GENERALIZED BALLOT SEQUENCES ARE ASCENT SEQUENCES

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Abstract. Ascent sequences were introduced by the author (in conjunction with others) to encode a class of permutations that avoid a single length-three bivincular pattern, and were the central object through which other combinatorial correspondences were discovered. In this note we prove the non-trivial fact that generalized ballot sequences are ascent sequences.

Ascent sequences were introduced in [1] to encode a collection of permutations that avoid the single length-three bivincular pattern $2|3\bar{1}$. These sequences were also shown to uniquely encode interval orders, Stoimenow matchings, and the set of upper triangular matrices whose entries are non-negative integers and which contain no all-zero rows or columns [2].

Generalized ballot sequences (also known as Yamanouchi words) are sequences of non-negative integers that encode election scenarios in which a prescribed set of candidates maintain their success-positions throughout the counting of the votes (i.e., reading of the ballot sequences from left to right). It is not immediately clear that generalized ballot sequences satisfy the defining property of ascent sequences and this is the purpose of this note.

Given a sequence of integers $x = (x_1, \ldots, x_n)$ we say there is an *ascent* at position i if $x_i < x_{i+1}$, and denote by $\mathsf{asc}(x)$ the number of ascents in x. We say that a sequence of positive integers $x = (x_1, \ldots, x_n)$ is an *ascent sequence* if $x_1 = 1$ and for all $1 \le i < n$ we have

$$x_{i+1} \in \{1, 2, \ldots, 2 + \mathsf{asc}(x_1, \ldots, x_i)\}.$$

This definition is equivalent to the definition given in [2] – the only difference being that unity has been added to every sequence entry. Let Asc_n be the set of ascent sequences of length n. For example

$$\mathsf{Asc}_3 = \{(1,1,1), (1,1,2), (1,2,1), (1,2,2), (1,2,3)\}.$$

Let $w = w_1 w_2 \ldots w_n$ be a word over the alphabet $\{1, 2, \ldots, n\}$. Let us write $\mathsf{asc}(w)$ for $\mathsf{asc}(w_1, \ldots, w_n)$. The word w is said to be a *generalized ballot sequence* or a *Yamanouchi word* (see e.g. Stanley [4, Prop. 7.10.3]) if for every left factor $w^{(k)} = w_1 \ldots w_k$ of w and for every i, the number of occurrences of i in $w^{(k)}$ is greater than or equal to the number of occurrences of $i+1$ in $w^{(k)}$. Let Ballot_n be the set of generalized ballot sequences of length n. For example, $\mathsf{Ballot}_3 = \{111, 112, 121, 123\}$.

Theorem 1. *If $x_1x_2\dots x_n$ is a generalized ballot sequence, then $(x_1,\dots,x_n)$ is an ascent sequence.*

Proof. Given $x = x_1\dots x_n \in \mathsf{Ballot}_n$, let $\max(x) = \max\{x_1,\dots,x_n\}$,

$$c_j^{(i)} = |\{k \ :\ x_k = j \text{ and } k \le i\}|, \text{ and}$$

$$\mathsf{Valid}(x) = \{1\} \cup \{j \ :\ c_{j-1}^{(n)} > c_j^{(n)} \text{ and } 1 < j \le \max(x)\} \cup \{\max(x)+1\}.$$

The set $\mathsf{Valid}(x)$ is the set of values that may be appended to x in order to yield a generalized ballot sequence of length $n+1$. We give a proof by induction.

The theorem is true for $n = 1$ since $\mathsf{Ballot}_1 = \{1\}$ and $\mathsf{Asc}_1 = \{(1)\}$. Suppose the claim to be true for $n = m$. Let $x = x_1\dots x_m \in \mathsf{Ballot}_m$. Since x is a generalized ballot sequence, all of the letters from $\{1,\dots,\max(x)\}$ appear at least once in x. Furthermore, the leftmost (i.e., first) occurrence of k in x means that the letter immediately to its left must be less than it (for otherwise the generalized ballot sequence property would be broken for the left factor of x that ends at the entry immediately preceding k). Let a_k be the smallest index j such that $x_j = k$. The above reasoning means we must have $x_{a_k-1} < x_{a_k}$ for all $k = 2,\dots,\max(x)$, in other words $(x_1,\dots,x_m)$ has ascents at positions $a_2-1,\dots,a_{\max(x)}-1$. This implies there are at least $\max(x)-1$ different ascents in x, i.e.,

$$\max(x_1,\dots,x_m) - 1 \le \mathsf{asc}(x_1,\dots,x_m). \tag{1}$$

Every $x' = x_1\dots x_m x_{m+1} \in \mathsf{Ballot}_{m+1}$ is uniquely formed from $x = x_1\dots x_m \in \mathsf{Ballot}_m$ and $x_{m+1} \in \mathsf{Valid}(x)$. From the induction hypothesis, $(x_1,\dots,x_m) \in \mathsf{Asc}_m$. Since $x_{m+1} \in \mathsf{Valid}(x)$, we have $x_{m+1} \in \{1,\dots,\max(x)+1\}$. Using the inequality in (1), since $\max(x) \le 1 + \mathsf{asc}(x)$ we have

$$x_{m+1} \in \{1,\dots,\max(x)+1\} \subseteq \{1,\dots,2+\mathsf{asc}(x)\}.$$

This condition implies $x' = (x_1,\dots,x_{m+1}) \in \mathsf{Asc}_{m+1}$. Therefore, by the principle of induction, the claim is true for all positive integers n. □

Ballot sequences are word-encodings of standard Young tableaux. We posit that emulating operations on standard Young tableaux, such as evacuation and flipping about the diagonal, will have some part to play in answering unsolved questions regarding ascent sequences. One such open problem is describing the 'dual' or 'flip' of an ascent sequence (see [2, Question 20]).

An intriguing problem is to classify the subsets of the four classes of combinatorial objects (pattern-avoiding permutations, Stoimenow matchings, restricted integer matrices, interval orders) that correspond to ballot sequences via the bijections given in the papers [1, 2, 3].

References

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