of big-choosers, who always take a match from a matchbox with the most number of matches remaining, and little-choosers who do the opposite. The choosers arrive according to a Bernoulli process. The first variant we consider is the setting of k matchboxes, each of arbitrary size. We show how to map the outcome of the chooser process to a standard Young tableau of corresponding shape. We completely classify the types of tableaux that emerge in this setting and study some of their properties.

Next we return to Knuth's original setting of two matchboxes, each initially equal in size, but now let the chooser process be a 2-state Markov chain. We determine exact expressions for the generating functions of the expected residue and first return, and study the asymptotic behaviour of their coefficients. The third variant is another twist on Knuth's original setting wherein we introduce an additional ('selfish') chooser that takes a match from each of the boxes. This shifts the problem from one of counting weighted Dyck paths in the plane to counting weighted Schröder paths. For this variant we give a closed form generating function for the expected first return to a diagonal state and study its asymptotic behaviour.

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1. Introduction

Banach's matchbox problem [3] considers the problem of two matchboxes, each initially containing the same number of matches. It asks about the probability distribution of the number of matches in one box once the other first empties. In this setting, at each time step a matchbox is chosen with equal probability and a match removed. Knuth [5] considered a variation of this problem in which, at each time step, the box containing the most (respectively, least) number of matches is selected with probability p (respectively, $1-p$) and a match is removed. This variation introduced the notions of *big-choosers* and *little-choosers* that describe the preferences of the chooser at each time step. His paper studied the expected number of matches remaining in one box once the other first empties.

Stirzaker's [10] thorough treatment of Knuth's problem used the tools of random walks to provide a clearer understanding of the process. He derived the distribution of the residue and its asymptotic behaviour in addition to proving similar results for the case of more general initial conditions. Recently Dukes and Mullins [2] extended Knuth's original setting to the case of k matchboxes each of which initially contain the same number of matches. They determined the expected value of the total number of matches remaining once a matchbox first empties. In addition to that, they also considered the return to diagonal states where a diagonal state is defined as a configuration in which all k matchboxes contain the same number of matches.

The present paper continues this line of research and considers three new aspects. First, it extends the setting for Knuth's matchbox problem to the case of k matchboxes that can each initially contain any number of matches. We study a combinatorial interpretation of this setting that we term a *big-chooser tableau* (BCT), which is a particular type of standard Young tableau (SYT), and consider statistics on these tableaux and how they relate to aspects of Knuth's matchbox problem. We present several results that include a characterization of chooser sequences that result in a given big-chooser tableau, how to discern whether a given SYT is in fact a BCT, and also see how the reflection of a BCT in its main diagonal leads to symmetries between the statistics. The appearance of standard Young tableaux in problems related to random processes is not new and examples can be found in Krattenthaler [6, 7] and Banderier [1].

Secondly, we revisit Knuth's process for the case of two matchboxes, each initially containing the same number of matches. However, instead of a Bernoulli big-chooser process we consider the setting in which the chooser process is a 2-state Markov chain. Such a Markov generalization for the classical 1D random walk is known as the *correlated random walk* and was studied by Renshaw and Henderson [9]. Under the assumption that the type of the next chooser to arrive will depend on the type of the most recent chooser, we derive generating functions for the expected number of steps until the first return to a diagonal state, and also the expected residue once one box has first emptied. We also determine the asymptotic behaviour of these expectations.

Finally, we consider another variant of Knuth's big-chooser 2-matchbox process in which a third type of chooser (a selfish chooser) now appears and the chooser process is once again Bernoulli. From a lattice paths perspective, this is a very natural generalization to consider as it moves the setting from one of Dyck paths (counted by the Catalan numbers) to the setting of Schröder paths (counted by the Schröder numbers). For this variant, we give an explicit expression for the generating function of the expected first return to a diagonal state and determine the asymptotics for this case.

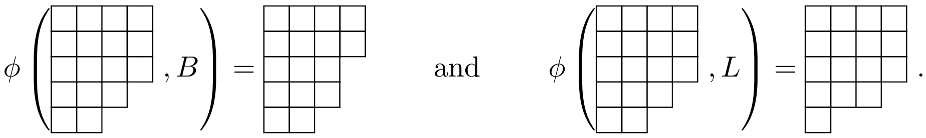
2. Big-chooser processes and Ferrers diagrams

In this section we consider a class of Young tableaux that arise from Knuth's big-chooser matchbox process [2] applied to Ferrers diagrams. Let $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k)$ be an integer partition of n. We picture λ as a Ferrers diagram with parts $\lambda_1, \lambda_2, \dots, \lambda_k$ from top to bottom. A *big-chooser process* is a length-n sequence $x = (x_1, \dots, x_n)$ whose entries take values in the set $\{B, L\}$. Here the event $x_i = B$ signifies that the ith chooser is a big-chooser, whereas $x_i = L$ signifies a little-chooser.

Definition 2.1. Given $\lambda = (\lambda_1, \dots, \lambda_k) \vdash n$, let us suppose that there are precisely a parts of λ with largest size λ_1, i.e. $\lambda_1 = \dots = \lambda_a > \lambda_{a+1}$. Let $c \in \{B, L\}$ and define $\phi(\lambda, c)$ to be the partition having size one less than λ with

$$\phi(\lambda, c) := \begin{cases} (\lambda_1, \dots, \lambda_k - 1) & \text{if } c = L \\ (\lambda_1, \dots, \lambda_a - 1, \dots, \lambda_k) & \text{if } c = B. \end{cases}$$

Example 2.2. For example, consider the integer partition $\lambda = (4, 4, 4, 3, 2)$. We have

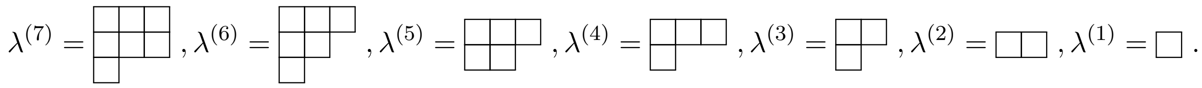


Definition 2.3. Let $\lambda \vdash n$ and consider the length-n big-chooser process $x = (x_1, \ldots, x_n)$. Define the sequence of Ferrers diagrams $\lambda^{(n)}, \lambda^{(n-1)}, \ldots, \lambda^{(1)}$ whereby $\lambda^{(n)} := \lambda$ and

$$\lambda^{(n-i)} = \phi(\lambda^{(n-i+1)}, x_i)$$

for all $i \in [1, n-1]$. Let $\Phi(\lambda, x)$ be the Young tableaux λ with entry 1 in $\lambda^{(1)}$ and entry i in cell $\lambda^{(i)} \backslash \lambda^{(i-1)}$ for all $i \in [2, n]$.

Example 2.4. Let $\lambda = (3, 3, 1)$ and consider the chooser process $x = (B, L, L, B, L, L, B)$. This results in the sequence of Ferrers diagrams:



This sequence of Ferrers diagrams can summarised by the labelled tableaux:

$$\Phi(\lambda, x) = \begin{array}{|c|c|c|} \hline 1 & 2 & 4 \\ \hline 3 & 5 & 7 \\ \hline 6 \\ \cline{1-1} \end{array}\ .$$

We call the outcome $\Phi(\lambda, x)$ a *big-chooser tableau*. Given a Ferrers shape λ we let

$$\mathrm{BCT}(\lambda) := \{\Phi(\lambda, x) \ : \ x \in \{B, L\}^n\},$$

the set of all big-chooser tableaux having shape λ. Our first result is a characterization of the set $\mathrm{BCT}(\lambda)$.

Definition 2.5. Let T be a SYT. Let us say that T contains a *bulge* at T_{ij} if there exist cells $I_{i'j'}$ and $T_{i''j''}$ where $T_{i'j'}$ is strictly southwest of T_{ij}, $T_{i''j''}$ is strictly northeast of T_{ij}, and

$$T_{i'j'} < T_{ij} > T_{i''j''}.$$

We will also say that T *bulges* at T_{ij} if T contains a bulge at T_{ij}.

For example, the SYT $T = \begin{array}{|c|c|c|} \hline 1 & 2 & 4 \\ \hline 3 & 6 \\ \cline{1-2} 5 \\ \cline{1-1} \end{array}$ contains a bulge at $T_{22} = 6$ due to the entries $T_{31} = 5$ and $T_{13} = 4$.

Proposition 2.6. *Let T be a SYT of shape λ. Then $T \in \mathrm{BCT}(\lambda)$ if, and only if, it does not contain a bulge.*

Proof. Suppose T does not contain a bulge. Then we can create big-chooser sequence s such that $\Phi(\lambda, s) = T$. To see this, set $T_n := T$. We construct the sequence $s = (s_1, \ldots, s_n)$ from right to left and remove cells from T as we go along. As i goes from n to 1, let

$$s_i := \begin{cases} B & \text{if } i \text{ is in the rightmost column of } T_i \\ L & \text{otherwise,} \end{cases}$$

and $T_{i-1} := T_i \backslash c_i$ where c_i is the cell in T_i containing the value i. Notice that c_i must either be at the bottom of the right-most column or at the end of the bottom row of T_{i-1} (by the properties of SYT and the fact that T does not have a bulge). This construction ensures that $\Phi(\lambda, s) = T$.

On the other hand, suppose T contains a bulge at cell c. Suppose cell c contains the value i. Then we can see that there is no big-chooser sequence s with $\Phi(\lambda, s) = T$ for the following reason. If there were, then when i was inserted into the tableau as a result of reading s_{n+1-i} (via Φ), there would still be empty cells north-east and south-west of c due to the bulge property. In the case that $s_{n+1-i} = B$ it would be the case that an empty cell north-east of c would have been chosen instead of c by a big-chooser, while in the case $s_{n+1-i} = L$ an empty cell south-west of c would have been chosen by a little-chooser instead of c. □

It is interesting to note that the property of a tableau containing a bulge can alternatively be expressed in terms of a corner-property of a sub-tableau.

Definition 2.7. Let T be a SYT. A cell c of T is called a *corner cell* if it has no adjacent cells to the south or east. A corner cell c is called an *internal corner cell* if there is a corner cell north-east of c and a corner cell south-west of c.

Proposition 2.8. *Let T be a SYT of size n and let c_i denote the cell of T containing the value i. Let $T^{(i)}$ be the tableau formed by restricting T to the entries $\{1, \ldots, i\}$. Then T has a bulge if, and only if, there is some $T^{(i)}$ in which c_i is an internal corner.*

Proof. Let us suppose that there is some i for which c_i is an internal corner of $T^{(i)}$. Then, by Definition 2.7, there are cells in $T^{(i)}$ lying strictly to the south-west and north-east c_i that both contain values smaller than i, since i is the largest value in $T^{(i)}$. Therefore, by Definition 2.5, T_i has a bulge at cell c_i. As $T^{(i)} \subset T$, we conclude that T has a bulge.

Conversely, suppose T has a bulge. Let $c \in T$ be a cell that T 'bulges' at and let a be its value. Then $c \in T^{(a)}$ is a corner cell of $T^{(a)}$, as a is the largest value of any cell in $T^{(a)}$. Furthermore, because T bulges at c, so too must $T^{(a)}$. Therefore there are cells lying strictly south-west and north-east of c in $T^{(a)}$. In particular, there must therefore be corner cells lying strictly south-west and north-east of c, and so c is an *internal corner* of $T^{(a)}$. □

2.1. **Sequences mapping to the same BCT.** A natural question to ask is, what collection of chooser sequences produce the same big-chooser tableau? In order to characterise these sequences, we now introduce the notion of a 'rectangular entry' in a BCT/SYT.

Definition 2.9. Let T be a SYT. We say that an entry $T_{ij} = a$ is a *rectangular entry* if $T_{ij} = ij$. An equivalent way to define this is that the subtableau of T achieved by restricting T to $[1, a]$ is a rectangular tableau. Let $\mathsf{Rect}(T)$ be the set of all rectangular entries in T, and let $\mathsf{rect}(T)$ be their number.

Consider the standard Young tableau T:

$$T = \begin{array}{|c|c|c|c|c|} \hline 1 & 3 & 4 & 10 & 15 \\ \hline 2 & 5 & 6 & 13 & 17 \\ \hline 7 & 8 & 9 \\ \cline{1-3} 11 & 12 \\ \cline{1-2} 14 \\ \cline{1-1} 16 \\ \cline{1-1} \end{array} \qquad (1)$$

We have $\mathsf{Rect}(T) = \{1, 2, 6, 9\}$ and $\mathsf{rect}(T) = 4$. Notice that two rectangular entries cannot lie strictly north-east/south-west of one another. To see this suppose that a and b are in $\mathsf{Rect}(T)$ for some T, with $a < b$. Then since b is a rectangular entry all entries smaller than it (and this includes a) must be in the rectangle whose bottom-right corner is at b. In this way rectangular entries appear, when one starts from the cell containing 1, in a weakly south-east direction. An equivalent way to say this is that a rectangular entry contains all smaller rectangular entries in its rectangle.

Definition 2.10. Given a standard Young tableau T, let us say that entry i is a *bubble* of T if all entries in rows beneath it are larger than it. Let $\text{Bubble}(T)$ be the set of bubbles of T.

For example, with T as in equation (1), we have

$$\text{Bubble}(T) = \{1, 2, 5, 6, 7, 8, 9, 11, 12, 14, 16\}.$$

For a tableau T, let $T^\top$ be the conjugate of T (i.e. the reflection in its main diagonal). In what follows it transpires that the sets $\text{Bubble}(T) \backslash \mathsf{Rect}(T)$ and $\text{Bubble}(T^\top) \backslash \mathsf{Rect}(T)$ are key features of our analysis. For this reason we give them the dedicated names

$$\mathsf{VBubble}(T) := \text{Bubble}(T)\backslash\mathsf{Rect}(T)$$
$$\mathsf{HBubble}(T) := \text{Bubble}(T^\top)\backslash\mathsf{Rect}(T).$$

Example 2.11. For the BCT T in equation (1), we have $\mathsf{Rect}(T) = \{1, 2, 6, 9\}$,

$$\text{Bubble}(T) = \{1, 2, 5, 6, 7, 8, 9, 11, 12, 14, 16\}$$
$$\text{Bubble}(T^\top) = \{1, 2, 3, 4, 6, 9, 10, 13, 15, 17\}.$$

We can see that the following three sets partition $[1, 17]$;

$$\mathsf{Rect}(T) = \{1, 2, 6, 9\}$$
$$\mathsf{VBubble}(T) = \text{Bubble}(T)\backslash\mathsf{Rect}(T) = \{5, 7, 8, 11, 12, 14, 16\}$$
$$\mathsf{HBubble}(T) = \text{Bubble}(T^\top)\backslash\mathsf{Rect}(T) = \{3, 4, 10, 13, 15, 17\}.$$

Lemma 2.12. *Let T be a BCT of shape $\lambda \vdash n$. Then the three sets $\mathsf{Rect}(T)$, $\mathsf{VBubble}(T)$, and $\mathsf{HBubble}(T)$ partition $\{1, \ldots, n\}$.*

Proof. For $n \geq 2$, the set $\mathsf{Rect}(T)$ is non-empty as it contains both 1 and 2. Notice that if $i \in \mathsf{Rect}(T)$, then all entries in the rows beneath i in T are larger than it, while all entries to the right of i are also larger than it. For this reason, we have $\mathsf{Rect}(T) \subseteq \text{Bubble}(T)$ and $\mathsf{Rect}(T) \subseteq \text{Bubble}(T^\top)$.

Let $i \notin \text{Bubble}(T)$. Then the entry i in T has some entry, a say, in a row beneath it which is less than i. This entry is necessarily below and to the left of i since T is a SYT. Suppose now that $i \notin \text{Bubble}(T^\top)$. Then the entry i in T has some entry, b say, in a column to its right which is less than i. This entry is necessarily in a row above i since T is a SYT. Then $a < i > b$ forms a bulge at i, which is a contradiction of T being a BCT and so i must be in $\text{Bubble}(T^\top)$.

The same argument shows that if $i \notin \text{Bubble}(T^\top)$ then i must be in $\text{Bubble}(T)$. As $\mathsf{Rect}(T) \subseteq \text{Bubble}(T), \text{Bubble}(T^\top)$, we have that the three sets $\mathsf{Rect}(T)$, $\mathsf{VBubble}(T) = \text{Bubble}(T)\backslash\mathsf{Rect}(T)$, and $\mathsf{HBubble}(T) = \text{Bubble}(T^\top)\backslash\mathsf{Rect}(T)$ partition $\{1, \ldots, n\}$. ☐

Proposition 2.13. *Let $T \in \text{BCT}(\lambda)$ with $\lambda \vdash n$. A big-chooser sequence $x = (x_1, \ldots, x_n)$ is such that $T = \Phi(\lambda, x)$ if and only if*

$$x_{n+1-i} = \begin{cases} B \textit{ or } L & \textit{if } i \in \mathsf{Rect}(T) \\ L & \textit{if } i \in \mathsf{VBubble}(T) \\ B & \textit{if } i \in \mathsf{HBubble}(T). \end{cases}$$

Proof. Suppose first that $\Phi(\lambda, x) = T$. We must show that for each $i \in [1, n]$,

$$x_{n+1-i} = \begin{cases} B \text{ or } L & \text{if } i \in \mathsf{Rect}(T) \\ L & \text{if } i \in \mathsf{VBubble}(T) \\ B & \text{if } i \in \mathsf{HBubble}(T). \end{cases}$$

Consider any $i \in [1, n]$. By Lemma 2.12, i is uniquely in one of $\mathsf{Rect}(T)$, $\mathsf{VBubble}(T)$, or $\mathsf{HBubble}(T)$. If $i \in \mathsf{Rect}(T)$ then trivially we have $x_{n-i+1} = B$ or L. If $i \in \mathsf{VBubble}(T)$ then by definition of $\Phi(\lambda, x)$, the entry i is in cell $\lambda^{(i)}\backslash\lambda^{(i-1)}$. Let $\lambda^{(i)} = (\lambda_1, \ldots, \lambda_k)$ and suppose that there are precisely a parts having length λ_1. As $i \notin \mathsf{Rect}(T)$ we have $a < k$. Notice that as i is the largest value in $T^{(i)}$ and, as $T^{(i)}$ does not have a bulge, i must either lie at the end of row a or row k, by Proposition 2.8. Since for this case $i \in \mathsf{VBubble}(T)$ we conclude that i is at the end of row k in $T^{(i)}$. By Definition 2.3, $\lambda^{(i-1)} = \Phi(\lambda^{(i)}, x_{n-i+1})$. Since $\Phi(\lambda, x) = T$ we must have $\Phi(\lambda^{(i)}, x_{n-i+1}) = (\lambda_1, \ldots, \lambda_k - 1)$ and therefore, by Definition 2.1, $x_{n-i+1} = L$, as required. Essentially the same argument applies for the case $i \in \mathsf{HBubble}(T)$ to show that one must have $x_{n-i+1} = B$.

Conversely, suppose that for each $i \in [1, n]$,

$$x_{n+1-i} = \begin{cases} B \text{ or } L & \text{if } i \in \mathsf{Rect}(T) \\ L & \text{if } i \in \mathsf{VBubble}(T) \\ B & \text{if } i \in \mathsf{HBubble}(T). \end{cases}$$

We must show that $\Phi(\lambda, x) = T$. Let us suppose that $\Phi(\lambda, x) \neq T$ and look for a contradiction. Let j be the largest value that is in a different cell in $\Phi(\lambda, x)$ and T. The tableaux $\Phi(\lambda, x)^{(j)}$ and $T^{(j)}$ have the same shape since the values $j+1$ through to n are in the same cells in $\Phi(\lambda, x)$ and T. If $j \in \mathsf{Rect}(T)$ then $T^{(j)}$ and $\Phi(\lambda, x)^{(j)}$ are rectangular and so both must necessarily have the value j in the bottom right cell. This contradicts the assumption that j is in different cells in the two tableaux. If $j \in \mathsf{VBubble}(T)$ then $T^{(j)}$ is not rectangular. Let $T^{(j)}$ have shape $(\lambda_1, \ldots, \lambda_k)$ with precisely a parts having size λ_1 for some $a < k$. Now since, for this case, $j \in \mathsf{VBubble}(T)$ we must have that j is in row k of $T^{(j)}$. But $j \in \mathsf{VBubble}(T)$ implies, by assumption, that $x_{n-j+1} = L$. From this it follows that $\Phi(\lambda^{(j)}, x_{n-j+1})$ has shape $(\lambda_1, \ldots, \lambda_k - 1)$ and so j must be in row k of $\Phi(\lambda, x)^{(j)}$. However, this contradicts the assumption that j is in different cells in the two tableaux. Again, essentially the same argument applies for the case $j \in \mathsf{HBubble}(T)$ to show that $\Phi(\lambda, x) = T$. □

Example 2.14. Consider the BCT T from equation (1). The three sets $\mathsf{Rect}(T)$, $\mathsf{VBubble}(T)$ and $\mathsf{HBubble}(T)$ were given in Example 2.11. Consequently we have the set of all 16 chooser sequences x that map to T:

$$\left\{ \left(B, L, B, L, B, L, L, B, \frac{B}{L}, L, L, \frac{B}{L}, L, B, B, \frac{B}{L}, \frac{B}{L} \right) \right\},$$

where we use the vincular notation $\frac{B}{L}$ to indicate an entry that could be either B or L.

Notice that we have a choice of either B or L for every entry of $\mathsf{Rect}(T)$, so the number of sequences that map to a given T is $2^{\mathsf{rect}(T)}$. Similarly, since every chooser sequence maps to a unique big-chooser tableau, the previous comment can be used in conjunction with this to give a simple identity between the number of chooser sequences and the number of rectangular entries in a given tableau.

Corollary 2.15.
 (i) *Let $T \in \mathrm{BCT}(\lambda)$ for some $\lambda \vdash n$. The number of x for which $T = \Phi(\lambda, x)$ is $2^{\mathsf{rect}(T)}$.*
 (ii) $\sum_{T \in \mathrm{BCT}(\lambda)} 2^{\mathsf{rect}(T)} = 2^n$.

2.2. **Expected values and statistics on BCT.** Let us now consider the Bernoulli probability distribution on the chooser process whereby $\mathbb{P}(x_i = B) = p = 1 - \mathbb{P}(x_i = L) \in (0, 1)$.

Proposition 2.16. *If the chooser process x is Bernoulli, then*

$$\mathbb{P}(\Phi(\lambda, x) = T) = p^{|\mathsf{HBubble}(T)|}(1-p)^{|\mathsf{VBubble}(T)|}. \tag{2}$$

Proof. From Proposition 2.13, we see that the set of chooser sequences x that map to a given BCT is dictated by whether an entry i is in one of the sets

$$\mathsf{Rect}(T), \ \mathrm{Bubble}(T^\top) \backslash \mathsf{Rect}(T), \ \text{or } \mathrm{Bubble}(T) \backslash \mathsf{Rect}(T).$$

More precisely, the entry $x_{n+1-i} = B$ if $i \in \mathrm{Bubble}(T^\top) \backslash \mathsf{Rect}(T)$ while the entry $x_{n+1-i} = L$ if $i \in \mathrm{Bubble}(T) \backslash \mathsf{Rect}(T)$. The other entries, which are in $\mathsf{Rect}(T)$, may be B or L. We have

$$\mathbb{P}(\Phi(\lambda, x) = T) = \prod_{i=1}^{n} f(x_i)$$

where

$$f(x_i) = \begin{cases} p & \text{if } n+1-i \in \mathsf{HBubble}(T) \\ 1-p & \text{if } n+1-i \in \mathsf{VBubble}(T) \\ 1 & \text{if } n+1-i \in \mathsf{Rect}(T). \end{cases}$$ □

As discussed in the introduction, the expected residue for the two matchbox case $\lambda = (n, n)$ was the focus of Knuth's original paper [5] that had a Bernoulli chooser process. Among the results proven by Stirzaker [10] in relation to Knuth's problem, he derived the generating function (equations 24 and 25) for the expected residue for the more general setting of $\lambda = (n, m)$ under that same Bernoulli measure. Dukes and Mullins considered the expected residue of the aggregate number of matches remaining once a box first empties for the case of k matchboxes each initially containing n matches.

In terms of BCT, each of the expected residues discussed above is the expected value of the statistic $\mathsf{sw}(T)-1$ over all big-chooser sequences with respect to a Bernoulli measure $\mathsf{Bernoulli}(p)$, where T has shape λ and $\mathsf{sw}(T)$ is the entry in the south-west most cell of T. Let $\mathsf{ne}(T)$ be the entry in the north-east most cell of T. In this subsection we show how the expected value of $\mathsf{sw}(T)$ for an arbitrary shape λ is related to the expected value of an associated (dual) problem: *Consider a collection of matchboxes and a Bernoulli big-chooser process on them. What is the number of matches remaining once all initially largest matchboxes have decreased in size?*

More formally, let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k) \vdash n$ be a fixed integer partition. Let $\mathsf{Residue}(\lambda)$ be the aggregate residue once the smallest matchbox has just emptied. Let $\mathsf{Residue}^*(\lambda)$ be the aggregate residue once all the initially largest matchboxes have decreased in size.

For example, in the case considered in Example 2.4, we have $sw(T) = 6$, $ne(T) = 4$, $\mathsf{Residue}(\lambda) = 5$, and $\mathsf{Residue}^*(\lambda) = 3$.

Theorem 2.17. *Let $\lambda \vdash n$ with conjugate shape λ', and let $i \in \mathbb{Z}_+$. Then*

$$\mathbb{E}_{\mathsf{Bern}(p)}(\mathsf{Residue}(\lambda)) = \mathbb{E}_{\mathsf{Bern}(1-p)}(\mathsf{Residue}^*(\lambda')).$$

Proof. Define

$$F_\lambda(x_1, x_2, y_1, y_2) := \sum_{T \in \mathrm{BCT}(\lambda)} x_1^{|\mathsf{VBubble}(T)|} x_2^{|\mathsf{HBubble}(T)|} y_1^{\mathsf{sw}(T)} y_2^{\mathsf{ne}(T)}.$$

Notice that if T is a BCT with $(|\mathsf{VBubble}(T)|, |\mathsf{HBubble}(T)|, \mathsf{sw}(T), \mathsf{ne}(T)) = (a, b, c, d)$ then the conjugate BCT T' has $(|\mathsf{VBubble}(T')|, |\mathsf{HBubble}(T')|, \mathsf{sw}(T'), \mathsf{ne}(T')) = (b, a, d, c)$. This shows that

$$F_\lambda(x_1, x_2, y_1, y_2) = F_{\lambda'}(x_2, x_1, y_2, y_1). \tag{3}$$

The expected value of $\mathsf{sw}(\lambda)$ is

$$\begin{aligned}
\mathbb{E}_{\mathsf{Bern}(p)}(\mathsf{sw}(\lambda)) &= \sum_{T \in \mathrm{BCT}(\lambda)} \mathsf{sw}(T) \mathbb{P}(\Phi(\lambda, x) = T) \\
&= \sum_{T \in \mathrm{BCT}(\lambda)} \mathsf{sw}(T)(1-p)^{|\mathsf{VBubble}(T)|} p^{|\mathsf{HBubble}(T)|} \\
&= \frac{\partial}{\partial y_1} \sum_{T \in \mathrm{BCT}(\lambda)} (1-p)^{|\mathsf{VBubble}(T)|} p^{|\mathsf{HBubble}(T)|} y_1^{\mathsf{sw}(T)} \\
&= \left. \frac{\partial F_\lambda(x_1, x_2, y_1, y_2)}{\partial y_1} \right|_{(1-p, p, 1, 1)} \\
&= \left. \frac{\partial F_{\lambda'}(x_2, x_1, y_2, y_1)}{\partial y_1} \right|_{(p, 1-p, 1, 1)} = \mathbb{E}_{\mathsf{Bern}(1-p)}(\mathsf{ne}(\lambda')),
\end{aligned}$$

where we applied equation (3) to obtain the penultimate expression. Since $\mathsf{Residue}(\lambda) = \mathsf{sw}(\lambda) - 1$ and $\mathsf{Residue}^*(\lambda') = \mathsf{ne}(\lambda') - 1$, the result follows. □

So the expected residue of the (classic) problem applied to matchboxes λ with a Bernoulli big-chooser process with parameter p is the same as the expected residue of the (dual) problem applied to matchboxes λ' with a Bernoulli big-chooser process with parameter $1 - p$.

We find it interesting to relate this to the original setting of Knuth's 2-roll toilet paper problem. The expected residue in Knuth's setting when the two rolls initially have n sheets

λ	$\lvert\mathrm{BCT}(\lambda)\rvert$	λ	$\lvert\mathrm{BCT}(\lambda)\rvert$	λ	$\lvert\mathrm{BCT}(\lambda)\rvert$
(1)	1	(5, 1, 1)	15	(2, 1, 1, 1, 1, 1, 1)	7
(2)	1	(4, 3)	14	(1, 1, 1, 1, 1, 1, 1, 1)	1
(1, 1)	1	(4, 2, 1)	19	(9)	1
(3)	1	(4, 1, 1, 1)	20	(8, 1)	8
(2, 1)	2	(3, 3, 1)	15	(7, 2)	27
(1, 1, 1)	1	(3, 2, 2)	15	(7, 1, 1)	28
(4)	1	(3, 2, 1, 1)	19	(6, 3)	48
(3, 1)	3	(3, 1, 1, 1, 1)	15	(6, 2, 1)	53
(2, 2)	2	(2, 2, 2, 1)	14	(6, 1, 1, 1)	56
(2, 1, 1)	3	(2, 2, 1, 1, 1)	14	(5, 4)	42
(1, 1, 1, 1)	1	(2, 1, 1, 1, 1, 1)	6	(5, 3, 1)	57
(5)	1	(1, 1, 1, 1, 1, 1, 1)	1	(5, 2, 2)	67
(4, 1)	4	(8)	1	(5, 2, 1, 1)	71
(3, 2)	5	(7, 1)	7	(5, 1, 1, 1, 1)	70
(3, 1, 1)	6	(6, 2)	20	(4, 4, 1)	43
(2, 2, 1)	5	(6, 1, 1)	21	(4, 3, 2)	59
(2, 1, 1, 1)	4	(5, 3)	28	(4, 3, 1, 1)	63
(1, 1, 1, 1, 1)	1	(5, 2, 1)	33	(4, 2, 2, 1)	63
(6)	1	(5, 1, 1, 1)	35	(4, 2, 1, 1, 1)	71
(5, 1)	5	(4, 4)	14	(4, 1, 1, 1, 1, 1)	56
(4, 2)	9	(4, 3, 1)	29	(3, 3, 3)	30
(4, 1, 1)	10	(4, 2, 2)	34	(3, 3, 2, 1)	59
(3, 3)	5	(4, 2, 1, 1)	38	(3, 3, 1, 1, 1)	67
(3, 2, 1)	10	(4, 1, 1, 1, 1)	35	(3, 2, 2, 2)	43
(3, 1, 1, 1)	10	(3, 3, 2)	30	(3, 2, 2, 1, 1)	57
(2, 2, 2)	5	(3, 3, 1, 1)	34	(3, 2, 1, 1, 1, 1)	53
(2, 2, 1, 1)	9	(3, 2, 2, 1)	29	(3, 1, 1, 1, 1, 1, 1)	28
(2, 1, 1, 1, 1)	5	(3, 2, 1, 1, 1)	33	(2, 2, 2, 2, 1)	42
(1, 1, 1, 1, 1, 1)	1	(3, 1, 1, 1, 1, 1)	21	(2, 2, 2, 1, 1, 1)	48
(7)	1	(2, 2, 2, 2)	14	(2, 2, 1, 1, 1, 1, 1)	27
(6, 1)	6	(2, 2, 2, 1, 1)	28	(2, 1, 1, 1, 1, 1, 1, 1)	8
(5, 2)	14	(2, 2, 1, 1, 1, 1)	20	(1, 1, 1, 1, 1, 1, 1, 1, 1)	1

TABLE 1. The number of big-chooser tableaux for a given shape λ for all partitions of numbers no greater than 9.

equals the expected value of an associated process that we now describe. Consider n matchboxes each containing two matches. A big chooser takes a match from a box containing the largest number with probability $1 - p$ and a match from a non-empty box containing the smallest number with probability p. Then the expected number of matches remaining once there are no longer any matchboxes containing two matches is the same as the expected residue of Knuth's problem.

2.3. **The number of BCT.** The number of BCT for a given shape λ is given in Table 1 for all $\lambda \vdash n$ with $n \in [1, 9]$. Since a standard Young tableaux must have at least three rows in order for there to be a possibility of a bulge, the set of standard Young tableaux having at most two rows equals the set of BCT having at most two rows. Consequently the number of BCT having shape $\lambda = (n, m)$ is easily given using the hook-length formula. Certain special cases can be enumerated without too much difficulty, e.g.

$$|\mathrm{BCT}(n-3,2,1)| = \binom{n-1}{3} - n + 6, \qquad \text{for all } n \geq 6 \tag{4}$$

$$|\mathrm{BCT}(n-4,2,2)| = \binom{n-1}{4} - \binom{n-5}{2} + n - 6, \qquad \text{for all } n \geq 7. \tag{5}$$

However, the sequence $|\mathrm{BCT}(n,n,n)|_{n \geq 1}$ begins 1, 5, 30, 191, 1261, 8542, 58976 and this does not appear to be a known sequence in the OEIS [8].

3. A Markov chooser process

In this section we consider Knuth's $\lambda = (n, n)$ big-chooser matchbox problem for the case of a Markov big-chooser process. We derive the expected number of steps until a first return to a diagonal state and also the expected residue.

Let $(X_i)_{i\geq 0}$ be a Markov chain with state space $\{B, L\}$. Suppose that initially

$$\mathbb{P}(X_0 = B) = p_1 = 1 - \mathbb{P}(X_0 = L)$$

and the transition probabilities are

$$\begin{aligned} \mathbb{P}(X_{i+1} = B | X_i = B) &= p_1 = 1 - \mathbb{P}(X_{i+1} = L | X_i = B) \\ \mathbb{P}(X_{i+1} = L | X_i = L) &= p_2 = 1 - \mathbb{P}(X_{i+1} = B | X_i = L), \end{aligned}$$

for all $i \geq 0$. For notational convenience we will set $q_1 := 1 - p_1$ and $q_2 := 1 - p_2$. We consider the two matchbox process with a Markov big-chooser process (X_i). The state of the two matchboxes at time t is the weakly decreasing pair $\mathsf{Box}_t = \left(\mathsf{box}_t^{(1)}, \mathsf{box}_t^{(2)}\right)$ with initial condition $\mathsf{Box}_0 = \left(\mathsf{box}_0^{(1)}, \mathsf{box}_0^{(2)}\right) = (n, n)$. The process $(\mathsf{Box}_i)_{0\leq i\leq 2n-1}$ evolves according to the following rules:

$$\mathsf{Box}_{i+1} = \begin{cases} \left(\mathsf{box}_i^{(1)}, \mathsf{box}_i^{(2)} - 1\right) & \text{if } \mathsf{box}_i^{(1)} = \mathsf{box}_i^{(2)} > 0 \\ \left(\mathsf{box}_i^{(1)}, \mathsf{box}_i^{(2)} - 1\right) & \text{if } \mathsf{box}_i^{(1)} > \mathsf{box}_i^{(2)} > 0 \text{ and } X_i = L \\ \left(\mathsf{box}_i^{(1)} - 1, \mathsf{box}_i^{(2)}\right) & \text{if } \mathsf{box}_i^{(1)} > \mathsf{box}_i^{(2)} > 0 \text{ and } X_i = B \\ \left(\mathsf{box}_i^{(1)} - 1, 0\right) & \text{if } \mathsf{box}_i^{(1)} > 0 \text{ and } \mathsf{box}_i^{(2)} = 0. \end{cases}$$

For such a process we will be interested in two random variables, the *residue* and the *first-return*, defined as follows:

$$\begin{aligned} \mathsf{Residue}_n &:= \mathsf{box}_{i^*}^{(1)}, \quad \text{where } i^* = \inf\left\{i \ : \ \mathsf{box}_i^{(2)} = 0\right\}, \text{ and} \\ \mathsf{FirstReturn}_n &:= \inf\left\{i > 0 \ : \ \mathsf{box}_i^{(1)} = \mathsf{box}_i^{(2)}\right\}. \end{aligned}$$

Example 3.1. Table 2 shows the probabilities $\mathbb{P}(\mathsf{Residue}_i = j)$ and $\mathbb{P}(\mathsf{FirstReturn}_i = j)$ for all $1 \leq j \leq i \leq 3$. From this we have that

$$\begin{aligned} \mathbb{E}(\mathsf{Residue}_1) &= 1 = \mathbb{E}(\mathsf{FirstReturn}_1) \\ \mathbb{E}(\mathsf{Residue}_2) &= p_1(2 - p_1) + (1 - p_1)(1 + p_2) = \mathbb{E}(\mathsf{FirstReturn}_2) \\ \mathbb{E}(\mathsf{Residue}_3) &= \left(p_1^2 + (1 - p_1)(1 - p_2)\right) \cdot (1 + (1 - p_1)(p_1 + p_2)) \\ &\quad + (p_1(1 - p_1) + (1 - p_1)p_2) \cdot (2 - p_1 + p_2 + p_1p_2) \\ \mathbb{E}(\mathsf{FirstReturn}_3) &= p_1^2 + (1 - p_1)(1 - p_2) \\ &\quad + (p_1(1 - p_1) + (1 - p_1)p_2) \cdot (2p_1(1 - p_2) + 3(1 - p_1)(1 - p_2) + 3p_2) \,. \end{aligned}$$

The difference

$$\mathbb{E}(\mathsf{Residue}_3) - \mathbb{E}(\mathsf{FirstReturn}_3) = (1 - p_1 - p_2) \cdot \left(p_1(p_1^2 + p_1p_2 - p_1 - 1) + p_1(1 - p_2)\right). \qquad (6)$$

When $p_1+p_2 = 1$ the right-hand side of equation (6) is 0 and the expectations are equal. This is so since the 2-state Markov chain with $p_2 = 1-p_1$ corresponds to the $(X_i)_{i\geq 0}$ being independent Bernoulli random variables and is therefore the original problem considered by Knuth.

3.1. The expected first return to a diagonal state.

Consider the event $\{\mathsf{FirstReturn}_n = i\}$, the first return to a diagonal state happens after $2i$ matches have been chosen, for some $i \in [1, n]$. This event occurs when

FR1: $X_0 = B$ or L,

i	$\mathbb{P}(\mathsf{Residue}_1 = i)$	$\mathbb{P}(\mathsf{FirstReturn}_1 = i)$
1	1	1
i	$\mathbb{P}(\mathsf{Residue}_2 = i)$	$\mathbb{P}(\mathsf{FirstReturn}_2 = i)$
1	$p_1^2 + (1-p_1)(1-p_2)$	$p_1^2 + (1-p_1)(1-p_2)$
2	$p_1(1-p_1) + (1-p_1)p_2$	$p_1(1-p_1) + (1-p_1)p_2$
i	$\mathbb{P}(\mathsf{Residue}_3 = i)$	$\mathbb{P}(\mathsf{FirstReturn}_3 = i)$
1	$(p_1^2 + (1-p_1)(1-p_2)) \cdot (p_1^2 + (1-p_1)(1-p_2))$ $+(p_1(1-p_1) + (1-p_1)p_2) \cdot (1-p_2)p_1$	$p_1^2 + (1-p_1)(1-p_2)$
2	$(p_1^2 + (1-p_1)(1-p_2)) \cdot (p_1(1-p_1) + (1-p_1)p_2)$ $+(p_1(1-p_1) + (1-p_1)p_2) \cdot (1-p_2)(1-p_1)$	$(p_1(1-p_1) + (1-p_1)p_2) \cdot (1-p_2)p_1$
3	$(p_1(1-p_1) + (1-p_1)p_2)p_2$	$p_1(1-p_1)(1-p_2)(1-p_1) + p_1(1-p_1)p_2$ $+(1-p_1)p_2((1-p_2)(1-p_1) + p_2)$

TABLE 2.

FR2: $X_1X_2\cdots X_{2i-2}$ is a Dyck word. In other words, the number of Ls in $X_1X_2\cdots X_{2i-2}$ is equal to the number of Bs, and the number of Ls in every prefix of the word is at least as many as the number of Bs. And,

FR3: $X_{2i-1} = B$.

Observe that for the event $\{\mathsf{FirstReturn}_n = 1\}$, the middle requirement is one about an empty word and so is trivially satisfied. Let us write

$$s_i^{\mathsf{Markov}} = \mathbb{P}(\mathsf{FirstReturn}_n(p_1, p_2) = i)$$

for all $1 \le i \le n$ and define

$$E_n^{\mathsf{Markov}}(p_1,p_2) := \mathbb{E}(\mathsf{FirstReturn}_n(p_1,p_2))$$
$$E^{\mathsf{Markov}}(z) := \sum_{n\ge 1} E_n^{\mathsf{Markov}}(p_1,p_2) z^n.$$

In this section we derive an expression for $E^{\mathsf{Markov}}(z)$. Before doing so we require a generating function of a 4-tuple overall all Dyck paths of semi-length n. A Dyck path of semi-length n is a path from $(0,0)$ to $(2n,0)$ that takes steps in the set $\{\mathsf{u} = (1,1), \mathsf{d}(1,-1)\}$ that never dips below the x-axis. For each vertex of such a Dyck path D except the first $(0,0)$ and last $(2n,0)$, let $\mathsf{peaks}(D)$, $\mathsf{valleys}(D)$, $\mathsf{ups}(D)$, and $\mathsf{downs}(D)$ be the number of vertices that form the centre vertices of peaks, valleys, double-ups, and double-downs, respectively. Let $\mathsf{slen}(D)$ be the semi-length of the Dyck path D. Define the generating function

$$F(z;u,v,x,y) := \sum_D z^{\mathsf{slen}(D)} u^{\mathsf{peaks}(D)} v^{\mathsf{valleys}(D)} x^{\mathsf{ups}(D)} y^{\mathsf{downs}(D)}. \tag{7}$$

where the sum is over all Dyck paths including the empty one. It begins

$$F(z;u,v,x,y) = 1 + zu + z^2(xuy + u^2v) + z^3(ux^2y^2 + 2u^2vxy + u^2vxy + u^3v^2) + \mathcal{O}(z^4).$$

Proposition 3.2.

$$F(z;u,v,x,y) = \frac{1 - z(uv + xy - 2xyv) - \sqrt{(z(uv+xy)-1)^2 - 4z^2uvxy}}{2zxyv}.$$

Proof. The set of Dyck paths $\mathcal{D}$ admits the factorization

$$\mathcal{D} = 1 + \mathsf{u}\mathcal{D}\mathsf{d}\mathcal{D}.$$

By conditioning on whether each of the $\mathcal{D}$'s on the right hand side represents the empty path, or not, we can rewrite this in a more useful way:

$$\mathcal{D} = 1 + \mathsf{ud} + \mathsf{ud}(\mathcal{D}-1) + \mathsf{u}(\mathcal{D}-1)\mathsf{d} + \mathsf{u}(\mathcal{D}-1)\mathsf{d}(\mathcal{D}-1).$$

Using this, we have

$$\begin{aligned}F(z;u,v,x,y) &= \sum_{D\in\mathcal{D}} z^{\mathsf{slen}(D)} u^{\mathsf{peaks}(D)} v^{\mathsf{valleys}(D)} x^{\mathsf{ups}(D)} y^{\mathsf{downs}(D)} \\ &= z^0u^0v^0x^0y^0 + z^{\mathsf{slen}(\mathsf{ud})} u^{\mathsf{peaks}(\mathsf{ud})} v^{\mathsf{valleys}(\mathsf{ud})} x^{\mathsf{ups}(\mathsf{ud})} y^{\mathsf{downs}(\mathsf{ud})} \\ &\quad + \sum_{D\in\{\mathsf{ud}(\mathcal{D}-1)\}} z^{\mathsf{slen}(D)} u^{\mathsf{peaks}(D)} v^{\mathsf{valleys}(D)} x^{\mathsf{ups}(D)} y^{\mathsf{downs}(D)} \\ &\quad + \sum_{D\in\{\mathsf{u}(\mathcal{D}-1)\mathsf{d}\}} z^{\mathsf{slen}(D)} u^{\mathsf{peaks}(D)} v^{\mathsf{valleys}(D)} x^{\mathsf{ups}(D)} y^{\mathsf{downs}(D)} \\ &\quad + \sum_{D\in\{\mathsf{u}(\mathcal{D}-1)\mathsf{d}(\mathcal{D}-1)\}} z^{\mathsf{slen}(D)} u^{\mathsf{peaks}(D)} v^{\mathsf{valleys}(D)} x^{\mathsf{ups}(D)} y^{\mathsf{downs}(D)}.\end{aligned}$$

The first term on the right-hand side is simply 1 while the second term (for $D = \mathsf{ud}$) is zu. The third term becomes $zuv(F(z;u,v,x,y)-1)$ and the remaining two terms are $zxy(F-1)$ and $zxyv(F-1)^2$. This gives the functional equation

$$F = F(z;u,v,x,y) = 1 + zu + zuv(F-1) + zxy(F-1) + zxyv(F-1)^2. \tag{8}$$

This quadratic in F has roots

$$F(z;u,v,x,y) = \frac{1 - z(uv+xy-2xyv) \pm \sqrt{(z(uv+xy)-1)^2 - 4z^2uvxy}}{2zxyv}.$$

Since the special case $u=v=x=y=1$ must result in the Catalan generating function $(1-\sqrt{1-4z})/2z$, the choice of root is clear and the result follows. □

Theorem 3.3. *The generating function for the expected first return is*

$$E^{\mathsf{Markov}}(z) = \frac{z + (c_2-c_1)z^2 - c_2z^2F(z;1-p_2,1-p_1,p_2,p_1)}{(1-z)^2},$$

where $c_1 := p_1^2 + q_1q_2$, $c_2 := p_1q_1(p_1+p_2)$, *and*

$$F(z;q_2,q_1,p_2,p_1) = \frac{1 - z(q_2q_1 + p_2p_1 - 2p_2p_1q_1) - \sqrt{1 - 2z(q_2q_1+p_2p_1) + z^2(q_2q_1-p_2p_1)^2}}{2zp_2p_1q_1}.$$

Proof. For $n \ge 2$ the expected value $\mathbb{E}(\mathsf{FirstReturn}_n(p_1,p_2)) = \sum_{i=1}^n i \cdot s_i^{\mathsf{Markov}}$. The first of these probabilities is

$$s_1^{\mathsf{Markov}} = p_1^2 + (1-p_1)(1-p_2).$$

For $i \in [2, n-1]$, the event $\{\mathsf{FirstReturn}_n(p_1,p_2) = i\}$ happens precisely when the three conditions (FR1–FR3) at the start of this section hold true. By conditioning on X_0 we have

$$\begin{aligned} s_i^{\mathsf{Markov}} =& \mathbb{P}(X_0 = B)\cdot\mathbb{P}(X_1 = L|X_0 = B)\cdot\mathbb{P}(X_1\cdots X_{2i-2} \text{ is a Dyck word}|X_1 = B) \\ &\cdot \mathbb{P}(X_{2i-1} = B|X_{2i-2} = B) \\ &+ \mathbb{P}(X_0 = L)\cdot\mathbb{P}(X_1 = L|X_0 = L)\cdot\mathbb{P}(X_1\cdots X_{2i-2} \text{ is a Dyck word}|X_1 = B) \\ &\cdot \mathbb{P}(X_{2i-1} = B|X_{2i-2} = B) \\ =& p_1(1-p_1)\left([z^{i-1}]F(z;1-p_2,1-p_1,p_2,p_1)\right)p_1 \\ &+ (1-p_1)p_2\left([z^{i-1}]F(z;1-p_2,1-p_1,p_2,p_1)\right)p_1 \\ =& p_1\left(p_1(1-p_1) + (1-p_1)p_2\right)[z^{i-1}]F(z;1-p_2,1-p_1,p_2,p_1), \end{aligned}$$

where F is given in Proposition 3.2. For $i = n$ we have

$$s_n^{\mathsf{Markov}} = 1 - (s_1^{\mathsf{Markov}} + s_2^{\mathsf{Markov}} + \ldots + s_{n-1}^{\mathsf{Markov}}).$$

In what follows we will abbreviate s_i^{Markov} as s_i. The above expressions for s_i allow us to write

$$\begin{aligned}
E^{\mathsf{Markov}}(z) &= z + \sum_{n\geq 2} E_n^{\mathsf{Markov}}(p_1,p_2) z^n \\
&= z + z^2\left(s_1 + 2s_2\right) + \sum_{n\geq 3} z^n \sum_{i=1}^{n} i s_i \\
&= z + z^2\left(s_1 + 2(1-s_1)\right) + \sum_{n\geq 3} z^n \left(s_1 + \left[\sum_{i=2}^{n-1} i s_i\right] + n\left[1-(s_1+\ldots+s_{n-1})\right]\right) \\
&= \left[z + \sum_{n\geq 2} z^n s_1\right] + \sum_{n\geq 2} z^n n\left[1-(s_1+\ldots+s_{n-1})\right] + \sum_{n\geq 3} z^n \sum_{i=2}^{n-1} i s_i \qquad (9)
\end{aligned}$$

The first term in (9) in the square brackets is

$$z + \sum_{n\geq 2} z^n s_1 = z + \left(p_1^2 + (1-p_1)(1-p_2)\right)\frac{z^2}{1-z}.$$

The second term in (9) is

$$\begin{aligned}
&\sum_{n\geq 2} z^n n\left[1-(s_1+\ldots+s_{n-1})\right] \\
&= \sum_{n\geq 2} n z^n \left(1-s_1\right) - \sum_{n\geq 3} z^n n \sum_{j=2}^{n-1} s_j \\
&= \left(1-\left(p_1^2+(1-p_1)(1-p_2)\right)\right) z \left(\frac{1}{(1-z)^2} - 1\right) - \sum_{n\geq 3} z^n n \sum_{j=2}^{n-1} s_j.
\end{aligned}$$

The final summation in this expression has the same form as the third term in (9). Setting $c_1 := p_1^2 + (1-p_1)(1-p_2)$ and bringing together these observations regarding (9) we have

$$\begin{aligned}
E^{\mathsf{Markov}}(z) &= z + \frac{c_1 z^2}{1-z} + (1-c_1)\, z \left(\frac{1}{(1-z)^2} - 1\right) - \sum_{n\geq 3} z^n n \sum_{j=2}^{n-1} s_j + \sum_{n\geq 3} z^n \sum_{i=2}^{n-1} i s_i \\
&= z + \frac{c_1 z^2}{1-z} + (1-c_1)\, z \left(\frac{1}{(1-z)^2} - 1\right) - \sum_{n\geq 3} z^n \sum_{i=2}^{n-1} (n-i) s_i. \qquad (10)
\end{aligned}$$

From before, we have that for $2 \leq i < n$ and $n \geq 3$,

$$s_i = c_2 [z^{i-1}] F(z; 1-p_2, 1-p_1, p_2, p_1),$$

where $c_2 := p_1\left(p_1(1-p_1) + (1-p_1)p_2\right)$. This means that the sum in (10) is

$$\begin{aligned}
\sum_{n\geq 3} z^n \sum_{i=2}^{n-1} (n-i) s_i &= \sum_{n\geq 3} z^n \sum_{i=2}^{n-1} (n-i) c_2 [z^{i-1}] F(z; 1-p_2, 1-p_1, p_2, p_1) \\
&= c_2 z \sum_{m\geq 2} z^m \sum_{j=1}^{m-1} (m-j) [z^j] F(z; 1-p_2, 1-p_1, p_2, p_1) \\
&= c_2 z \cdot \frac{z}{(1-z)^2} \left(F(z; 1-p_2, 1-p_1, p_2, p_1) - 1\right).
\end{aligned}$$

Replacing this in equation (10) and simplifying gives

$$E^{\mathsf{Markov}}(z) = z + \frac{c_1 z^2}{1-z} + (1-c_1)\, z\left(\frac{1}{(1-z)^2} - 1\right) - \frac{c_2 z^2\left(F(z;1-p_2,1-p_1,p_2,p_1)-1\right)}{(1-z)^2}$$

$$= \frac{z + (c_2-c_1)z^2 - c_2 z^2 F(z;1-p_2,1-p_1,p_2,p_1)}{(1-z)^2}. \qquad \square$$

From this proof, since $s_i^{\mathsf{Markov}} = p_1\,(p_1 q_1 + p_2 q_1)\,[z^{i-1}]F(z;q_2,q_1,p_2,p_1)$, we can give a generating function for the probability of first returning to a diagonal state after $2n$ steps.

Corollary 3.4. *Let $S^{\mathsf{Markov}}(z) = \sum_{n\geq 1} s_n^{\mathsf{Markov}} z^n$ be the generating function for the sequence of probabilities s_n^{Markov}, where s_n^{Markov} is the probability of first returning to a diagonal state after $2n$ steps. Then*

$$S^{\mathsf{Markov}}(z) = (q_1 q_2 + p_1^2)z + p_1(p_1 q_1 + p_2 q_1)z(F(z;q_2,q_1,p_2,p_1)-1).$$

3.2. **The expected residue.** Let $M_n^{\mathsf{Markov}}(p_1,p_2) = \mathbb{E}(\mathsf{Residue}_n)$ be the expected residue for this case and set

$$M^{\mathsf{Markov}}(z;p_1,q_1) = \sum_{n\geq 1} M_n^{\mathsf{Markov}}(p_1,p_2)z^n.$$

Theorem 3.5.

$$M^{\mathsf{Markov}}(z;p_1,p_2) = \frac{p_1 z\left(1-(p_1+p_2)z + (p_1+p_2-1)z^2\right)}{q_1(1-z)^2(1-S^{\mathsf{Markov}}(z))} - \frac{z((p_1-q_1)+(q_2-p_1)z)}{q_1(1-z)^2}.$$

Proof. As discussed in Example 3.1, $M_1^{\mathsf{Markov}}(p_1,p_2) = 1$. For all $n \geq 2$, we condition on the first return to the diagonal, if any, to write

$$M_n^{\mathsf{Markov}}(p_1,p_2) = \sum_{0<i<n} s_i^{\mathsf{Markov}} M_{n-i}^{\mathsf{Markov}}(p_1,p_2) + L_n^{\mathsf{Markov}}(p_1,p_2)$$

The probability s_i^{Markov} is defined in the previous section. We define now the probability $p_{n,j}$ to denote the probability of a path that starts at (n,n) and never returns to the boundary line but instead first touches the x-axis at $(j,0)$. Then the L^{Markov} values are $L_1^{\mathsf{Markov}}(p_1,p_2) = 1$ and for $n \geq 2$, $L_n^{\mathsf{Markov}}(p_1,p_2) = \sum_{j\geq 2} j \cdot p_{n,j}$ is the contribution to the expected residue from those cases where the path never returns to the boundary line. By setting $L^{\mathsf{Markov}}(z) = \sum_{n\geq 1} L_n^{\mathsf{Markov}}(p_1,q_1)z^n$ we have the equation $M^{\mathsf{Markov}}(z) = S^{\mathsf{Markov}}(z)M^{\mathsf{Markov}}(z) + L^{\mathsf{Markov}}(z)$ which gives

$$M^{\mathsf{Markov}}(z) = \frac{L^{\mathsf{Markov}}(z)}{1-S^{\mathsf{Markov}}(z)}.$$

The denominator $1 - S^{\mathsf{Markov}}(z)$ becomes the term mentioned in Corollary 3.4. To determine the numerator $L^{\mathsf{Markov}}(z)$ requires some work.

Let $P_n(x) = \sum_{j\geq 1} p_{n,j}x^j$ and $P(x,y) = \sum_{n\geq 1} P_n(x)y^n$. Notice that we can easily calculate $p_{1,1} = 1$, $p_{2,2} = p_1 q_1 + q_1 p_2$ and for $n \geq 2$, $p_{n,0} = p_{n,1} = 0$. From this it is clear that

$$P_1(x) = x \text{ and } P_2(x) = (p_1 q_1 + q_1 p_2)x^2.$$

For $n \geq 3$ we condition on which node a path enters row 1 at to write:

$$p_{n,j} = p_{n-1,j-1}p_2 + \sum_{i=0}^{n-j-1} p_{n-1,j+i}q_2(p_1)^i q_1.$$

Multiply both sides by x^j, sum over all $j \geq 2$, and simplify to find:

$$P_n(x) - p_{n,1}x = xp_2 P_{n-1}(x) + \frac{q_1 q_2 x^2}{p_1(p_1-x)} \cdot (P_{n-1}(p_1) - p_{n-1,1}p_1) - \frac{xq_1q_2}{p_1-x}\left(P_{n-1}(x) - p_{n-1,1}x\right).$$

A simple argument shows that $P_{n-1}(p_1) = p_1 s_{n-1}^{\mathsf{Markov}}/q_2$ for all $n \geq 3$. For $n = 1$ we have $P_1(p_1) = p_1$. Using this and rearranging gives

$$P_n(x) = \frac{q_1 x^2}{p_1 - x} s_{n-1}^{\mathsf{Markov}} + \frac{x(p_2(p_1 - x) - q_1 q_2)}{p_1 - x} P_{n-1}(x),$$

for all $n \geq 3$. By multiplying this simple recurrence by y^n and summing over all $n \geq 3$, we now have

$$P(x, y) = \frac{(p_1 - x)(xy + (p_1 q_1 - p_1 p_2)x^2 y^2) + (p_1 q_1^2 - q_1 p_1 p_2)x^2 y^2 + q_1 x^2 y S^{\mathsf{Markov}}(y)}{(p_1 - x)(1 - p_2 xy) + q_1 q_2 xy}.$$

Differentiating P with respect to x and substituting $(x, y) \leftarrow (1, z)$ gives

$$L^{\mathsf{Markov}}(z) = \frac{\partial P(x,y)}{\partial x}\bigg|_{\substack{x \leftarrow 1 \\ y \leftarrow z}} = \frac{z}{q_1(1-z)^2}\big[q_1 + (p_1 q_1 - q_2 - p_1 p_2)z + (p_1 p_2 - p_1 q_1)z^2 \\ + (p_1 - q_1)S^{\mathsf{Markov}}(z) + (q_2 - p_1)zS^{\mathsf{Markov}}(z)\big].$$

Thus $M^{\mathsf{Markov}}(z; p_1, p_2)$ equals

$$\frac{z\left(q_1 + (p_1 q_1 - q_2 - p_1 p_2)z + (p_1 p_2 - p_1 q_1)z^2 + (p_1 - q_1)S^{\mathsf{Markov}}(z) + (q_2 - p_1)zS^{\mathsf{Markov}}(z)\right)}{q_1(1-z)^2\left(1 - S^{\mathsf{Markov}}(z)\right)}.$$

Dividing $1 - S^{\mathsf{Markov}}(z)$ into the numerator and simplifying results in the stated expression. □

4. Asymptotics for the Markov-chooser process

4.1. **Asymptotics of the expected first return to a diagonal state.** In this section we consider the asymptotic behaviour of $[z^n]E^{\mathsf{Markov}}(z)$. First let us notice that we can rearrange the radicand in the statement of Theorem 3.3 to give

$$F(z; 1 - p_2, 1 - p_1, p_2, p_1) - 1 = \frac{1 - (q_1 q_2 + p_1 p_2)z - \sqrt{(1 - \alpha z)(1 - \beta z)}}{2 p_1 p_2 q_1 z}, \tag{11}$$

where $\alpha := (\sqrt{p_1 p_2} - \sqrt{q_1 q_2})^2$ and $\beta := (\sqrt{p_1 p_2} + \sqrt{q_1 q_2})^2$.

Lemma 4.1. *We have $0 \leq \alpha < \beta \leq 1$. Moreover, $\alpha = 0$ iff $p_1 + p_2 = 1$ and $\beta = 1$ iff $p_1 = p_2$.*

Proof. That $0 \leq \alpha < \beta$ for all $0 < p_1, p_2 < 1$ is clear. We have that $\alpha = (\sqrt{p_1 p_2} - \sqrt{q_1 q_2})^2 = 0$ if and only if $\sqrt{p_1 p_2} = \sqrt{q_1 q_2} = \sqrt{(1 - p_1)(1 - p_2)}$, which occurs iff $p_1 p_2 = (1 - p_1)(1 - p_2)$ and this is equivalent to $p_1 + p_2 = 1$.

Secondly, the equality $\beta = (\sqrt{p_1 p_2} + \sqrt{q_1 q_2})^2 = 1$ holds iff $\sqrt{p_1 p_2} = 1 - \sqrt{q_1 q_2}$. Squaring both sides and simplifying results in $2\sqrt{q_1 q_2} = 1 + q_1 q_2 - p_1 p_2$. Squaring both sides again and simplifying results in $(q_1 - q_2)^2 = 0$, which is equivalent to $p_1 = p_2$.

A similar argument can be used to show that we never have $\beta > 1$. □

Proposition 4.2. *The radius of convergence of $S^{\mathsf{Markov}}(z)$ is $\rho = \dfrac{1}{(\sqrt{p_1 p_2} + \sqrt{q_1 q_2})^2}$. Moreover $\rho \geq 1$ for all $0 < p_1, p_2 < 1$ and $\rho = 1$ iff $p_1 = p_2$*

Proof. By writing

$$S^{\mathsf{Markov}}(z) = c_1 z + c_2 z \left(\frac{1 - (q_1 q_2 + p_1 p_2)z - \sqrt{(1 - \alpha z)(1 - \beta z)}}{2 p_1 p_2 q_1 z}\right) \\ = c_1 z + \frac{c_2}{2 p_1 p_2 q_1}\left(1 - (q_1 q_2 + p_1 p_2)z - \sqrt{(1 - \alpha z)(1 - \beta z)}\right),$$

and recalling from Proposition 4.1 that $0 \leq \alpha < \beta \leq 1$, it follows that the radius of convergence of $S^{\mathsf{Markov}}(z)$ is $\rho = 1/\beta$. That $\rho \geq 1$ follows directly from Lemma 4.1 since $\beta \leq 1$. That $\rho = 1$ iff $p_1 = p_2$ is the last statement of Lemma 4.1. □

Proposition 4.3. *The probability of an eventual return to a diagonal state is*

$$S^{\mathsf{Markov}}(1) = \begin{cases} 1 & \text{if } p_1 \geq p_2 \\ 1 - \dfrac{p_2^2 - p_1^2}{p_2} & \text{if } p_1 < p_2. \end{cases}$$

Proof. Recall that $c_1 = p_1^2 + q_1 q_2$, $c_2 = p_1 q_1 (p_1 + p_2)$. Setting $z = 1$ in Corollary 3.4 gives

$$\begin{aligned} S^{\mathsf{Markov}}(1) &= c_1 + c_2 \frac{1 - (q_1 q_2 + p_1 p_2) - \sqrt{(p_1 p_2 + q_1 q_2 - 1)^2 - 4 p_1 p_2 q_1 q_2}}{2 p_1 p_2 q_1} \\ &= c_1 + c_2 \frac{p_1 + p_2 - 2p_1 p_2 - \sqrt{p_1^2 + p_2^2 - 2 p_1 p_2}}{2 p_1 p_2 q_1} \\ &= q_1 q_2 + p_1^2 + p_1 (p_1 q_1 + q_1 p_2) \frac{p_1 + p_2 - 2 p_1 p_2 - |p_1 - p_2|}{2 p_1 p_2 q_1} \\ &= q_1 q_2 + p_1^2 - p_1 (p_1 + p_2) + (p_1 + p_2) \frac{p_1 + p_2 - |p_1 - p_2|}{2 p_2} \\ &= 1 - p_1 - p_2 + (p_1 + p_2) \frac{p_1 + p_2 - |p_1 - p_2|}{2 p_2}. \end{aligned}$$

The result follows. □

We will require $\frac{d}{dz} S^{\mathsf{Markov}}(z)$ at $z = 1$ for $p_1 \neq p_2$ so we determine that now.

Proposition 4.4.

(a) $$\frac{d}{dz} S^{\mathsf{Markov}}(z)\Big|_{z=1} = \begin{cases} p_1 \dfrac{q_1 + q_2}{p_1 - p_2} & \text{if } p_1 > p_2, \\ \dfrac{p_1^2 q_1 + p_2^2 q_2}{p_2 (p_2 - p_1)} & \text{if } p_1 < p_2. \end{cases}$$

(b) *For* $p_1 > p_2$, $$\frac{d^2}{dz^2} S^{\mathsf{Markov}}(z)\Big|_{z=1} = \frac{2 q_1 (p_1 + p_2)(2 p_1 - p_1^2 - p_2^2)}{(p_1 - p_2)^3}.$$

Proof. (a) Differentiating $S(z) := S^{\mathsf{Markov}}(z)$ in Corollary 3.4 gives $S'(z) = c_1 + c_2 (F(z) - 1) + c_2 z F'(z)$. To differentiate $F(z)$ we use the functional equation in Equation 8:

$$(F(z) - 1) = z[q_2 + (q_1 q_2 + p_1 p_2)(F(z) - 1) + p_1 p_2 q_1 (F(z) - 1)^2].$$

Differentiate both sides to find

$$\begin{aligned} F'(z) &= \left[q_2 + (q_1 q_2 + p_1 p_2)(F(z) - 1) + p_1 p_2 q_1 (F(z) - 1)^2\right] \\ &\quad + z \left[(q_1 q_2 + p_1 p_2) F'(z) + 2 p_1 p_2 q_1 (F(z) - 1) F'(z)\right], \end{aligned} \tag{12}$$

and so

$$\begin{aligned} F'(1) &= \left[q_2 + (q_1 q_2 + p_1 p_2)(F(1) - 1) + p_1 p_2 q_1 (F(1) - 1)^2\right] \\ &\quad + \left[(q_1 q_2 + p_1 p_2) F'(1) + 2 p_1 p_2 q_1 (F(1) - 1) F'(1)\right]. \end{aligned}$$

Note that for $F(1)$ we have

$$\begin{aligned} F(1) - 1 &= \frac{1 - (q_1 q_2 + p_1 p_2) - \sqrt{((p_1 p_2 + q_1 q_2) - 1)^2 - 4 p_1 p_2 q_1 q_2}}{2 p_1 p_2 q_1} \\ &= \frac{p_1 + p_2 - 2 p_1 p_2 - |p_1 - p_2|}{2 p_1 p_2 q_1} = \begin{cases} \frac{1}{p_1} & \text{if } p_1 > p_2 \\ \frac{q_2}{p_2 q_1} & \text{if } p_1 < p_2. \end{cases} \end{aligned}$$

We consider the two cases separately.

Case $p_1 > p_2$: For this case we have

$$F'(1) = \left[q_2 + (q_1q_2 + p_1p_2)\frac{1}{p_1} + p_1p_2q_1\frac{1}{p_1^2}\right] \\ + \left[(q_1q_2 + p_1p_2)F'(1) + 2p_1p_2q_1\frac{1}{p_1}F'(1)\right].$$

Solving for $F'(1)$ gives $F'(1) = \dfrac{1}{p_1(p_1 - p_2)}$. Therefore

$$S'(1) = c_1 + c_2\frac{1}{p_1} + c_2\frac{1}{p_1(p_1 - p_2)} = p_1\frac{q_1 + q_2}{p_1 - p_2}.$$

Case $p_1 < p_2$: In this case we have

$$F'(1) = \left[q_2 + (q_1q_2 + p_1p_2)(\frac{q_2}{p_2q_1}) + p_1p_2q_1(\frac{q_2}{p_2q_1})^2\right] \\ + \left[(q_1q_2 + p_1p_2)F'(1) + 2p_1p_2q_1\frac{q_2}{p_2q_1}F'(1)\right].$$

Solving for $F'(1)$ gives $F'(1) = \dfrac{q_2}{p_2q_1(p_2 - p_1)}$. Therefore

$$S'(1) = c_1 + c_2\frac{q_2}{p_2q_1} + c_2\frac{q_2}{p_2q_1(p_2 - p_1)} = \frac{p_1^2q_1 + p_2^2q_2}{p_2(p_2 - p_1)}.$$

(b) Differentiate $S'(z) = c_1 + c_2(F(z) - 1) + c_2zF'(z)$ from part (a) to get $S''(z) = 2c_2F'(z) + c_2zF''(z)$. From the proof of part (a) with $p_1 > p_2$ we have $F'(1) = \dfrac{1}{p_1(p_1 - p_2)}$ and we now require $F''(1)$. For notational convenience within this part of the proof let $\lambda := q_1q_2 + p_1p_2$ and $\mu := p_1p_2q_1$. Then differentiating $F'(z)$ in equation (12) gives

$$F''(z) = \left[\lambda F'(z) + 2\mu(F(z) - 1)F'(z)\right] + \left[\lambda F'(z) + 2\mu(F(z) - 1)F'(z)\right] \\ + z\left[\lambda F''(z) + 2\mu(F'(z))^2 + 2\mu(F(z) - 1)F''(z)\right].$$

This gives

$$F''(1) = \left[\frac{\lambda}{p_1(p_1 - p_2)} + \frac{2\mu}{p_1^2(p_1 - p_2)}\right] + \left[\frac{\lambda}{p_1(p_1 - p_2)} + \frac{2\mu}{p_1^2(p_1 - p_2)}\right] \\ + \left[\lambda F''(1) + \frac{2\mu}{p_1^2(p_1 - p_2)^2} + \frac{2\mu}{p_1}F''(1)\right],$$

where we used $F(1) - 1 = 1/p_1$. Solving the above equation for $F''(1)$ and simplifying gives

$$F''(1) = \frac{2(p_1q_1 + p_2(p_1 - p_2))}{p_1(p_1 - p_2)^3}.$$

Therefore

$$\begin{aligned} S''(1) &= 2c_2\left(\frac{1}{p_1(p_1 - p_2)} + \frac{(p_1q_1 + p_2(p_1 - p_2))}{p_1(p_1 - p_2)^3}\right) \\ &= \frac{2q_1(p_1 + p_2)}{(p_1 - p_2)^3}\left[(p_1 - p_2)^2 + p_1q_1 + p_1p_2 - p_2^2\right] \\ &= \frac{2p_1q_1q_2(p_1 + p_2)}{(p_1 - p_2)^3}. \end{aligned}$$

□

Since $S^{\mathsf{Markov}}(z)$ is a linear function of $F(z)$ (from Corollary 3.4), we can rewrite the expression for $E^{\mathsf{Markov}}(z)$ given in Theorem 3.3 as

$$E^{\mathsf{Markov}}(z) = \frac{z}{(1 - z)^2}\left(1 - S^{\mathsf{Markov}}(z)\right). \tag{13}$$

A more direct way to see this is to note that the coefficient of z^n of the right-hand side is $n - \sum_{i=1}^{n}(n-i)s_i$.

We are now ready to state our main result of this section.

Theorem 4.5. *Let r be any value greater than $\frac{1}{\rho}$. Then*

$$E_n^{\mathsf{Markov}}(p_1,p_2) = \begin{cases} p_1\dfrac{q_1+q_2}{p_1-p_2} + \mathcal{O}(r^n) & \text{if } p_1 > p_2 \\ \sqrt{\dfrac{16p_1q_1}{\pi}}\sqrt{n} \;+\; \dfrac{1-6p_1q_1}{\sqrt{16p_1q_1\pi}}\dfrac{1}{\sqrt{n}} \;+\; \mathcal{O}(\dfrac{1}{\sqrt{n^3}}) & \text{if } p_1 = p_2 \\ (\dfrac{p_2^2-p_1^2}{p_2})n + \dfrac{p_1^2q_1+p_2^2q_2}{p_2(p_2-p_1)} + \mathcal{O}(r^n) & \text{if } p_1 < p_2. \end{cases}$$

Proof. **Case $p_1 = p_2$.** For notational convenience we will let $p := p_1$ and $q := q_1$ for this part of the proof. From Lemma 4.1, since $p_1 = p_2$ we have $\beta = 1$ and $\alpha = (p-q)^2$. By combining (13) with the expression for $S^{\mathsf{Markov}}(z)$ used in Proposition 4.2, we can write

$$\begin{aligned} E^{\mathsf{Markov}}(z) =& \frac{z}{(1-z)^2}\left(1 - c_1 z - \frac{p_1+p_2}{2p_2}\left[1-(q_1q_2+p_1p_2)z - \sqrt{(1-\alpha z)(1-\beta z)}\right]\right) \\ =& \frac{z}{(1-z)^2}\left[1-(p^2+q^2)z - 1 + (p^2+q^2)z + \sqrt{1-\alpha z}\sqrt{1-z}\right] \\ =& (1-z)^{-\frac{3}{2}}(z\sqrt{1-\alpha z}). \end{aligned}$$

Thus $E^{\mathsf{Markov}}(z)$ has a (dominant) singularity at $z = 1$. We proceed by using the methods of singularity analysis of generating functions, as explained in Flajolet and Sedgewick [4, Chpt. VI]. Set $g(z) = z\sqrt{1-\alpha z}$ and write $g(z) = \sum_{n\geq 0} g_n(1-z)^n$. The first two coefficients g_n are

$$\begin{aligned} g_0 &= g(1) = \sqrt{1-\alpha} = \sqrt{1-(p-q)^2} = 2\sqrt{pq}, \text{ and} \\ g_1 &= -g'(1) = -\left.\left[(1-\alpha z)^{\frac{1}{2}} + \frac{1}{2}z(1-\alpha z)^{-\frac{1}{2}}(-\alpha)\right]\right|_{z=1} \\ &= -2\sqrt{pq} + \frac{\alpha}{4\sqrt{pq}} = \frac{p^2-10pq+q^2}{4\sqrt{pq}}. \end{aligned}$$

From this we have:

$$\begin{aligned} E^{\mathsf{Markov}}(z) &= (1-z)^{-\frac{3}{2}}(z\sqrt{1-\alpha z}) \\ &= (1-z)^{-\frac{3}{2}}[2\sqrt{pq} + \frac{p^2-10pq+q^2}{4\sqrt{pq}}(1-z) + \mathcal{O}((1-z)^2) \\ &= 2\sqrt{pq}(1-z)^{-\frac{3}{2}} + \frac{p^2-10pq+q^2}{4\sqrt{pq}}(1-z)^{-\frac{1}{2}} + \mathcal{O}((1-z)^{\frac{1}{2}}). \end{aligned}$$

We quote the following identities from [4, p.388];

$$[z^n](1-z)^{-3/2} = \sqrt{\frac{n}{\pi}}\left(2 + \frac{3}{4n} + \mathcal{O}\left(\frac{1}{n^2}\right)\right), \tag{14}$$

$$[z^n](1-z)^{-1/2} = \frac{1}{\sqrt{n\pi}}\left(1 - \frac{1}{8n} + \mathcal{O}\left(\frac{1}{n^2}\right)\right), \text{ and} \tag{15}$$

$$[z^n](1-z)^{1/2} = \mathcal{O}\left(\frac{1}{\sqrt{n^3}}\right). \tag{16}$$

This allows us to write

$$\begin{aligned}[z^n]E^{\mathsf{Markov}}(z) =& 2\sqrt{pq}\sqrt{\frac{n}{\pi}}\left(2+\frac{3}{4n}+\mathcal{O}\left(\frac{1}{n^2}\right)\right) \\ &+ \frac{p^2-10pq+q^2}{4\sqrt{pq}}\frac{1}{\sqrt{n\pi}}\left(1-\frac{1}{8n}+\mathcal{O}\left(\frac{1}{n^2}\right)\right) + \mathcal{O}\left(\frac{1}{\sqrt{n^3}}\right) \\ =& \sqrt{\frac{16pq}{\pi}}\sqrt{n} + \frac{p^2-4pq+q^2}{\sqrt{16pq\pi}}\frac{1}{\sqrt{n}} + \mathcal{O}\left(\frac{1}{\sqrt{n^3}}\right),\end{aligned}$$

which completes the proof for the case $p_1 = p_2$.

Case $p_1 \neq p_2$**.** Consider $E^{\mathsf{Markov}}(z) = \frac{z}{(1-z)^2}(1-S^{\mathsf{Markov}}(z))$. As $1-S^{\mathsf{Markov}}(z)$ is holomorphic on a disc centered at the origin of radius $\rho > 1$ (since $p_1 \neq p_2$), the function $E^{\mathsf{Markov}}(z) = z\sum_{n\geq -2} a_n(1-z)^n$ for some constants $(a_n)_{n\geq -2}$. Since $E^{\mathsf{Markov}}(z)$ has a double pole at $z=1$, we will use the asymptotic analysis of functions with poles outlined in Wilf [11, Section 5.2]. By [11, Theorem 5.5], we have that as $n\to\infty$,

$$[z^n]E(z) = a_{-2}n + a_{-1} + \mathcal{O}\left(\left(\frac{1}{\rho}+\epsilon\right)^n\right),$$

for any $\epsilon > 0$. Let $1-S(z) = \sum_{n\geq 0} b_n(1-z)^n$, so that $a_{-2} = b_0$ and $a_{-1} = b_1$. Then

$$b_0 = 1-S(z)\Big|_{z=1} = 1-S(1) = \begin{cases} 0 & \text{if } p_1 > p_2 \\ \dfrac{p_2^2-p_1^2}{p_2} & \text{if } p_1 < p_2. \end{cases}$$

Similarly,

$$b_1 = -\frac{d}{dz}[1-S(z)]\Big|_{z=1} = S'(1) = \begin{cases} p_1\dfrac{q_1+q_2}{p_1-p_2} & \text{if } p_1 > p_2 \\ \dfrac{p_1^2q_1+p_2^2q_2}{p_2(p_2-p_1)} & \text{if } p_1 < p_2, \end{cases}$$

by Proposition 4.4. □

4.2. **Asymptotics of the expected residue** $M_n^{\mathsf{Markov}}(p_1,p_2)$**.** In this section we consider the asymptotic behaviour of $[z^n]M^{\mathsf{Markov}}(z) = M_n^{\mathsf{Markov}}(p_1,p_2)$ for n large. From Theorem 3.5 we have

$$M^{\mathsf{Markov}}(z;p_1,p_2) = \frac{p_1 z\left(1-(p_1+p_2)z+(p_1+p_2-1)z^2\right)}{q_1(1-z)^2(1-S^{\mathsf{Markov}}(z))} - \frac{z((p_1-q_1)+(q_2-p_1)z)}{q_1(1-z)^2},$$

and

$$S^{\mathsf{Markov}}(z) = (q_1q_2+p_1^2)z + \frac{q_1(p_1+p_2)}{2p_2q_1}\left(1-(q_1q_2+p_1p_2)z-\sqrt{(1-\alpha z)(1-\beta z)}\right),$$

by Corollary 3.4 combined with section 4.1. In order to analyse $M^{\mathsf{Markov}}(z)$ it is necessary to determine when $S^{\mathsf{Markov}}(z) = 1$.

Lemma 4.6. *For $|z|\leq 1$, if $S^{\mathsf{Markov}}(z) = 1$ then $z=1$.*

Proof. Recall that $S^{\mathsf{Markov}}(z)$ is convergent on a disc (centered at the origin) of radius $\rho \geq 1$, and note that we have $0 < s_n \in \mathbb{R}$ for all n. Suppose that we have $z_0 \in \mathbb{C}\backslash\{1\}$ with $|z_0| \leq 1$ and satisfying $S^{\mathsf{Markov}}(z_0) = 1$. Then $|z_0| \le 1$ implies $\mathrm{Re}(s_n z_0^n) \le s_n$ for all n. In particular $z_0 \ne 1$ implies $\mathrm{Re}(s_1 z_0) < s_1$. From this we have

$$1 = S^{\mathsf{Markov}}(z_0) = \sum_{n\ge 1} s_n z_0^n = \mathrm{Re}\left(\sum_{n\ge 1} s_n z_0^n\right) = \sum_{n\ge 1} s_n \mathrm{Re}(s_n z_0^n) < \sum_{n\ge 1} s_n \le 1,$$

where the last inequality comes from Proposition 4.3. But this is a contradiction. □

In the analysis of $[z^n]M^{\mathsf{Markov}}(z)$, there are therefore three cases to consider:

Case 1: $p_1 < p_2$ – in this case $S^{\mathsf{Markov}}(1) < 1$ and $\beta < 1$
Case 2: $p_1 = p_2$ – in this case $S^{\mathsf{Markov}}(1) = 1$ and $\beta = 1$
Case 3: $p_1 > p_2$ – in this case $S^{\mathsf{Markov}}(1) = 1$ and $\beta < 1$.

These are distinct cases since if $S^{\mathsf{Markov}}(1) = 1$ then it gives $M^{\mathsf{Markov}}(z)$ a zero in the denominator at $z = 1$, while if $\beta = 1$ then $S^{\mathsf{Markov}}(z)$ has an (isolated) singularity at $z = 1$ due to a $\sqrt{1-z}$ term. We consider the three cases in turn.

Proposition 4.7. *Let $p_1 < p_2$, and let r be any number greater than $(\sqrt{p_1 p_2} + \sqrt{q_1 q_2})^2$. Then*

$$[z^n]M^{\mathit{Markov}}(z) = \frac{p_2 - p_1}{q_1} \cdot n + \left(\frac{q_2 - p_1}{q_1} + \frac{p_1 p_2 (q_1 + q_2)}{q_1 (p_2^2 - p_1^2)}\right) + \mathcal{O}(r^n).$$

Proof. Let $M^{\mathsf{Markov}}(z) = \dfrac{f(z)}{(1-z)^2}$ where

$$f(z) := \frac{1}{q_1} \frac{z\left[q_1 + (p_1 q_1 - q_2 - p_1 p_2) z + p_1 (p_2 - q_1) z^2 + (p_1 - q_1) S(z) + (q_2 - p_1) z S(z)\right]}{1 - S(z)}$$

and $S(z) := S^{\mathsf{Markov}}(z)$. Since $S(z)$ has radius of convergence $\rho > 1$ (see Proposition 4.2), and since $S(z) \ne 1$ for any $z \in \bar{D}(0;1)$ (see Lemma 4.6 and Proposition 4.3), we can let $f(z) = \sum_{n\ge 0} a_n (1-z)^n$ for some constants $(a_n)_{n\ge 0}$. The first of these coefficients is

$$a_0 = f(1) = \frac{1}{q_1} \frac{q_1 - q_2 + (q_2 - q_1) S(1)}{1 - S(1)} = \frac{p_2 - p_1}{q_1}.$$

The next coefficient, with a purposeful sign change due to our choice to expand f about $1 - z$, is derived as follows. (Note we will use $c_3 := p_1 q_1 - q_2 - p_1 p_2$ to make the equations more manageable.))

$$
\begin{aligned}
-a_1 =& \frac{d}{dz} f(z)\Big|_{z=1} \\
=& \frac{1}{q_1} \frac{\left[q_1 + c_3 z + p_1(p_2 - q_1)z^2 + (p_1 - q_1)S(z) + (q_2 - p_1)zS(z)\right]}{1 - S(z)}\Big|_{z=1} \\
&+ \frac{1}{q_1} \frac{z\left[c_3 + 2p_1(p_2 - q_1)z + (p_1 - q_1)S'(z) + (q_2 - p_1)S(z) + (q_2 - p_1)zS'(z)\right]}{1 - S(z)}\Big|_{z=1} \\
&+ \frac{S'(z)}{q_1} \frac{z\left[q_1 + c_3 z + p_1(p_2 - q_1)z^2 + (p_1 - q_1)S(z) + (q_2 - p_1)zS(z)\right]}{(1 - S(z))^2}\Big|_{z=1} \\
=& \frac{1}{q_1} \frac{q_1 - q_2 + (q_2 - q_1)S(1)}{1 - S(1)} + \frac{1}{q_1} \frac{-q_2 + p_1(p_2 - q_1) + (q_2 - q_1)S'(1) + (q_2 - p_1)S(1)}{1 - S(1)} \\
&+ \frac{S'(1)}{q_1} \frac{\left[q_1 - q_2 + (q_2 - q_1)S(1)\right]}{(1 - S(1))^2} \\
=& \frac{1}{q_1(1 - S(1))} \left[q_1 - 2q_2 + p_1(p_2 - q_1) + (2q_2 - 1)S(1)\right] \\
=& \frac{(1 - 2q_2)(1 - S(1))}{q_1(1 - S(1))} + \frac{q_1 + p_1(p_2 - q_1) - 1}{q_1(1 - S(1))} \\
=& \frac{p_2 - q_2}{q_1} - \frac{p_1}{q_1} \frac{p_2(q_2 + q_1)}{p_2^2 - p_1^2},
\end{aligned}
$$

which follows since $1 - S(1) = (p_2^2 - p_1^2)/p_2$ by Proposition 4.3. Applying Wilf [11, Theorem 5.5] we have that as $n \to \infty$,

$$
\begin{aligned}
[z^n]M^{\mathsf{Markov}}(z) =& [z^n]\frac{a_0}{(1 - z)^2} + [z^n]\frac{a_1}{1 - z} + \mathcal{O}(r^n) \\
=& \frac{p_2 - p_1}{q_1}(n + 1) + \frac{p_1}{q_1} \frac{p_2(q_2 + q_1)}{p_2^2 - p_1^2} - \frac{p_2 - q_2}{q_1} + \mathcal{O}(r^n).
\end{aligned}
$$

□

Proposition 4.8. *Let $p_1 = p_2 = p$. Then*

$$
[z^n]M^{\mathsf{Markov}}(z) = \sqrt{\frac{4pn}{q\pi}} + \frac{2q - 1}{q} + \left(\frac{3}{4}\sqrt{\frac{p}{q\pi}} + \frac{12p^2 - 8p - 1}{8q\sqrt{pq\pi}}\right)\frac{1}{\sqrt{n}} + \mathcal{O}\left(\frac{1}{\sqrt{n^3}}\right).
$$

Proof. First notice that $\alpha = (\sqrt{p_1 p_2} - \sqrt{q_1 q_2})^2 = (p - q)^2$ and $\beta = (\sqrt{p_1 p_2} + \sqrt{q_1 q_2})^2 = 1$. By using equation (11), and letting $S(z) := S^{\mathsf{Markov}}(z)$, we have

$$
\begin{aligned}
1 - S(z) =& 1 - (q_1 q_2 + p_1^2)z - (p_1 q_1(p_1 + p_2))z\left(\frac{1 - (q_1 q_2 + p_1 p_2)z - \sqrt{(1 - \alpha z)(1 - \beta z)}}{2p_1 p_2 q_1 z}\right) \\
=& 1 - (q^2 + p^2)z - \left(1 - (q^2 + p^2)z - \sqrt{(1 - (p - q)^2 z)}\sqrt{(1 - z)}\right) \\
=& \sqrt{(1 - (p - q)^2 z)}\sqrt{(1 - z)}.
\end{aligned}
$$

So

$$
\begin{aligned}
M^{\mathsf{Markov}}(z) =& \frac{z}{(1-z)^2(1-S(z))} \cdot \left[1 + \left(p_1 - \frac{q_2}{q_1} - \frac{1}{q_1}p_1p_2\right) z + \left(\frac{1}{q_1}p_1p_2 - p_1\right) z^2 \right. \\
&\left. + \left(\frac{1}{q_1} - 2\right) S(z) + \left(1 - \frac{1}{q_1} + \frac{q_2}{q_1}\right) zS(z)\right] \\
=& \frac{z}{(1-z)^2} \cdot \frac{1 + \left(p - 1 - \frac{p^2}{q}\right) z + \left(\frac{p^2}{q} - p\right) z^2 + \left(\frac{1-2q}{q}\right) S(z) + \left(\frac{2q-1}{q}\right) zS(z)}{1 - S(z)} \\
=& \frac{z}{(1-z)^2} \cdot \frac{(1-z) + z(1-z)\left(\frac{pq - p^2}{q}\right) + (1-z)\left(\frac{1-2q}{q}\right) S(z)}{1 - S(z)} \\
=& \frac{z}{1-z} \cdot \frac{1}{q} \cdot \frac{q + p(q-p)z + (1-2q)S(z)}{1 - S(z)} \\
=& \frac{z}{1-z} \cdot \frac{1}{q} \cdot \frac{q + p(q-p)z + (1-2q)(1 - \sqrt{(1-(p-q)^2 z)}\sqrt{(1-z)})}{\sqrt{(1-(p-q)^2 z)}\sqrt{(1-z)}} \\
=& \frac{z}{1-z} \cdot \frac{2q-1}{q} + \frac{f(z)}{(1-z)^{3/2}},
\end{aligned}
$$

where $f(z) := \frac{z}{q} \cdot \frac{p + p(q-p)z}{\sqrt{(1-(p-q)^2 z)}}$. The coefficient of z^n coming from the first term is $2 - \frac{1}{q}$. To find the coefficient of z^n in the second term, set $f(z) = \sum_{n\geq 0} a_n (1-z)^n$ for some constants $(a_n)_{n\geq 0}$. Then

$$
a_0 = f(1) = \frac{1}{q}\frac{p + p(q-p)}{\sqrt{(1-(p-q)^2)}} = \frac{p}{q}\frac{2q}{2\sqrt{pq}} = \sqrt{\frac{p}{q}}.
$$

The second coefficient is

$$
\begin{aligned}
-a_1 &= f'(1) \\
&= \frac{p + 2p(q-p)}{q\sqrt{(1-(p-q)^2)}} - \frac{1}{2q}\frac{p + p(q-p)}{(1-(p-q)^2)^{\frac{3}{2}}}(-(p-q)^2) \\
&= \frac{p + 2pq - 2p^2}{2q\sqrt{pq}} + \frac{p}{2q}\frac{2q(p-q)^2}{8pq\sqrt{pq}} \\
&= \frac{4(p + 2pq - 2p^2) + (p-q)^2}{8q\sqrt{pq}} = \frac{-12p^2 + 8p + 1}{8q\sqrt{pq}}.
\end{aligned}
$$

Again, making use of equations (14) and (15) from the proof of Theorem 4.5, we have

$$
[z^n]M^{\mathsf{Markov}}(z) = \left(2 - \frac{1}{q}\right) + \sqrt{\frac{p}{q}}\sqrt{\frac{n}{\pi}}\left(2 + \frac{3}{4n}\right) + \frac{12p^2 - 8p - 1}{8q\sqrt{pq}} \cdot \frac{1}{\sqrt{n\pi}}\left(1 - \frac{1}{8n}\right) + \mathcal{O}\left(\frac{1}{\sqrt{n^3}}\right),
$$

from which the result follows. □

Proposition 4.9. *Let $p_1 > p_2$, and let r be any number greater than $(\sqrt{p_1p_2} + \sqrt{q_1q_2})^2$. Then*

$$
[z^n]M^{\mathsf{Markov}}(z) = \frac{2p_1q_1^2 + (p_1q_1 - p_2q_2) + p_1p_2(p_1 - p_2)}{q_1(q_1 + q_2)(p_1 - p_2)} + \mathcal{O}(r^n).
$$

Proof. Let $S(z) := S^{\mathsf{Markov}}(z)$ and let $M^{\mathsf{Markov}}(z) = \frac{1}{(1-z)^2} \cdot \frac{f(z)}{1-S(z)}$ where

$$f(z) := z\left[1 + \left(p_1 - \frac{q_2}{q_1} - \frac{1}{q_1}p_1p_2\right)z + \left(\frac{1}{q_1}p_1p_2 - p_1\right)z^2 + \left(\frac{1}{q_1} - 2\right)S(z) + \left(1 - \frac{1}{q_1} + \frac{q_2}{q_1}\right)zS(z)\right].$$

The functions $1-S(z)$ and $f(z)$ are holomorphic on a disc centered at the origin of radius $\rho > 1$, so the asymptotics of $[z^n]M^{\mathsf{Markov}}(z)$ are dictated by the behaviour of $M^{\mathsf{Markov}}(z)$ at $z = 1$. It will be essential to understand the zeros of both $f(z)$ and $1 - S(z)$ at that point. In what follows, we will require $f(1)$, $f'(1)$ and $f''(1)$.

- For $f(1)$:

$$\begin{aligned} f(1) &= 1 + p_1 - \frac{q_2}{q_1} - \frac{1}{q_1}p_1p_2 + \frac{1}{q_1}p_1p_2 - p_1 + (\frac{1}{q_1} - 2)S(1) + (1 - \frac{1}{q_1} + \frac{q_2}{q_1})S(1) \\ &= 1 - \frac{q_2}{q_1} + (\frac{1}{q_1} - 2) + (1 - \frac{1}{q_1} + \frac{q_2}{q_1}) \\ &= 0. \end{aligned} \tag{17}$$

- To find $f'(1)$ we have

$$\begin{aligned} f'(z) &= \left[1 + \left(p_1 - \frac{q_2}{q_1} - \frac{1}{q_1}p_1p_2\right)z + \left(\frac{1}{q_1}p_1p_2 - p_1\right)z^2 + \left(\frac{1}{q_1} - 2\right)S(z)\right. \\ &\quad \left. + \left(1 - \frac{1}{q_1} + \frac{q_2}{q_1}\right)zS(z)\right] + z\left[\left(p_1 - \frac{q_2}{q_1} - \frac{1}{q_1}p_1p_2\right) + 2\left(\frac{1}{q_1}p_1p_2 - p_1\right)z\right. \\ &\quad \left. + \left(\frac{1}{q_1} - 2\right)S'(z) + \left(1 - \frac{1}{q_1} + \frac{q_2}{q_1}\right)S(z) + \left(1 - \frac{1}{q_1} + \frac{q_2}{q_1}\right)zS'(z)\right] \\ &= \frac{1}{q_1}\left[q_1 + 2(p_1q_1 - q_2 - p_1p_2)z + 3(p_1p_2 - p_1q_1)z^2 + (p_1 - q_1)S(z)\right. \\ &\quad \left. + 2(q_2 - p_1)zS(z) + (p_1 - q_1)zS'(z) + (q_2 - p_1)z^2S'(z)\right]. \end{aligned} \tag{18}$$

From this, and by using $S'(1) = p_1\frac{q_1+q_2}{p_1-p_2})$ from Proposition 4.4, we have

$$\begin{aligned} f'(1) &= \frac{1}{q_1}\left[\; q_1 + 2(p_1q_1 - q_2 - p_1p_2) + 3(p_1p_2 - p_1q_1) + (p_1 - q_1) + 2(q_2 - p_1)\right. \\ &\quad \left. + (p_1 - q_1)p_1 \cdot \frac{q_1+q_2}{p_1-p_2} + (q_2 - p_1)p_1 \cdot \frac{q_1+q_2}{p_1-p_2}\right] \\ &= \frac{1}{q_1}\left[p_1p_2 - p_1q_1 - p_1 + (q_2 - q_1)p_1 \cdot \frac{q_1+q_2}{p_1-p_2}\right] \\ &= \frac{p_1}{q_1}\left[-(q_1+q_2) + (q_2 - q_1)\frac{q_1+q_2}{p_1-p_2}\right] \\ &= \frac{p_1}{q_1}\left[-(q_1+q_2) + (q_1+q_2)\right] \\ &= 0. \end{aligned}$$

- To determine $f''(1)$ simply differentiate equation (18):

$$\begin{aligned} f''(z) &= \frac{1}{q_1}\left[2(p_1q_1 - q_2 - p_1p_2) + 6(p_1p_2 - p_1q_1)z + (p_1 - q_1)S'(z)\right. \\ &\quad + 2(q_2 - p_1)S(z) + 2(q_2 - p_1)zS'(z) + (p_1 - q_1)S'(z) \\ &\quad \left. + (p_1 - q_1)zS''(z) + 2(q_2 - p_1)zS'(z) + (q_2 - p_1)z^2S''(z)\right]. \end{aligned}$$

This now gives

$$\begin{aligned}
f''(1) &= \frac{1}{q_1}\big[2(p_1q_1 - q_2 - p_1p_2) + 6(p_1p_2 - p_1q_1) + (p_1 - q_1)S'(1) \\
&\qquad +2(q_2 - p_1) + 2(q_2 - p_1)S'(1) + (p_1 - q_1)S'(1) \\
&\qquad +(p_1 - q_1)S''(1) + 2(q_2 - p_1)S'(1) + (q_2 - p_1)S''(1)\big] \\
&= \frac{1}{q_1}\left[4p_1p_2 - 4p_1q_1 - 2p_1 + 2(q_2 - p_2)S'(1) + (q_2 - q_1)S''(1)\right] \\
&= \frac{1}{q_1}\left[2p_1(2p_2 - 2q_1 - 1) + 2(q_2 - p_2)p_1\frac{q_1 + q_2}{p_1 - p_2} + (q_2 - q_1)\frac{2p_1q_1q_2(p_1 + p_2)}{(p_1 - p_2)^3}\right],
\end{aligned}$$

by using $S''(1) = \dfrac{2p_1q_1q_2(p_1 + p_2)}{(p_1 - p_2)^3}$ from Prop. 4.4 and $S'(1) = p_1\dfrac{q_1 + q_2}{p_1 - p_2}$ from Prop. 4.4. Thus

$$\begin{aligned}
f''(1) &= \frac{2p_1}{q_1(p_1 - p_2)^2}[(2p_2 - 2q_1 - 1)(p_1 - p_2)^2 \\
&\qquad +(q_2 - p_2)(q_1 + q_2)(p_1 - p_2) + q_1q_2(p_1 + p_2)] \\
&= \frac{2p_1}{q_1(p_1 - p_2)^2}[2p_1q_1^2 + (p_1q_1 - p_2q_2) + p_1p_2(p_1 - p_2)].
\end{aligned}$$

Returning now to the equation $M^{\mathsf{Markov}}(z) = \dfrac{1}{(1-z)^2}\dfrac{f(z)}{1 - S(z)}$, we have seen that $f(z)$ has a zero of order 2 at $z = 1$, while $1 - S(z)$ has a zero of order 1. We can therefore let $\dfrac{f(z)}{1 - S(z)} = \sum_{n\geq 1} a_n(1 - z)^n$ for some constants $(a_n)_{n\geq 1}$. The first coefficient is

$$a_1 = -\frac{d}{dz}\left[\frac{f(z)}{1 - S(z)}\right]\Bigg|_{z=1} = -\left[\frac{f'(z)}{1 - S(z)} - \frac{f(z)}{(1 - S(z))^2}(-S'(z))\right]\Bigg|_{z=1}.$$

By l'Hôpital's rule this is

$$\begin{aligned}
a_1 &= -\left[\frac{\lim_{z\to 1} f''(z)}{\lim_{z\to 1} -S'(z)} + \frac{\lim_{z\to 1} f'(z)S'(z) + f(z)S''(z)}{\lim_{z\to 1} 2(1 - S(z))(-S'(z))})\right] \\
&= -\left[\frac{f''(1)}{-S'(1)} + \frac{\lim_{z\to 1} f''(z)S'(z) + f'(z)S''(z) + f'(z)S''(z) + f(z)S'''(z)}{\lim_{z\to 1} 2(S'(z))^2 - 2(1 - S(z))S''(z)})\right] \\
&= -\left[\frac{f''(1)}{-S'(1)} + \frac{f''(1)S'(1)}{2(S'(1))^2}\right] \\
&= \frac{f''(1)}{2S'(1)} \\
&= \frac{\dfrac{2p_1}{q_1(p_1 - p_2)^2}[2p_1q_1^2 + (p_1q_1 - p_2q_2) + p_1p_2(p_1 - p_2)]}{2p_1\dfrac{q_1 + q_2}{p_1 - p_2}} \\
&= \frac{2p_1q_1^2 + (p_1q_1 - p_2q_2) + p_1p_2(p_1 - p_2)}{q_1(q_1 + q_2)(p_1 - p_2)}.
\end{aligned}$$

The result now follows by Wilf [11, Theorem 5.5] which in this case states $[z^n]M^{\mathsf{Markov}}(z) = a_1 + \mathcal{O}(r^n)$. □

5. Knuth's 2-matchbox problem with a third selfish chooser

In this section we consider a generalization of Knuth's matchbox process in which a new type of user is introduced. This generalization has the effect of moving the path-counting considerations from Catalan paths to Schröder paths and consequently requires a delicate analysis of probabilities associated with such paths. We consider the expected number of choosers until a first return to a diagonal state and also the expected residue. In Proposition 5.1 we give an explicit recurrence for the expected residue in which all quantities have explicit sums. The binomial form that appears in one of the sums does not allow us to give a closed form expression for the associated generating function. In Proposition 5.2 we prove a closed form generating function for the probability generating function for the first return to a diagonal state. This allows us to give an expression for the expected first return, see Proposition 5.3, along with an asymptotic analysis of the expected first return (Propositions 5.7 and 5.8).

5.1. **Generating functions for the expected first return and expected residue.** Consider Knuth's 2-matchbox problem for the case that there are big-choosers, little-choosers, and *selfish* choosers. The selfish chooser chooses a match from both of the two matchboxes. We assume a Bernoulli probability measure on the chooser process whereby the ith chooser is a big-, little-, or selfish-chooser with probability p, q, and $r := 1 - p - q$, respectively. In terms of the lattice path representation of this problem, this corresponds to a lattice walk from (n, n) to $(0, 0)$ in the region $\{(x, y) : 0 \leq y \leq x\}$ that takes steps in the set $\{\mathsf{s} = (0, -1), \mathsf{w} = (-1, 0), \mathsf{d} = (-1, -1)\}$. The precise weighting for each step depends on the top/right entry of the step.

The selfish-chooser steps d have a probability r of occurring except when they are on the x-axis (i.e. come from a position $(a, 0)$), in which case they have a probability 0 of occurring. The big-chooser steps w have a probability p of occurring, except when they come from a position (a, a) (then probability 0) or when they are on the x-axis (then probability 1). The little-chooser steps s have a probability q of occurring, except when they come from a position (a, a) (then probability $p + q$) or when they are on the x-axis (then probability 0).

We can now write down expressions for the expected residue $\mathbb{E}\mathsf{Residue}_n(p, q)$ conditioning on the types of lattices paths which contribute to the process. Let $M_n^{\mathsf{Schröder}} := M_{n,n}^{\mathsf{Schröder}}(p, q) = \mathbb{E}\mathsf{Residue}_n(p, q)$ be the expected residue. Note that the first few values for this quantity are

$$\begin{aligned} M_0^{\mathsf{Schröder}} &= 0 \\ M_1^{\mathsf{Schröder}} &= p + q \\ M_2^{\mathsf{Schröder}} &= (p + q)(2 + p^2 - 2p + pq). \end{aligned}$$

Proposition 5.1. *For all $n \geq 2$ we have*

$$M_n^{\mathit{Schröder}}(p, q) = \sum_{i=1}^{n-1} s_i^{\mathit{Schröder}}(p, q) \cdot M_{n-i}^{\mathit{Schröder}}(p, q) \;+\; L_n^{\mathit{Schröder}}(p, q),$$

where

$$\begin{aligned} s_i^{\mathit{Schröder}}(p, q) &:= \frac{(pq)^i (p + q)}{q} \sum_{k=0}^{i-1} \frac{1}{2i - 2k - 1} \binom{2i - k - 1}{k, i - k, i - k - 1} \left(\frac{1 - p - q}{pq}\right)^k, \\ L_n^{\mathit{Schröder}}(p, q) &:= \sum_{j=1}^{n} \sum_{k=0}^{n-j} j d_{n,j,k} (p + q) q^{n-k-1} p^{n-k-j} (1 - p - q)^k, \end{aligned}$$

with

$$d_{n,j,k} := \begin{cases} \Gamma_n(j, 1) & \text{if } k = 0 \\ \Gamma_n(j + k, 1 + k) \binom{2n - j - k - 1}{k} + \Gamma_n(j + k, k) \binom{2n - j - k - 1}{k - 1} & \text{if } 1 \leq k \leq n - j, \end{cases}$$

and $\Gamma_n(n, n) := 1$, $\Gamma_n(a, a) := 0$ for $a < n$, and for $b < a \leq n$,

$$\Gamma_n(a, b) := \frac{a - b}{n - b} \binom{2n - 1 - a - b}{n - b - 1}.$$

Proof. Every path that results from this random walk falls into one of two cases

(a) The path leaves (n,n) and passes through at least one non-diagonal point $(n-a,n-a-1)$ before returning to the diagonal for the first time at $(n-i,n-i)$, where $0 \le a < i$.
(b) The path initially takes $m \ge 0$ steps d and then leaves the diagonal (if $m < n$) and does not return before reaching the x axis for the first time at $(j,0)$.

By separating the paths into these two cases we have a recursion for the expected residue

$$M_n^{\mathsf{Schröder}}(p,q) = \sum_{i=1}^{n-1} s_i^{\mathsf{Schröder}}(p,q) \cdot M_{n-i}^{\mathsf{Schröder}}(p,q) \; + \; \sum_{j=1}^{n} j D_{n,j}(p,q),$$

for all $n \ge 2$, where $s_i^{\mathsf{Schröder}}(p,q)$ is the probability that the initial part of a path is covered by case (a) and $D_{n,j}(p,q)$ is the probability of a path described in case (b). To give an expression for $s_i^{\mathsf{Schröder}}(p,q)$ we will condition on the number, k, of d steps in the initial part of the path represented by (a) before the first return to the diagonal. Note that the end point of each such initial part before it returns to the diagonal must be a w step. We must therefore have $k \in [0,i-1]$ since $k=i$ would correspond to k diagonal steps, and this would imply the initial part of the path would not have left the diagonal, contradicting (a). The number of ways of arranging the $i-k$ s steps and the $i-k$ w steps relative to each other (i.e. ignoring the k d steps for the moment) is given by $c_{i-k} = \frac{1}{i-k}\binom{2(i-k-1)}{i-k-1}$, the $(i-k)$th Catalan number. A path counted by c_{i-k} has $1+2(i-k)$ vertices, and we can place the k d steps at any of the vertices except the last one. Thus the problem is to count the number of solutions to $y_1 + \ldots + y_{2(i-k)} = k$ where $y_j \ge 0$. A stars and bars consideration shows this to be $\binom{2(i-k)+k-1}{k} = \binom{2i-k-1}{k}$[1] Hence

$$s_i^{\mathsf{Schröder}}(p,q) = \sum_{k=0}^{i-1} c_{i-k}\binom{2i-k-1}{k}(p+q)q^{i-k-1}p^{i-k}r^k.$$

Notice that $s_1^{\mathsf{Schröder}}(p,q) = c_1(p+q)p = p(p+q)$. The product of the Catalan number and binomial coefficient can be replaced with

$$c_{i-k}\binom{2i-k-1}{k} \quad = \quad \frac{1}{2i-2k-1}\binom{2i-k-1}{k,i-k,i-k-1}.$$

Consider now the quantity $D_{n,j}(p,q)$, the probability of a path that does not return to the diagonal, once it leaves it, before reaching the x axis (for the first time) at $(j,0)$. We will need to be able to count paths between pairs of points that do not touch the main diagonal after leaving it. Let $\Gamma_n(a,b)$ be the number of non-diagonal (other than at the initial point) touching paths from (n,n) to (a,b) that take steps in $\{\mathsf{s},\mathsf{w}\}$, and do not go above the line $y=x$. Two special cases are $\Gamma_n(n,n)=1$ (achieved by the empty path) and $\Gamma_n(a,a)=0$ for $a<n$ since the endpoint is a diagonal one. The other cases correspond to the number of paths from $(n,n-1)$ to (a,b) where $n \ge a > b$ that take steps in $\{\mathsf{s},\mathsf{w}\}$ and which do not touch the main diagonal. By a simple counting argument, this number is

$$\Gamma_n(a,b) := \frac{a-b}{n-b}\binom{2n-1-a-b}{n-b-1}.$$

The paths associated with $D_{n,j}(p,q)$ are those paths P that leave (n,n), take diagonal, vertical, and horizontal steps, do not return to the diagonal after leaving it, and then meet the x axis at the endpoint $(j,0)$. Suppose that such a path P consists of k diagonal steps d where $k \in [0,n-j]$. The probability weighting associated with these steps is r^k. Notice that P, on touching the x-axis, ends with a s step or a d step. Let P' be the path P with the d steps removed. Then P' can be considered as a path from $(n,n) \to (n,n-1) \to \cdots \to (j+k,k)$ that takes steps in the set $\{\mathsf{s},\mathsf{w}\}$ and does not touch the diagonal after leaving it.

(i) If P ends in a s step, then every P' that ends in a south step can form such a P. Notice that

$$P' : \; (n,n) \to (n,n-1) \to \cdots \to (j+k,k+1) \to (j+k,k),$$

[1] Sometimes denoted by the multiset binomial $\left(\!\binom{2(i-k)}{k}\!\right)$.

is a path that consists of $1+(n-(j+k))+(n-k)=2n+1-j-2k$ vertices. The only condition on reintroducing the d steps is that d steps cannot replace the final vertex of P', but can replace all $2n-j$ others. Notice that P' is the set of all paths from $(n,n-1)$ to $(j+k,1+k)$ with a s both prepended and appended to each such path. The number of such paths P' is $\Gamma_n(j+k,1+k)$. Again, by a stars and bars consideration the number of original paths P that contain k d steps is $\binom{2n-j-2k+k-1}{k}=\binom{2n-j-k-1}{k}$. The probability weight associated with P is $(p+q)q^{n-k-1}p^{n-k-j}r^k$. Therefore we have the contribution to $D_{n,j}$ from this case as

$$\begin{aligned} D^{(1)}_{n,j}(p,q) &= \sum_{k=0}^{n-j}\Gamma_n(j+k,1+k)\binom{2n-j-k-1}{k}(p+q)q^{n-k-1}p^{n-k-j}r^k \\ &=: \sum_{k=0}^{n-j} d^{(1)}_{n,j,k}(p+q)q^{n-k-1}p^{n-k-j}r^k. \end{aligned}$$

(ii) If P ends in a d step, then every P' can form such a P, regardless of whether P' ends in a s step or w step. Note however that P' must start with a s step. The number of such paths is the number of paths from $(n,n-1)$ to $(j+k,k)$ that do not touch the diagonal and is $\Gamma_n(j+k,k)$. Note that k must be in $[1,n-j]$ since it has at least one d step (the final one). The number of original paths P that contain k d steps and for which the last one is always a d step is the number of solutions to $x_1+x_2+\ldots+x_{2n-2k-j+1}=k$ where $x_\ell\geq 0$ for all $\ell<2n-2k-j+1$ while $x_{2n-2k-j+1}\geq 1$. This is the number of solutions to the equation $x'_1+x'_2+\ldots+x'_{2n-2k-j+1}=k-1$ where each $x'_\ell\geq 0$. By a stars and bars consideration, this equals $\binom{2n-2k-j+1+k-1-1}{k-1}=\binom{2n-k-j-1}{k-1}$. The probability weight associated with such a path P is $(p+q)q^{n-k-1}p^{n-k-j}r^k$ and so the contribution to $D_{n,j}$ from this case is

$$\begin{aligned} D^{(2)}_{n,j}(p,q) &= \sum_{k=1}^{n-j}\Gamma_n(j+k,k)\binom{2n-j-k-2}{k-1}(p+q)q^{n-k-1}p^{n-k-j}r^k \\ &=: \sum_{k=1}^{n-j} d^{(2)}_{n,j,k}(p+q)q^{n-k-1}p^{n-k-j}r^k \end{aligned}$$

Bringing these together we have

$$D_{n,j}(p,q)=D^{(1)}_{n,j}(p,q)+D^{(2)}_{n,j}(p,q)=\sum_{k=0}^{n-j}d_{n,j,k}(p+q)q^{n-k-1}p^{n-k-j}r^k$$

where

$$d_{n,j,k}=\begin{cases} d^{(1)}_{n,j,0}=\Gamma_n(j,1) & \text{if } k=0\\ d^{(1)}_{n,j,k}+d^{(2)}_{n,j,k} & \text{if } 1\leq k\leq n-j.\end{cases}$$

Finally, for $k\geq 1$ we have

$$d^{(1)}_{n,j,k}+d^{(2)}_{n,j,k}=\Gamma_n(j+k,1+k)\tbinom{2n-j-k-1}{k}+\Gamma_n(j+k,k)\tbinom{2n-j-k-1}{k-1}. \qquad \square$$

Define

$$M^{\mathsf{Schröder}}(z;p,q) := \sum_{n\geq 1} M_n^{\mathsf{Schröder}}(p,q)z^n$$
$$S^{\mathsf{Schröder}}(z;p,q) := \sum_{n\geq 1} s_n^{\mathsf{Schröder}}(p,q)z^n$$
$$L^{\mathsf{Schröder}}(z;p,q) := \sum_{n\geq 1} L_n^{\mathsf{Schröder}}(p,q)z^n.$$

By multiplying the recursion in Proposition 5.1 by z^n and summing over all $n \geq 2$ we find $(M^{\mathsf{Schröder}}(z;p,q) - M_1^{\mathsf{Schröder}}(p,q)) = S^{\mathsf{Schröder}}(z)M^{\mathsf{Schröder}}(z;p,q) + (L^{\mathsf{Schröder}}(z;p,q) - L_1^{\mathsf{Schröder}}(p,q)$. Since $M_1^{\mathsf{Schröder}}(p,q) = p+q = L_1^{\mathsf{Schröder}}(p,q)$, after rearranging this equation becomes

$$M^{\mathsf{Schröder}}(z;p,q) = \frac{L^{\mathsf{Schröder}}(z;p,q)}{1 - S^{\mathsf{Schröder}}(z;p,q)}. \tag{19}$$

Proposition 5.2.

$$S^{\mathit{Schröder}}(z;p,q) = \frac{p+q}{2q} \cdot \frac{1-(1-p-q)z - \sqrt{1-(2(1-p-q)+4pq)\,z + (1-p-q)^2z^2}}{1-(1-p-q)z}.$$

Proof. Let $S_{i,k}^{(1)}$ be the number of Schröder paths from (i,i) to $(0,0)$ that consist of exactly k d steps, have a (possibly empty) initial run of d steps before leaving the diagonal and then return to the diagonal for the first time at $(0,0)$. Set $S_i^{(1)}(x) = \sum_{k=0}^{i-1} S_{i,k}^{(1)} x^k$ and $S^{(1)}(x,z) = \sum_{i\geq 1} S_i^{(1)}(x)z^i$.

Let Schr_i be the set of all Schröder paths from (i,i) to $(0,0)$ that take steps in $\{\mathsf{s},\mathsf{w},\mathsf{d}\}$ and which never go above the main diagonal. Let $\mathsf{Schr} = \cup_{i\geq 0}\mathsf{Schr}_i$ where $\mathsf{Schr}_0 = \{\epsilon\}$ is the empty path. Define

$$S^{(2)}(x,z) := \sum_{P\in\mathsf{Schr}} x^{\mathsf{d}(P)} z^{\ell(P)},$$

where $\mathsf{d}(P)$ is the number of diagonal d steps in P and $\ell(P)$ is the semi-length of P.[2] By conditioning on whether the first step of a non-empty Schröder path is s or d, we find

- If $P = \mathsf{d}P'$ where $P' \in \mathsf{Schr}$ then $\mathsf{d}(P) = 1 + \mathsf{d}(P')$ and $\ell(P) = 1 + \ell(P')$.
- If $P = \mathsf{s}P'\mathsf{w}P''$ then $\mathsf{d}(P) = \mathsf{d}(P') + \mathsf{d}(P'')$ and $\ell(P) = 1 + \ell(P') + \ell(P'')$.

Using this, we have $S^{(2)}(x,z) - 1 = xzS^{(2)}(x,z) + z(S^{(2)}(x,z))^2$, a quadratic in $S^{(2)}$ which has solution

$$S^{(2)}(x,z) = \frac{1 - xz - \sqrt{1-(2x+4)z + x^2z^2}}{2z}.$$

The paths that are counted by $S_i^{(1)}(1)$ are Schröder paths that begin at (i,i), can remain on the diagonal until leaving and then must not return until $(0,0)$. Note that they cannot remain on the diagonal so they must leave and return at some point. Let MSchr_i be the set of such paths from (i,i) to $(0,0)$ and let $\mathsf{MSchr} = \cup_{i\geq 1}\mathsf{MSchr}_i$. Every $P \in \mathsf{MSchr}_i$ is such that $P = \mathsf{d}^j\mathsf{s}P'\mathsf{w}$ where d^* is a length-j (possibly empty) run of d steps and $P' \in \mathsf{Schr}$. Moreover $\mathsf{d}(P) = j + \mathsf{d}(P')$ and $\ell(P) = j + 1 + \ell(P')$. We have $S^{(1)}(x,z) = \sum_{i\geq 1} S_i^{(1)}(x)z^i = \sum_{i\geq 1}\sum_{k=0}^{i-1} S_{i,k}^{(1)}x^kz^i = \sum_{P\in\mathsf{MSchr}} x^{\mathsf{d}(P)}z^{\ell(P)}$, so now

$$S^{(1)}(x,z) = \frac{1}{1-xz} \cdot zS^{(2)}(x,z) = \frac{1 - xz - \sqrt{1-(2x+4)z+x^2z^2}}{2(1-xz)}.$$

[2]Note that for a Schröder path, we have $\ell(\mathsf{d}) = 2\ell(\mathsf{s}) = 2\ell(\mathsf{w}) = 1$.

In light of this we now have:

$$\begin{aligned}
S^{\mathsf{Schröder}}(z;p,q) &:= \sum_{i\geq 1} s_i^{\mathsf{Schröder}}(p,q) z^i \\
&= \sum_{i\geq 1} \frac{(pq)^i(p+q)}{q} \sum_{k=0}^{i-1} S_{i,k}^{(1)} \left(\frac{1-p-q}{pq}\right)^k z^i \\
&= \frac{p+q}{q} \sum_{i\geq 1} (pqz)^i S_i^{(1)} \left(\frac{1-p-q}{pq}\right) \\
&= \frac{p+q}{q} S^{(1)} \left(\frac{1-p-q}{pq}, pqz\right).
\end{aligned}$$

□

Proposition 5.3. *Let $E^{\mathsf{Schröder}}(z;p,q)$ be the generating function for the sequence of expected first returns to a diagonal state. Then*

$$E^{\mathsf{Schröder}}(z;p,q) = \frac{z}{(1-z)^2}\left[1 - S^{\mathsf{Schröder}}(z;p,q)\right].$$

Proof. Let $E_n^{\mathsf{Schröder}}(p,q)$ denote the expected value of the first return, where, if a path starts at (n,n), leaves the diagonal and subsequently returns to the diagonal for the first time at $(n-i, n-i)$, then the value of the first return is i. The special case of the path that starts at (n,n) and takes n d steps (so never leaves the diagonal before arriving at $(0,0)$) is deemed to have a first return of n.

Then we have $E_1^{\mathsf{Schröder}}(p,q) = 1$ and, for all $n \geq 2$,

$$E_n^{\mathsf{Schröder}}(p,q) = \sum_{i=1}^{n-1} i \cdot s_i^{\mathsf{Schröder}}(p,q) \; + \; n\left(1 - s_1^{\mathsf{Schröder}}(p,q) - s_2^{\mathsf{Schröder}}(p,q) - \ldots - s_{n-1}^{\mathsf{Schröder}}(p,q)\right).$$

So

$$\begin{aligned}
E^{\mathsf{Schröder}}(z;p,q) =& \sum_{n\geq 1} E_n^{\mathsf{Schröder}}(p,q) z^n \\
=& z + \sum_{n\geq 2} z^n \left(\sum_{i=1}^{n-1} i \cdot s_i^{\mathsf{Schröder}}(p,q) \; + \; n\left(1 - \sum_{j=1}^{n-1} s_j^{\mathsf{Schröder}}(p,q)\right)\right) \\
=& \sum_{n\geq 1} nz^n - \sum_{n\geq 2} z^n \left(\sum_{i=1}^{n-1} (n-i) s_i^{\mathsf{Schröder}}(p,q)\right) \\
=& \frac{z}{(1-z)^2} - \sum_{n\geq 2}\left(\sum_{i=1}^{n-1} (n-i) z^{n-i} \cdot s_i^{\mathsf{Schröder}}(p,q) z^i\right) \\
=& \frac{z}{(1-z)^2} - \left(\sum_{m\geq 1} mz^m\right)\left(\sum_{i\geq 1} s_i^{\mathsf{Schröder}}(p,q) z^i\right) \\
=& \frac{z}{(1-z)^2}\left(1 - S(z;p,q)\right).
\end{aligned}$$

□

5.2. **Asymptotics of $[z^n]E^{\mathsf{Schröder}}(z;p,q)$.** In this section we let $S(z) := S^{\mathsf{Schröder}}(z;p,q)$. From Proposition 5.2 we have

$$S(z) = \frac{p+q}{2q}\left(1 - \frac{\sqrt{(1-(\alpha+\beta)z)(1-(\alpha-\beta)z)}}{1-rz}\right). \tag{20}$$

where $\alpha = (1-p)(1-q) + pq$ and $\beta = 2\sqrt{p(1-p)}\sqrt{q(1-q)}$.

Proposition 5.4. *$S(z)$ has radius of convergence $\rho := \dfrac{1}{\alpha+\beta}$. Furthermore, $\rho \geq 1$ with equality if and only if $p = q$.*

Proof. It is clear that $\alpha, \beta > 0$. Notice also that $\alpha^2 = (r + 2pq)^2 = r^2 + 4pqr + 4p^2q^2 = r^2 + 4pq(r + pq) > 4pq(r + pq) = \beta^2 \implies \alpha > \beta$. So $\alpha + \beta > \alpha - \beta > 0$. Moreover, $\alpha = 1 - p - q + 2pq = r + 2pq$ shows that $\alpha > r$, and hence $\alpha + \beta > r$. Therefore $S(z)$ has radius of convergence $\dfrac{1}{\alpha+\beta}$. Finally, notice that α and β both (uniquely) take maximum values at $p = q$ and that in this case we have $\alpha + \beta = 1$. □

From Proposition 5.3, we let

$$E(z) := E^{\mathsf{Schröder}}(z; p, q) = \frac{z}{(1-z)^2}\left(1 - S^{\mathsf{Schröder}}(z; p, q)\right).$$

In order to derive the asymptotics of $[z^n]E(z)$ it will be necessary to know $S(1)$ and $S'(1)$.

Proposition 5.5. *We have*

$$S(1) = \begin{cases} 1 & \text{if } p > q \\ \dfrac{p}{q} & \text{if } p < q. \end{cases}$$

Proof. Consider $S(z)$ as given in Proposition 5.2. The radicand in $S(z)$ with $z = 1$ is $1 - (2r + 4pq) + r^2 = (p - q)^2$. So $S(1) = \dfrac{1}{2q}(p + q - |p - q|)$, as $p \neq q$, and the result follows. □

Proposition 5.6. *We have*

$$S'(1) = \begin{cases} \dfrac{p}{p-q} - \dfrac{r}{q}\dfrac{p^2+q^2}{p^2-q^2} & \text{if } p > q \\ \dfrac{p}{q-p} - \dfrac{r}{q}\dfrac{p^2+q^2}{q^2-p^2} & \text{if } p < q. \end{cases}$$

Proof. Differentiate $S(z)$ as given in Proposition 5.2 with respect to z to find

$$\begin{aligned} S'(z) =& \frac{p+q}{2q}\left(-\frac{1}{2}\frac{[1-(2r+4pq)z+r^2z^2]^{-\frac{1}{2}}(-(2r+4pq)+2r^2z)}{1-rz}\right) \\ &+ \frac{p+q}{2q}\left(\frac{\sqrt{1-(2r+4pq)z+r^2z^2}}{(1-rz)^2}(-r)\right). \end{aligned}$$

From this, and using the fact that $\sqrt{1-(2r+4pq)+r^2} = |p-q|$ (as we saw in Proposition 5.5), we have

$$\begin{aligned} S'(1) =& \frac{p+q}{2q}\left(\frac{-1}{|p-q|}\frac{(r^2-(r+2pq))}{1-r}\right) - \frac{p+q}{2q}\left(\frac{|p-q|r}{(1-r)^2}\right) \\ =& \frac{1}{2q}\left(\frac{2pq - r(p+q)}{|p-q|} - \frac{|p-q|r}{p+q}\right) \\ =& \frac{p}{|p-q|} - \frac{r}{2q}\left(\frac{p+q}{|p-q|} + \frac{|p-q|}{p+q}\right), \end{aligned}$$

from which the result follows by considering the cases $p \gtrless q$. □

Proposition 5.7. *Let τ be any number greater than $\rho = \frac{1}{\alpha+\beta}$ and let $p \neq q$. Then*

$$[z^n]E(z) = \begin{cases} \dfrac{p}{p-q} - \dfrac{r}{q}\dfrac{p^2+q^2}{p^2-q^2} + \mathcal{O}(\tau^n) & \text{if } p > q \\ \dfrac{q-p}{q}\cdot n + \dfrac{p}{q-p} - \dfrac{r}{q}\dfrac{p^2+q^2}{q^2-p^2} + \mathcal{O}(\tau^n) & \text{if } p < q. \end{cases}$$

Proof. Recall that

$$E(z) = \frac{z}{(1-z)^2}\,(1 - S(z))$$

and $S(z)$ has radius of convergence $\rho > 1$ (since $p \neq q$). Let $E(z) = \frac{1}{(1-z)^2} f(z)$, where $f(z) := z(1 - S(z))$, and we know that $f(z) = \sum_{n\geq 0} a_n(1-z)^n$ for some constants $(a_n)_{n\geq 0}$. The first coefficient a_0 is $a_0 = f(1) = 1 - S(1)$. The (negation of the) second coefficient is $-a_1 = f'(1)$, where

$$f'(1) = 1 - S(1) - S'(1) = a_0 - S'(1).$$

By Wilf [11, Theorem 5.5] we have

$$\begin{aligned}[z^n]E(z) =& [z^n]\frac{a_0}{(1-z)^2} + [z^n]\frac{a_1}{1-z} + \mathcal{O}(\tau^n)\\ =& (n+1)a_0 + a_1 + \mathcal{O}(\tau^n)\\ =& (1 - S(1))n + S'(1) + \mathcal{O}(\tau^n).\end{aligned}$$

The result now follows by using Propositions 5.5 and 5.6. □

Proposition 5.8. *For the case $p = q$ we have*

$$[z^n]E(z) = 2\sqrt{\frac{1+r}{1-r}}\sqrt{\frac{n}{\pi}} + \mathcal{O}\left(\frac{1}{\sqrt{n}}\right).$$

Proof. Recall from equation (20) we have

$$S(z) = \frac{p+q}{2q}\left(1 - \frac{\sqrt{(1-(\alpha+\beta)z)(1-(\alpha-\beta)z)}}{1-rz}\right),$$

where $\alpha = (1-p)(1-q) + pq$ and $\beta = 2\sqrt{p(1-p)}\sqrt{q(1-q)}$. When $p = q$, we have $\alpha + \beta = 1$ and $\alpha - \beta = (1-2p)^2 = r^2$, and so

$$S(z) = \left(1 - \sqrt{1-z}\frac{\sqrt{1-r^2z}}{1-rz}\right).$$

Using this we have

$$E(z) = \frac{z}{(1-z)^{\frac{3}{2}}}\frac{\sqrt{1-r^2z}}{1-rz}.$$

Now let $f(z) = z\frac{\sqrt{1-r^2z}}{1-rz}$ and write $f(z) = \sum_{n\geq 0} a_n(1-z)^n$ for some constants $(a_n)_{n\geq 0}$. The first coefficient is

$$a_0 = f(1) = \frac{\sqrt{1-r^2}}{1-r} = \sqrt{\frac{1+r}{1-r}}.$$

The result now follows by using the identity from equation (14). □

The outstanding matter is an expression for $L^{\mathsf{Schröder}}(z) := \sum_{n\geq 2} z^n \sum_{j=1}^{n} jD_{n,j}(p,q)$. However the form for $D_{n,j}$ involves a product of binomial coefficients and which does not seem to have a closed form generating function.

References

[1] C. Banderier, P. Marchal and M. Wallner. Periodic Polya Urns, the Density Method and Asymptotics of Young Tableaux. *The Annals of Probability* **48(4)**, January 2019.

[2] M. Dukes and A. Mullins. Knuth's big-chooser matchbox process: the case of many matchboxes. *Advances in Applied Mathematics* **173B** (2026), 102996.

[3] W. Feller. *An Introduction to Probability Theory And Its Applications.* 3rd edition, Wiley, 1968, Chapter VI, Section 8.

[4] P. Flajolet and R. Sedgewick. *Analytic combinatorics.* Cambridge University Press, 2009.

[5] D. E. Knuth. The Toilet Paper Problem. *The American Mathematical Monthly* **91** (1984), no. 8, 465–470.
[6] C. Krattenthaler, A. J. Guttmann, and X. G. Viennot. Vicious Walkers, Friendly Walkers and Young Tableaux II: with a wall. *Journal of Physics A: Mathematical and General.* **33** (2000), 8835–8866.
[7] C. Krattenthaler. Refined enumeration of two-rowed set-valued standard tableaux via two-coloured Motzkin paths. Preprint arXiv:2501.09208.
[8] OEIS Foundation Inc. (2026), The On-Line Encyclopedia of Integer Sequences, Published electronically at https://oeis.org.
[9] E. Renshaw and R. Henderson. The Correlated Random Walk. *Journal of Applied Probability* **18** (1981), no. 2, 403–14.
[10] D. Stirzaker. A generalization of the matchbox problem. *The Mathematical Scientist* **13** (1988), 104–114.
[11] H. S. Wilf. *Generatingfunctionology*. Academic Press, 1990.