

Restricted classes of Fishburn structures and their associated polynomials

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ABSTRACT. Interval orders are in bijective correspondence with several other combinatorial objects that include ascent sequences, Fishburn permutations, and Fishburn matrices. We consider the six different subsets (classes) of interval orders that emerge by further imposing some subset of the following three additional constraints: (a) the interval order is $(3+1)$ free, (b) the interval order is N -free, and (c) the interval order has a covering relation that is N -free. The problem of characterizing the sets of ascent sequences, Fishburn permutations and Fishburn matrices in this framework through established bijections has been touched upon by several authors. We present an almost complete characterization of this framework for five of the six classes. For the outstanding class we are able to give an ascent sequence characterization via a bijection of Chen, Yan and Zhou.

We then build on this framework by considering two fundamental polynomials associated to each class. Since the ascent statistic on ascent sequences corresponds to the dimension statistic on Fishburn matrices, which in turn corresponds to the magnitude statistic on interval orders, the generating function of the ascent statistic in each class captures the same information. For Fishburn permutations we consider the classical Eulerian distribution for each of the classes. We complete the characterization of the ascent polynomials for each of the six classes, and also do this for five of the six Eulerian distributions. The coefficients of many of these polynomials appear to have log-concave and unimodal properties and we prove this for many. We conjecture unimodality of the ascent polynomials corresponding to ascent sequences and have verified the unimodality conjecture up to $n = 300$.

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1. INTRODUCTION

Interval orders have recently been the topic of numerous studies owing to their connection to other Fishburn structures. The papers of Bousquet-Melou et al. [7] and Dukes & Parviainen [17] proved bijections between interval orders, ascent sequences, Fishburn permutations, Fishburn matrices, and Stoimenow matchings. These bijections were shown to preserve several natural

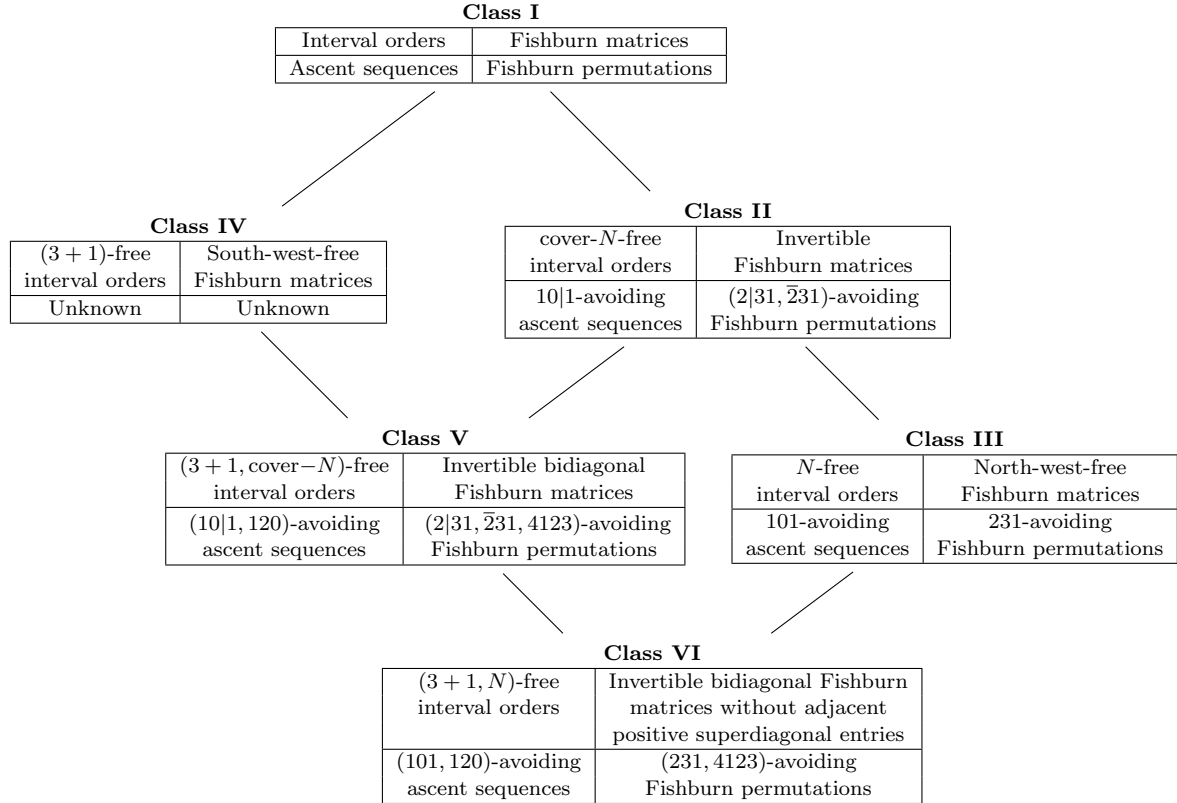


FIGURE 1. An overview of the restricted subsets of Fishburn structures considered in this paper.

properties of each of the structures. Further work on this topic has shown that the images of natural subsets of these structures admit descriptions in terms of substructure-avoidance concepts, see Dukes and McNamara [16], Jelínek [22], Lv, Kitaev and Zhang [28], and Duncan and Steingrímsson [19]. We extend this work by investigating these and other subsets more deeply. We also consider a number of other natural subsets of Fishburn structures, and give new descriptions for their images, as well as determining their counting sequences. Figure 1 gives an overview of the restricted subsets that we consider, and also the four different types of Fishburn structures we study. In each box, the top left quadrant describes a particular restricted subset of interval orders. The remaining three quadrants describe the restricted subsets of Fishburn structures which essentially correspond to interval orders through established bijections from the literature. A line between boxes indicates the lower interval order class is contained in the upper one.

A main motivation for this paper was the open question of [16] which asked for a characterization of ascent sequences that correspond to *series-parallel semiorders*, also known as $(3 + 1, N)$ -free interval orders, under the classical bijection from [7]. In that work it was noticed that series-parallel semiorders are equinumerous with certain sets of pattern avoiding permutations which were enumerated by Baxter and Pudwell in [1]. The discovery that ascent sequences and Fishburn permutations that correspond to series-parallel semiorders may each be described in terms of pattern-avoidance prompted the question of whether other natural subsets of Fishburn structures could be consolidated into this larger picture of characterization through substructure-avoidance. The set of *self-modified ascent sequences* [7, 16] was an obvious candidate, and the corresponding Fishburn structures fit surprisingly elegantly into our picture,

as we will show. The main contribution of this paper is constructing a framework of characterizations through substructure-avoidance for natural subsets of Fishburn structures. We hope it will be possible to expand this even further in future work.

The Catalan numbers are known to enumerate both the set of *semiorders*, also known as $(3 + 1)$ -free interval orders, and the set of series-parallel interval orders, also known as N -free interval orders. These sets featured in the construction of *Catalan pairs* by Disanto, Ferrari, Pinzani and Rinaldi [13], and also Jelínek [22]. Consequently another motivation for this work is to investigate these particular types of interval orders, and their connection to each other more deeply.

A separate motivation for some of the work in this paper comes from a recent paper of the authors [18]. In that paper, the authors considered a system of activities which are related in their occurrence by constraints termed *precedence constraints* and *response constraints*. The ways in which such a system can evolve relate to the set of all linear extensions of an underlying poset. When one instead constrains activities by insisting that each activity occurs at some point during a given interval of time, the underlying posets become interval orders and the challenge is then to characterize their linear extensions. Such a system has been considered, for example, in [30].

We build on the aforementioned framework by considering two fundamental polynomials associated to each class. Since the ascent statistic on ascent sequences corresponds to the dimension statistic on Fishburn matrices, which in turn corresponds to the magnitude statistic on interval orders, the generating function of the ascent statistic (which we call the ascent polynomial) in each class captures the same information. We complete the characterization of the ascent polynomials for each of the six classes.

The generating function of the descent statistic on permutations is called the Eulerian polynomial [6]. The phrase *Eulerian distribution* is often used in the literature to denote the generating function of the descent statistic on a restricted set of permutations. The Eulerian distribution on involutions was considered in [14], while the Eulerian distribution on 231-avoiding permutations is known to be given by the Narayana polynomials [21]. Other work in this area includes [4, 10]. We characterize the Eulerian distribution for five of the six classes of Fishburn permutations in the framework.

The unimodal and log-concave properties of the sequences of coefficients of all of these polynomials is thoroughly investigated. Proofs of unimodality and log-concavity are established for many of the cases and conjectures made for those we were unable to prove. The large values for which we were able to verify these conjectures provide strong supporting evidence. In particular, we conjecture unimodality of the ascent polynomials corresponding to ascent sequences and have verified the unimodality conjecture up to $n = 300$.

In Section 2 we review the various Fishburn structures to be considered in this paper, namely interval orders, Fishburn matrices, ascent sequences and Fishburn permutations, as well as some established bijections between them. We also define the notion of pattern-avoidance which is necessary to describe the restrictions we study.

In Section 3 we introduce five restricted subsets of Fishburn structures (classes II–VI), each one corresponding to interval orders avoiding a specific substructures under established bijections [7, 17]. Note that class I corresponds to the case of no restrictions and is simply the set of all interval orders. The main results of Sections 4 to 7 are new characterizations of each of the other types of Fishburn structures in each class, as outlined in Figure 1. Also, in Section 5.1 we investigate the correspondence between Dyck paths and the two Catalan-enumerated classes, and demonstrate the translation of a number of statistics.

In Section 8 we derive the Eulerian distributions on our restricted sets of Fishburn permutations, as well as the ascent distributions on ascent sequences. We show that the two turn out to be related in most cases. We also consider the properties of unimodality and log-concavity for these distributions/polynomials.

In Section 9 we consider the major index distribution on Fishburn permutations, and the *ascent major index* on ascent sequences. We present a result regarding these distributions for the general class of Fishburn structures, and show that it also holds for class III.

In Section 10 we return to the case of semiorders whose corresponding Fishburn structures elude characterization (except for Fishburn matrices). In an attempt to shed some light on this class, we investigate the interaction of these interval orders with an alternative bijection defined by Chen et al. in [11].

2. FISHBURN STRUCTURES

In this section we define interval orders, Fishburn matrices, ascent sequences, and Fishburn permutations. We also recall several of the established bijections from the literature that we will use later in the paper. These definitions can be found in the papers [7, 17, 11, 20]. Given natural numbers a and b , we will use the notation $[a, b] := \{a, a + 1, \dots, b\}$ and set $[a] := [1, a]$.

2.1. Interval Orders and Fishburn matrices. A *poset* P is a pair $(P, \leq)$ consisting of a set P and a binary relation $\leq$ on P , called a *partial order*, which is reflexive, transitive and antisymmetric. Every partial order has an associated *strict partial order* $<$, which is the reflexive reduction of $\leq$, and an associated covering relation $\lessdot$, which is just the transitive reduction of $<$ when P is finite.

A poset P is Q -free if it contains no induced subposet which is isomorphic to the poset Q . In this paper we are interested in three types of forbidden subposet, namely $(2 + 2)$, $(3 + 1)$ and N , whose Hasse diagrams are shown in Figure 2.

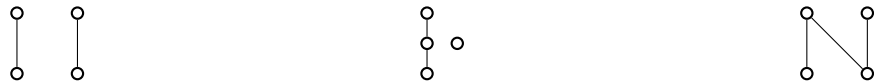


FIGURE 2. Hasse diagrams for the three posets $(2 + 2)$, $(3 + 1)$ and N from left to right.

We will also consider posets whose covering relation is N -free, that is, whose covering relation has no subrelation which is isomorphic to $N^\dagger$. More precisely, the covering relation of P is N -free if there does not exist $a, b, c, d \in P$ such that $a \lessdot b \lessdot c \lessdot d$, with no other covering relations between a, b, c and d . We also refer to such posets as *cover- N -free*. To make clear the distinction, the Hasse diagram of the smallest poset which is not N -free but whose covering relation is N -free is given in Figure 3.



FIGURE 3. The smallest poset which is not N -free but is cover- N -free.

We present two equivalent definitions of an interval order. A poset P is called an *interval order* if it is $(2 + 2)$ -free. Equivalently, P is an interval order if the *strict downsets* of P can be linearly ordered by inclusion. That is, if $P = \{p_1, p_2, \dots, p_n\}$ and there exists a permutation $\pi \in S_n$ such that

$$D(p_{\pi(1)}) \subseteq D(p_{\pi(2)}) \subseteq \dots \subseteq D(p_{\pi(n)}),$$

where $D(p) := \{q \in P : q < p\}$ is the *strict downset* of the element p . The number of unique strict downsets of P is known as the *magnitude* of P , and is denoted $\text{mag}(P)$. Given the characterization of an interval order P in terms of its downsets, its structure can be uniquely

[†]Unfortunately, in the literature [2, 24] on the type of posets which we refer to here as having N -free covering relation, they are simply called N -free. We make this note to avoid confusion.

encoded by listing each of the unique strict downsets together with the set of elements of p that have that strict downset. Let $D_i(P)$ be the i th unique strict downset of P , so that

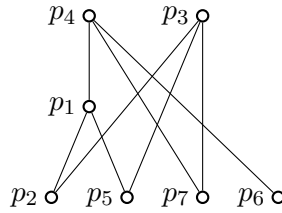
$$\emptyset = D_0(P) \subset D_1(P) \subset \cdots \subset D_{\text{mag}(P)-1}(P).$$

Define the i th level set of P to be $L_i(P) := \{p \in P : D(p) = D_i(P)\}$, the set of elements of P that each have strict downset $D_i(P)$.

We may represent P as a $\text{mag}(P) \times \text{mag}(P)$ matrix $M(P)$ whose entries are given by $[M(P)]_{i,j} = L_{i-1}(P) \cap (D_j(P) \setminus D_{j-1}(P))$ where $D_{\text{mag}(P)}(P) = P$ by convention. The structure of this matrix gives rise to the following condition for two elements of P being comparable under $<$.

Remark 2.1. Let $p, q \in P$ such that $p \in [M(P)]_{i,j}$ and $q \in [M(P)]_{i',j'}$. Then $p < q$ if and only if $j < i'$.

Example 2.2. Consider the interval order P with Hasse diagram:



The strict downsets are

$$D_0(P) = \emptyset, D_1(P) = \{p_2, p_5\}, D_2(P) = \{p_2, p_5, p_7\}, D_3(P) = \{p_1, p_2, p_5, p_6, p_7\},$$

while the level sets are

$$L_0(P) = \{p_2, p_5, p_6, p_7\}, L_1(P) = \{p_1\}, L_2(P) = \{p_3\}, L_3(P) = \{p_4\}.$$

This results in the upper-triangular matrix

$$M(P) = \begin{pmatrix} \{p_2, p_5\} & \{p_7\} & \{p_6\} & \emptyset \\ & \emptyset & \{p_1\} & \emptyset \\ & & \emptyset & \{p_3\} \\ & & & \{p_4\} \end{pmatrix}.$$

In this paper, our focus is on *unlabelled* interval orders, meaning that we do not distinguish between posets which differ only in the labelling of their elements. Formally, an unlabelled poset may be thought of as an equivalence class of order-isomorphic labelled posets. Whenever we refer to interval orders or posets in the remainder of this paper, they are implicitly unlabelled. It is natural then to consider a matrix $|M(P)|$ that records cardinalities of the entries of $M(P)$.

Example 2.3. Let $M(P)$ be the matrix from Example 2.2. Then

$$|M(P)| = \begin{pmatrix} 2 & 1 & 1 & 0 \\ & 0 & 1 & 0 \\ & & 0 & 1 \\ & & & 1 \end{pmatrix}.$$

Any relabelling of the elements of P will result in the same $|M(P)|$.

Matrices corresponding to interval orders in this way are called *Fishburn matrices*. The *weight* of an integer matrix is the sum of its entries, and in the case of Fishburn matrices, it is obviously equal to the cardinality of the corresponding interval order. Given the above discussion, it can be seen that Fishburn matrices may be defined by the following property.

Definition 2.4. A *Fishburn matrix* is an upper triangular matrix of non-negative entries which has neither rows nor columns of only zeros.

More details can be found in Dukes, Jelínek, and Kubitzke [15]. We denote by FPosets_n the set of all interval orders on n elements and by FMatrices_n the set of all Fishburn matrices of weight n . We name these sets separately for mathematical clarity, though a Fishburn matrix should be viewed as a representation of an interval order rather than as a distinct object. The *interval order magnitude* statistic, which equals the *matrix dimension* statistic via the Fishburn matrix representation just described, is the natural statistic we will consider for elements of FPosets_n . Given a Fishburn matrix M , we denote its dimension by $\dim(M)$. We also denote the sum over the i th row and over the j th column of M by $\text{rowsum}_i(M)$ and $\text{colsum}_j(M)$, respectively. Lastly, we denote by $\text{mindex}(M)$ the minimum index of a row of M whose rightmost entry is non-zero.

2.2. Ascent sequences. Given a sequence of numbers $x = (x_1, \dots, x_n)$, we say that a has a *descent* (resp. *ascent*) at position i if $x_i > x_{i+1}$ (resp. $x_i < x_{i+1}$). Let us define

$$\begin{aligned}\text{Des}(x) &:= \{i : x_i > x_{i+1}\} \\ \text{Asc}(x) &:= \{i : x_i < x_{i+1}\}.\end{aligned}$$

In addition to this, let us write $\text{des}(x) := |\text{Des}(x)|$, $\text{asc}(x) := |\text{Asc}(x)|$, $\text{maj}(x) := \sum_{i \in \text{Des}(x)} i$ and $\text{amaj}(x) := \sum_{i \in \text{Asc}(x)} i$.

Definition 2.5. A sequence $a = (a_1, \dots, a_n)$ of non-negative integers is called an *ascent sequence* if $a_1 = 0$ and

$$a_{i+1} \in \{0, \dots, 1 + \text{asc}((a_1, \dots, a_i))\}$$

for all $i \in [n-1]$. Denote by $\mathcal{A}_n$ the set of all ascent sequences of length n .

Next we introduce the notion of pattern avoidance in ascent sequences as done in Duncan and Steingrímsson [19]. Consider an ascent sequence $a = (a_1, \dots, a_n)$ and a word $w = w_1 \dots w_k$ of non-negative integers where $k \leq n$, which we call a (classical) *pattern*. An *occurrence* of w in a is a subsequence of a of length k , whose entries are in the same relative position as the entries in w . More precisely, a subsequence $(a_{i_1}, \dots, a_{i_k})$ of a such that $a_{i_s} \leq a_{i_t}$ iff $w_s \leq w_t$ for all $s, t \in [k]$. In this case, we say that a *contains* w , and otherwise, that a *avoids* w .

A *vincular pattern* is a pair (w, X) consisting of a classical pattern w of length k and a set $X \subseteq [k-1]$. An occurrence of the vincular pattern (w, X) in a is a subsequence $(a_{i_1}, \dots, a_{i_k})$ of a such that

- (i) $(a_{i_1}, \dots, a_{i_k})$ is an occurrence of the classical pattern w in a and
- (ii) $i_{x+1} = i_x + 1$ for all $x \in X$.

Again we say that a *contains* (w, X) in this case, and otherwise that a *avoids* (w, X) . We will adopt a more concise notation to represent a vincular pattern (w, X) , which is obtained by inserting a vertical bar between w_x and w_{x+1} for all $x \in X$. For example the pattern $(101, \{2\})$ may be written simply as $10|1$.

Example 2.6. The ascent sequence $(0, 1, 2, 3, 4, 2, 4, 0, 3)$ has three occurrences of 101 , namely $(3, 2, 3)$, $(3, 0, 3)$ and $(4, 2, 4)$. The subsequences $(3, 0, 3)$ and $(4, 2, 4)$ are additionally occurrences of $10|1$, and $(4, 2, 4)$ is in fact an occurrence of $1|0|1$.

Naturally any classical pattern may be written as a vincular pattern with $X = \emptyset$, and an occurrence of (w, X) is necessarily an occurrence of (w, X') if $X' \subseteq X$. The set of ascent sequences of length n which avoids a pattern (w, X) is denoted by $\mathcal{A}_n((w, X))$.

2.3. Fishburn permutations. Denote by S_n the set of all permutations of the set $[n]$. We will write elements of S_n as words over the alphabet $[n]$. For permutations, descents, ascents, Des , Asc , des , asc , maj and amaj are defined as in the previous section. Pattern avoidance is also defined analogously to the previous section, the only difference being that patterns with repeated entries are nonsensical in the context of permutations, so any pattern may itself be written as a permutation. For permutations, it makes sense to consider the concept of *bivincular pattern avoidance*, which was introduced in [7], and generalizes vincular pattern avoidance by

considering “vertical adjacency” in the same way that vincular pattern avoidance generalizes classical pattern avoidance by considering “horizontal adjacency”. A *bivincular pattern* is a triple (w, X, Y) consisting of a permutation $w = w_1 \dots w_k \in S_k$ and sets $X, Y \subseteq [k-1]$. An occurrence of the bivincular pattern (w, X, Y) in a permutation $\pi = \pi_1 \dots \pi_n \in S_n$ is a subword $\pi_{i_1} \dots \pi_{i_k}$ of π such that

- (i) $\pi_{i_1} \dots \pi_{i_k}$ is an occurrence of the classical pattern w in π ,
- (ii) $i_{x+1} = i_x + 1$ for all $x \in X$ and
- (iii) $j_{y+1} = j_y + 1$ for all $y \in Y$

where $\{\pi_{i_1} \dots \pi_{i_k}\} = \{j_1, \dots, j_k\}$ with $j_1 < j_2 < \dots < j_k$. Again we say that x *contains* (w, X, Y) in this case, and otherwise that x *avoids* (w, X, Y) . Similarly to before, an equivalent notation for representing a bivincular pattern (w, X, Y) is obtained by inserting a vertical bar between w_x and w_{x+1} for all $x \in X$, as well as placing a horizontal bar over each w_y for all $y \in Y$. For example, the pattern $(231, \{1\}, \{1\})$ may be written as $2|3\bar{1}$.

Any vincular pattern may be written as a bivincular pattern with $Y = \emptyset$, and an occurrence of (w, X, Y) is automatically an occurrence of (w, X', Y') if $X' \subseteq X$ and $Y' \subseteq Y$. The set of permutations avoiding a pattern (w, X, Y) is denoted by $S_n((w, X, Y))$. For more details, see [7, 5].

A *Fishburn permutation* is a permutation which avoids the bivincular pattern $2|3\bar{1}$. We denote by $\mathcal{F}_n$ the set $S_n(2|3\bar{1})$ of all Fishburn permutations of length n . Fishburn permutations may alternatively be characterized as follows.

Definition 2.7. A permutation $\pi \in S_n$ is a *Fishburn permutation* if for every $i \in \text{Asc}(\pi)$, either $\pi_i = 1$ or the value of $\pi_i - 1$ is to the left of π_i in π .

Notice that deleting the entry n from an element of $\mathcal{F}_n$ will always result in an element of $\mathcal{F}_{n-1}$ since the deletion of n cannot produce a new ascent. It follows that we may construct the set $\mathcal{F}_n$ recursively from $\mathcal{F}_{n-1}$, by considering each $\pi \in \mathcal{F}_{n-1}$ and then inserting n into certain sites between entries of π . Given $\pi \in \mathcal{F}_{n-1}$, we call a site of π *active* if inserting n into the site results in an element of $\mathcal{F}_n$. It is not difficult to see that the site before π_1 and the site after π_{n-1} are always active. The site between π_i and π_{i+1} is active if and only if $\pi_i = 1$ or $\pi_i - 1$ is to the left of π_i in π .

2.4. Bijections between Fishburn structures. In this section we recall two bijections, called Φ_{MA} and Ψ_{MA} , from Fishburn matrices to ascent sequences and also two bijections, called Φ_{AS} and Ψ_{SA} , between ascent sequences and Fishburn permutations. The subscript of each bijection indicates its domain and codomain, with M , A and S standing for Fishburn matrix, ascent sequence and Fishburn permutation respectively. These bijections have been used to prove equidistribution results on the various Fishburn structures and, moreover, to establish several symmetry results [11, 20]. Our focus in this paper will be on the classical bijections introduced in [7] and [17], which we denote by Φ_{MA} and Φ_{AS} , and which will be used to define the various restricted sets of Fishburn structures studied in this paper. The bijection Ψ_{SA} will be used to prove the translation of a statistic in the restricted case of series-parallel interval orders, which follows from the fact that the image of Ψ_{SA} agrees with that of the classical bijection in this case. Our main interest in the bijection Ψ_{MA} is in its interaction with the restricted case of semiorders, which proves to be the most difficult of the restricted cases. An overview of the bijections defined in this section is given in Figure 4. The following bijection was presented and proven in [17].

Definition 2.8 ($\Phi_{MA} : \text{FMatrices}_n \rightarrow \mathcal{A}_n$). Consider the following operation f on a given matrix $M \in \text{FMatrices}_n$.

- (Rem1) If $\text{rowsum}_{\text{mindex}(M)}(M) > 1$, then let $f(M)$ be the matrix M with the entry $M_{\text{mindex}(M), \text{dim}(M)}$ reduced by 1.
- (Rem2) If $\text{rowsum}_{\text{mindex}(M)}(M) = 1$ and $\text{mindex}(M) = \text{dim}(M)$, then let $f(M)$ be the matrix M with row $\text{dim}(M)$ and column $\text{dim}(M)$ removed.

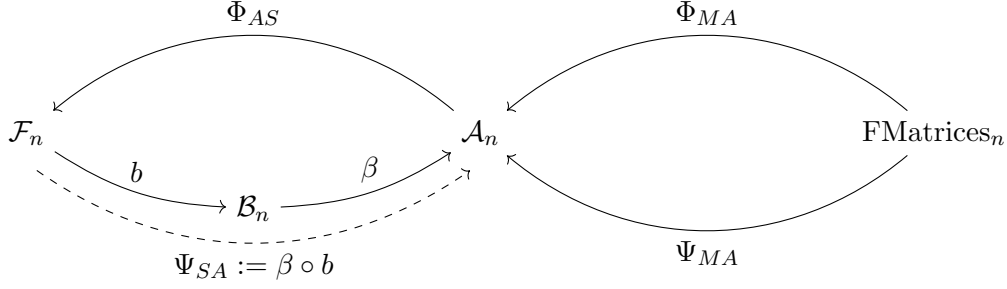


FIGURE 4. An overview of the bijections we review in Section 2.4. Note that this diagram does not commute.

(Rem3) If $\text{rowsum}_{\text{mindex}(M)}(M) = 1$ and $\text{mindex}(M) < \dim(M)$, then we construct $f(M)$ in the following way. Set $m_{i,\dim(M)} := m_{i,\text{mindex}(M)}$ for all $i \in [\text{mindex}(M) - 1]$. Now simultaneously delete row $\text{mindex}(M)$ and column $\text{mindex}(M)$. Let the resulting $(\dim(M) - 1) \times (\dim(M) - 1)$ matrix be $f(M)$.

The bijection $\Phi_{MA} : \text{FMatrices}_n \rightarrow \mathcal{A}_n$ is defined by

$$\Phi_{MA}(M) := (\text{mindex}(f^{n-1}(M)) - 1, \text{mindex}(f^{n-2}(M)) - 1, \dots, \text{mindex}(f(M)) - 1, \text{mindex}(M) - 1).$$

More details, as well as the reverse construction, can be found in [17]. The following bijection was introduced by Chen et. al. [11] to prove a conjecture of [22] regarding the equidistribution of a number of statistics between Fishburn structures.

Definition 2.9 ($\Psi_{MA} : \text{FMatrices}_n \rightarrow \mathcal{A}_n$). Consider the following operation g on a given matrix $M \in \text{FMatrices}_n$.

- (Rem1) If $\text{rowsum}_{\text{mindex}(M)}(M) > 1$, then let $g(M)$ be the matrix M with the entry $M_{\text{mindex}(M), \dim(M)}$ reduced by 1.
- (Rem2) If $\text{rowsum}_{\text{mindex}(M)}(M) = 1$ and $\text{mindex}(M) = \dim(M)$, then let $g(M)$ be the matrix M with row $\dim(M)$ and column $\dim(M)$ removed.
- (Rem3) If $\text{rowsum}_{\text{mindex}(M)}(M) = 1$ and $\text{mindex}(M) < \dim(M)$, then we construct $g(M)$ in the following way. Set $m_{i,j-1} := m_{i,j}$ for all $i, j \in [\text{mindex}(M) + 1, \dim(M)]$. Now simultaneously delete row $\text{mindex}(M)$ and column $\dim(M)$. Let the resulting $(\dim(M) - 1) \times (\dim(M) - 1)$ matrix be $g(M)$.

The bijection $\Psi_{MA} : \text{FMatrices}_n \rightarrow \mathcal{A}_n$ is defined by

$$\Psi_{MA}(M) := (\text{mindex}(g^{n-1}(M)) - 1, \text{mindex}(g^{n-2}(M)) - 1, \dots, \text{mindex}(g(M)) - 1, \text{mindex}(M) - 1).$$

Further details including the inverse $\Psi_{MA}^{-1} = \Psi_{AM}$ can be found in [11].

Proposition 2.10. If $M \in \text{FMatrices}_n$ then

$$\text{asc}(\Phi_{MA}(M)) = \dim(M) - 1 = \text{asc}(\Psi_{MA}(M)).$$

Proof. The first equality is proven in [17, Theorem 7], and the second in [11, Lemma 3.3]. $\square$

The following definition is given in [7].

Definition 2.11 ($\Phi_{AS} : \mathcal{A}_n \rightarrow \mathcal{F}_n$). The bijection Φ_{AS} is defined recursively on n as follows. For $n = 1$, we set $\Phi_{AS}((0)) = 1$. For $n \geq 2$, label the active sites of $\Phi_{AS}((a_1, \dots, a_{n-1}))$ with labels $0, 1, 2$ and so on from left to right. We define $\Phi_{AS}((a_1, \dots, a_n))$ to be the permutation obtained by inserting n into the active site of $\Phi_{AS}((a_1, \dots, a_{n-1}))$ which is labelled a_n .

The following property of Φ_{AS} facilitates the proof that it is indeed a bijection, which we omit here.

Remark 2.12. If a is an ascent sequence then the rightmost site of $\Phi_{AS}(a)$ is the active site labelled $\text{asc}(a) + 1$.

Suppose that τ is the result of deleting the entry k from the permutation $\pi \in \mathcal{F}_k$. Then the active site of τ into which we insert k to form π has label a_k , or equivalently, there are a_k other active sites to its left. The insertion of k into τ cannot make any other active site not active, or a non-active site active since, for every $\pi_i \neq 1$, we have that $\pi_i - 1$ is already placed in relation to π_i . For the same reason, the newly created site immediately to the left of k must be active. As a consequence we have the following remark.

Remark 2.13. Suppose that we are at iteration k of constructing the Fishburn permutation $\Phi_{AS}(a)$ for some $a \in \mathcal{A}_n$, so we insert k into the active site of $\Phi_{AS}((a_1, \dots, a_{k-1}))$ which is labelled a_k . Then

- (i) the site which is now immediately to the left of k is active and inherits the label a_k ,
- (ii) the site which is now immediately to the right of k may or may not be active depending on the position of $k - 1$ relative to k , and
- (iii) the status of all other sites as active or not active remains unchanged.

Fu et al. [20] presented a bijection from the set of Fishburn permutations to the set of ascent sequences. Their bijection was the composition of an earlier bijection due to Baril and Vajnovski [3] that mapped from permutations to inversion tables with a mapping from a restricted subset of inversion tables to ascent sequences. To define these bijections requires several notions to be first introduced.

An *inversion sequence* of length n is an integer sequence $(b_1, \dots, b_n)$ satisfying $b_i \in [0, i - 1]$ for all $i \in [n]$. We denote the set of inversion sequences by $\mathcal{I}_n$. A *labelled interval* is a pair (I, ℓ) where I is an interval and ℓ is an integer which we call the *label* of I .

Definition 2.14. For a permutation $\pi = \pi_1\pi_2 \dots \pi_n \in S_n$ and an $i \in [0, n - 1]$, the i th *slice* of π is the sequence of labelled intervals $U_i(\pi) = (I_1, \ell_1), (I_2, \ell_2), \dots, (I_k, \ell_k)$ defined recursively as follows. Set $U_0(\pi) = ([0, n], 0)$. For $i \in [n - 1]$, let $U_{i-1}(\pi) = (I_1, \ell_1), (I_2, \ell_2), \dots, (I_k, \ell_k)$ and let $v \in [k]$ such that $\pi_i \in I_v$. Now $U_i(\pi)$ is defined according to one of four cases.

(Slice1) If $\min I_v < \pi_i < \max I_v$ then

$$U_i(\pi) = (I_1, \ell_1), \dots, (I_{v-1}, \ell_{v-1}), (H, \ell_v), (J, \ell_{v+1}), (I_{v+1}, \ell_{v+2}), \dots, (I_{k-1}, \ell_k), (I_k, \ell_k + 1),$$

where $H = [\pi_i + 1, \max I_v]$ and $J = [\min I_v, \pi_i - 1]$.

(Slice2) If $\min I_v < \max I_v = \pi_i$ then

$$U_i(\pi) = (I_1, \ell_1), \dots, (I_{v-1}, \ell_{v-1}), (J, \ell_{v+1}), (I_{v+1}, \ell_{v+2}), \dots, (I_{k-1}, \ell_k), (I_k, \ell_k + 1),$$

where $J = [\min I_v, \pi_i - 1]$.

(Slice3) If $\min I_v = \pi_i < \max I_v$ then

$$U_i(\pi) = (I_1, \ell_1), \dots, (I_{v-1}, \ell_{v-1}), (J, \ell_v), (I_{v+1}, \ell_{v+1}), \dots, (I_{k-1}, \ell_{k-1}), (I_k, \ell_k + 1),$$

where $J = [\pi_i + 1, \max I_v]$.

(Slice4) If $\min I_v = \max I_v = \pi_i$ then

$$U_i(\pi) = (I_1, \ell_1), \dots, (I_{v-1}, \ell_{v-1}), (I_{v+1}, \ell_{v+1}), \dots, (I_{k-1}, \ell_{k-1}), (I_k, \ell_k + 1).$$

Please consult Figure 5 taken from [3] for a visualization of the above definition. Also see [3] for more details and an example.

Definition 2.15. Let $\pi = \pi_1\pi_2 \dots \pi_n \in S_n$. For each $i \in [n]$, if (I_v, ℓ_v) is the labelled interval in the $(i - 1)$ th slice of π such that $\pi_i \in I_v$, we define $b_i = \ell_v$. Denote by $b(\pi)$ the sequence $(b_1, b_2, \dots, b_n)$.

Baril and Vajnovszki [3] showed that $b : S_n \rightarrow \mathcal{I}_n$ is a bijection. Fu et al. [20] restricted this bijection to $\mathcal{F}_n$ and showed that it maps bijectively onto a certain subset $\mathcal{B}_n$ of inversion sequences. We include the definition of the set $\mathcal{B}_n$ and the bijection β for the sake of completeness, despite the fact that we will not explicitly make use of them in this paper. Let $\mathcal{B}_n$ be the set of inversion sequences $(b_1, \dots, b_n) \in \mathcal{I}_n$ satisfying:

- (i) If $b_i \geq b_{i+1}$ and $b_i = i - 1$, then $b_j < j - 1$ for all $j > i$.

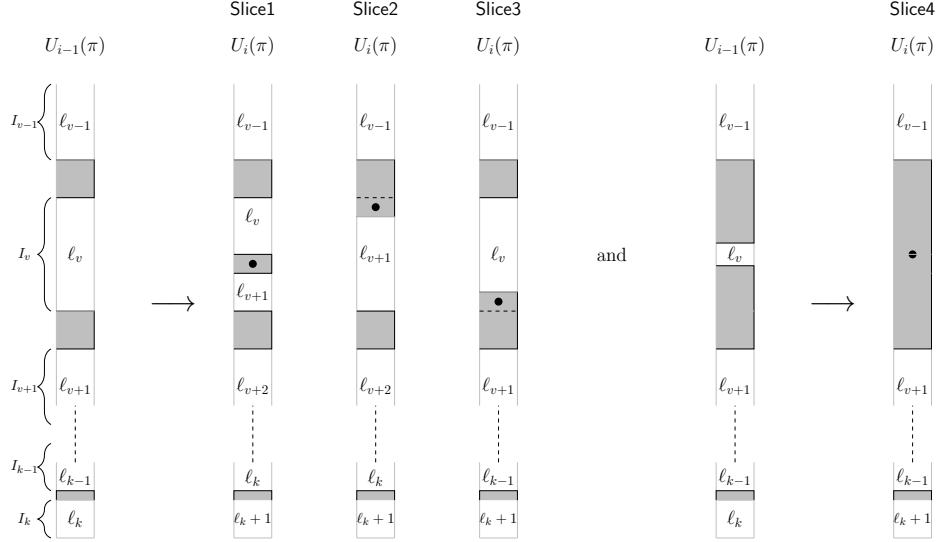


FIGURE 5. The four cases in Definition 2.14 from [3, Definition 1].

(ii) If $b_i \geq b_{i+1}$ and $b_i < i - 1$, then $i - 1$ does not occur in $(b_{i+1}, b_{i+2}, \dots, b_n)$.

Let $b \in \mathcal{B}_n$ be an input sequence and suppose that $[n - 1] \setminus \text{Asc}(b) = \{i_1, i_2, \dots, i_k\}$ with $i_1 > i_2 > \dots > i_k$. Now apply the following procedure:

```

for  $i = i_1, i_2, \dots, i_k$  do:
  for  $j = i + 1, i + 2, \dots, n$  do:
    if  $b_i < i - 1$  and  $b_j > i - 1$  then  $b_j \leftarrow b_j - 1$ 

```

and denote the resulting sequence by $\beta(b)$.

Fu et al. proved that $\beta : \mathcal{B}_n \rightarrow \mathcal{A}_n$ is a bijection and make the following observation.

Remark 2.16. The entries in $b \in \mathcal{B}_n$ and the entries in $\beta(b)$ are in the same relative order.

Finally we define another bijection from Fishburn permutations to ascent sequences.

Definition 2.17 ($\Psi_{SA} : \mathcal{F}_n \rightarrow \mathcal{A}_n$). The bijection Ψ_{SA} is defined by $\Psi_{SA} := \beta \circ b|_{\mathcal{F}_n}$.

In the main theorem of [20], it is shown that Ψ_{SA} translates a number of properties between ascent sequences and Fishburn permutations. We are interested in one in particular.

Proposition 2.18. [20, Theorem 12] If $\pi \in \mathcal{F}_n$ then

$$\text{Des}(\pi) = \text{Asc}(\Psi_{SA}(\pi)).$$

It follows also that $\text{maj}(\pi) = \text{amaj}(\Psi_{SA}(\pi))$ and $\text{des}(\pi) = \text{asc}(\Psi_{SA}(\pi))$.

3. RESTRICTED FISHBURN STRUCTURES

We will consider six distinct classes of Fishburn structures, which we present here in terms of the relevant interval orders:

- I.** Interval orders: $(2 + 2)$ -free posets.
- II.** Interval orders which are cover- N -free.
- III.** *Series parallel interval orders*: Interval orders which are N -free.
- IV.** *Semiororders*, also known as *unit interval orders*: Interval orders which are $(3 + 1)$ -free.
- V.** Interval orders which are $(3 + 1)$ -free and cover- N -free.
- VI.** *Series parallel semiororders*: Interval orders which are $(3 + 1)$ -free and N -free.

The absence of interval orders whose covering relation is $(3 + 1)$ -free as defining a separate class is explained by the fact that a poset is $(3 + 1)$ -free if and only if its covering relation is $(3 + 1)$ -free.

Figure 1 shows the six classes ordered by containment. It is straightforward to check that an N -free poset must also have an N -free covering relation. From this we see that class III is a special case of class II. The containment between all other classes given in the diagram is immediate from their definitions. Note also that, by definition, class V is the intersection of classes II and IV, and that class VI is the intersection of classes III and IV (or equivalently III and V).

For $R \in \{I, II, III, IV, V, VI\}$, we denote by $\text{FPosets}_n^{(R)}$ the set of all interval orders on n which belong to class R , and by $\text{FMatrices}_n^{(R)}$ the set of corresponding Fishburn matrix representations. We also denote by $\mathcal{A}_n^{(R)}$ and $\mathcal{F}_n^{(R)}$ the corresponding sets of ascent sequences and Fishburn permutations respectively via Φ_{AS} and Φ_{MA} :

$$\begin{aligned}\mathcal{A}_n^{(R)} &:= \Phi_{MA}(\text{FMatrices}_n^{(R)}) \\ \mathcal{F}_n^{(R)} &:= \Phi_{AS}(\mathcal{A}_n^{(R)}).\end{aligned}$$

Since a Fishburn matrix is a unique representation of an interval order, and Φ_{AS} and Φ_{MA} are bijections, our discussion of containment and intersection holds for every kind of Fishburn structure.

Table 1 gives a summary of the characterizations of the Fishburn structures in each class, as well as the enumeration of each class. It consists of a mixture of known results and results proven in the following sections. Those cells that refer to a proposition represent new results.

I	Interval orders	Fishburn matrices	Ascent sequences	Fishburn permutations	A022493
II	Cover- N -free $\equiv$ max cardinality chain (Prop. 4.6)	Invertible	10 1-avoiding $\equiv$ self-modified	2 31, $\bar{2}31$ -avoiding $\equiv$ $3\bar{1}52\bar{4}$ -avoiding (Prop. 4.8)	A098569
III	N -free	North-west-free	101-avoiding	231-avoiding	A000108 Catalan
IV	$(3+1)$ -free	South-west-free	Unknown	Unknown	A000108 Catalan
V	$(3+1, \text{cover}-N)$ -free	Invertible, bidiagonal (Prop. 6.1)	10 1, 120-avoiding (Prop. 6.4)	2 31, $\bar{2}31$, 4123-avoiding (Prop. 6.5)	A001519 (Prop. 6.6)
VI	$(3+1, N)$ -free	Invertible, bidiagonal, no adjacent positive superdiagonal entries (Prop. 7.1)	101, 120-avoiding (Prop. 7.3)	231, 4123-avoiding (Prop. 7.4)	A116703

TABLE 1. Characterizations of the Fishburn structures via Φ_{AS} and Φ_{MA} in classes I to VI, and their counting sequences from OEIS [29]. Note that the terms north-west-free and south-west-free appear in Definition 5.1.

The unrestricted class I is enumerated by the Fishburn numbers 1, 2, 5, 15, 53, 217, 1014, 5335, $\dots$, which is sequence [29, A022493], and this was proven in [7].

4. CLASS II: COVER- N -FREE INTERVAL ORDERS

In this section, we consider the restricted class of Fishburn structures which correspond to cover- N -free interval orders. Cover- N -free interval orders were first considered in [2] where the authors derived a generating function for their number. This class of Fishburn structure is already well-known and studied under a different characterization. The earliest of these structures to be considered was the set of *self-modified ascent sequences* in [7]. The corresponding interval orders are characterized by the existence of a chain whose length is equal to the magnitude of the interval order, or in terms of their Fishburn matrix representation by the condition that each diagonal entry is non-zero [16], i.e. the invertible Fishburn matrices. The corresponding Fishburn permutations are also known to be characterized by avoidance of the *barred pattern* $3\bar{1}52\bar{4}$. The ordinary generating function for these permutations was derived in [7], and the corresponding counting sequence is 1, 2, 5, 14, 43, 143, 510, 1936, $\dots$ [29, A098569].

In Lemmas 4.5 and 4.7, and Proposition 4.6, we present alternative characterizations of the ascent sequences, Fishburn permutations and interval orders in this class. We also show, in Proposition 4.9, that the ascent statistic is well-behaved under the action of the bijection Φ_{AS} .

Definition 4.1 ([7]). An ascent sequence $a = (a_1 \dots, a_n)$ is *self-modified* if

$$a_k \in [0, a_{k-1}] \cup \{1 + \text{asc}(a_1, \dots, a_{k-1})\}$$

for all $k > 1$.

Definition 4.2. A permutation π avoids the *barred pattern* $3\bar{1}52\bar{4}$ if every occurrence of the (classical) pattern 231 in π plays the role of 352 in an occurrence of the (classical) pattern 31524 in π . In other words, for every occurrence $\pi_i\pi_j\pi_k$ of 231 in π , there exists l and m such that $\pi_i\pi_l\pi_j\pi_k\pi_m$ is an occurrence of 31524 in π .

This bold bar notation is not to be confused with the bar notation used for bivincular patterns.

Proposition 4.3. [16, Corollary 3.8] The following sets are in 1-1 correspondence via the classical bijections Φ_{MA} and Φ_{AS} .

- (i) The set of interval orders of size n which have a chain whose cardinality is equal to the magnitude of the interval order.
- (ii) The set of Fishburn matrices of weight n which have all non-zero diagonal entries.
- (iii) The set of self-modified ascent sequences of length n .
- (iv) The set of permutations of $[n]$ which avoid the barred pattern $3\bar{1}52\bar{4}$.

We will need to introduce at this point some notation to write sequences with repeated entries. We write x^k to denote $k \geq 0$ consecutive occurrences of the term x . For example, $(0^2, 1^0, 2^1, 3^2) = (0, 0, 2, 3, 3)$.

Lemma 4.4. [16, Lemma 3.4] Let M be a Fishburn matrix of weight n and dimension d having all non-zero diagonal entries. Then

$$\Phi_{MA}(M) = (0^{m_{1,1}}, 1^{m_{2,2}}, 0^{m_{1,2}}, 2^{m_{3,3}}, 1^{m_{2,3}}, 0^{m_{1,3}}, \dots, (d-1)^{m_{d,d}}, (d-2)^{m_{d-1,d}}, \dots, 0^{m_{1,d}}).$$

It follows then from Proposition 4.3 that a is a self-modified ascent sequence of length n iff a is of the form

$$a = (0^{m_{1,1}}, 1^{m_{2,2}}, 0^{m_{1,2}}, 2^{m_{3,3}}, 1^{m_{2,3}}, 0^{m_{1,3}}, \dots, (d-1)^{m_{d,d}}, (d-2)^{m_{d-1,d}}, \dots, 0^{m_{1,d}}),$$

for some $d \in [n]$, where the $m_{i,j}$ are non-negative integers and $m_{i,i} \neq 0$ for all $i \in [d]$.

In other words, a self-modified ascent sequence consists of d weakly descending runs, the i th of which has first term $i - 1$. Lin and Yan [27, Remark 3.3] found that the self-modified ascent sequences are exactly the 10|1-avoiding ascent sequences. We give a proof of this fact.

Lemma 4.5. $\mathcal{A}_n(10|1)$ is the set of self-modified ascent sequences of length n .

Proof. We prove this by induction on n . The base case $n = 1$ is immediate since (0) is the only ascent sequence of length 1. An ascent sequence which contains 10|1 cannot be made to avoid 10|1 by appending terms, nor can an ascent sequence which is not self-modified be made to be self-modified by appending terms. Hence, we may suppose that $a = (a_1, \dots, a_{n-1})$ is a self-modified and 10|1-avoiding ascent sequence and show that appending the term a_n to a completes an occurrence of 10|1 if and only if $a_n \notin [0, a_{n-1}] \cup \{1 + \text{asc}(a)\}$. It is clear from the characterization of a given in Lemma 4.4 that $\text{asc}(a)$ is equal to the maximum term of a , and that every value less than or equal to $\text{asc}(a)$ occurs in a . Hence a_n completes an occurrence of 10|1 iff $a_{n-1} < a_n \leq \text{asc}(a)$. $\square$

Proposition 4.6. Let $P \in \text{FPosets}_n$. Then P is cover- N -free iff every diagonal entry of its Fishburn matrix representation is non-zero.

Proof. We prove the following equivalent statement. The labelled interval order P contains a cover- N pattern iff $M(P)$ has at least one empty diagonal entry.

$\implies$: Suppose that there exist $a, b, c, d \in P$ with $a < b > c < d$ and no other covering relations between a, b, c and d . Suppose also that the entries of $M(P)$ which contain a and c are in columns j_a and j_c respectively, and that the entry of $M(P)$ which contains b is in row i_b . By Remark 2.1, we have $j_a < i_b$ and $j_c < i_b$. The element d ensures that a and c necessarily have

distinct strict upsets, that is $\{p \in P : p > a\} \neq \{p \in P : p > c\}$, and it follows that $j_a \neq j_c$. So we must have either $j_a < i_b - 1$ or $j_c < i_b - 1$. If we have $j_a < i_b - 1$ then the diagonal entry of column $i_b - 1$ must be empty, since otherwise an element p in said entry would satisfy $a < p < b$ by Remark 2.1, contradicting $a \lessdot b$. Similarly for the case $j_c < i_b - 1$.

$\Leftarrow$: Suppose that the (i, i) entry of $M(P)$ is the empty set. To avoid empty rows or columns, we cannot have $i = 1$ or $i = \dim(M)$. Let c (resp. a) be an element of some entry of column $i - 1$ (resp. i), and let d (resp. b) be an element of some entry of row i (resp. $i + 1$). Using Remark 2.1, it is straightforward to check that $a \lessdot b \succ c \lessdot d$ and that no other covering relations between a, b, c and d . $\square$

We note that the avoidance of the barred pattern $3\bar{1}52\bar{4}$ is equivalent to the avoiding of a pair of bivincular patterns.

Lemma 4.7. We have

$$S_n(2|31, \bar{2}31) = S_n(3\bar{1}52\bar{4})$$

Proof. In [5, Section 6] it was noted that a permutation avoids the barred pattern $\bar{4}25\bar{1}3$ iff it avoids both the bivincular patterns $13|2$ and $13\bar{2}$. This follows from a more general result of Pudwell [31]. The claim follows by reversing each of these patterns. $\square$

While it was already known that every 231-avoiding permutation is also a $3\bar{1}52\bar{4}$ -avoiding permutation, and every $3\bar{1}52\bar{4}$ -avoiding permutations is also a Fishburn permutation (i.e. $2|3\bar{1}$ -avoiding), the description in Lemma 4.7 in terms of bivincular pattern avoidance makes this fact more explicit.

Proposition 4.8. We have

$$\mathcal{A}_n^{(\text{II})} = \mathcal{A}_n(10|1) \text{ and } \mathcal{F}_n^{(\text{II})} = S_n(2|31, \bar{2}31).$$

Proof. This follows directly from Propositions 4.3, 4.6, and Lemmas 4.5, 4.7. $\square$

We proceed to prove that the bijection Φ_{AS} preserves the ascent statistic on ascent sequences in class II. This result will be relevant in Section 8 where we consider the ascent polynomial over the various restricted sets of ascent sequences.

Proposition 4.9. If $a \in \mathcal{A}_n^{(\text{II})}$ then

$$\text{asc}(a) = \text{asc}(\Phi_{AS}(a)).$$

Proof. We prove the claim by induction on n . For the base case $n = 1$ we have $\mathcal{A}_1^{(\text{II})} = \{(0)\}$ and $\Phi_{AS}((0)) = 1$ so the result follows. Let $a' \in \mathcal{A}_{n+1}^{(\text{II})}$ be the result of appending the integer a_{n+1} to $a = (a_1, \dots, a_n)$ (note that necessarily $a \in \mathcal{A}_n^{(\text{II})}$), and let $\pi' = \Phi_{AS}(a')$. Recall that $\mathcal{A}_n^{(\text{II})}$ is the set of self-modified ascent sequences of length n , so we may consider two cases.

- (i) $a_{n+1} \in [0, a_n]$: In this case we have $\text{asc}(a') = \text{asc}(a)$. We wish to show that $\text{asc}(\pi') = \text{asc}(\pi)$. We have $a_{n+1} \leq a_n \leq \text{asc}(a) < \text{asc}(a) + 1$, so a_{n+1} is not the label of the rightmost site of π by Remark 2.12. If $a_{n+1} = 0$ then $n + 1$ is inserted into the leftmost site of π and we are done. Otherwise, suppose that a_{n+1} is the label of the active site between π_i and π_{i+1} . Since the site is active, $\pi_i - 1$ is to the left of π_i . Consider the π_i th iteration of the construction of π : in $\Phi_{AS}((a_1, \dots, a_{\pi_i-1}))$ the active site labelled a_{π_i-1} is immediately to the left of $\pi_i - 1$ by Remark 2.13, so the fact that π_i is inserted to the right of $\pi_i - 1$ implies that $a_{\pi_i-1} < a_{\pi_i}$. Hence $a_{\pi_i} = \text{asc}(a_1, \dots, a_{\pi_i-1}) + 1$ by the definition of a self-modified ascent sequence, meaning that in the construction of $\Phi_{AS}((a_1, \dots, a_{\pi_i}))$, π_i is inserted into the rightmost site by Remark 2.12. Certainly π_{i+1} , being to the right of π_i , cannot yet have been inserted into the permutation at this iteration of the construction of π , so $\pi_i < \pi_{i+1}$. Hence the insertion of $n + 1$ between π_i and π_{i+1} leaves the number of ascents unchanged, and we are done.
- (ii) $a_{n+1} = \text{asc}(a) + 1$: In this case, $a_n < a_{n+1}$ so $\text{asc}(a') = \text{asc}(a) + 1$. In the construction of π' , we insert $n + 1$ into the rightmost site of π by Remark 2.12, so $\text{asc}(\pi') = \text{asc}(\pi) + 1$.

□

Another consequence of the well-behaved nature of class II is that the two bijections between Fishburn matrices and ascent sequences, Φ_{MA} and Ψ_{MA} , are in fact identical when restricted to this class.

Proposition 4.10. If $M \in \text{FMatrices}_n^{(\text{II})}$ then

$$\Phi_{MA}(M) = \Psi_{MA}(M).$$

Proof. From [16] we have that $\text{FMatrices}_n^{(\text{II})}$ is exactly the subset of FMatrices_n of matrices M which do not invoke the use **Rem3** in the construction of $\Phi_{MA}(M)$. The result follows since Φ_{MA} and Ψ_{MA} only differ in **Rem3**. □

Recall that $\text{FMatrices}_n^{(\text{III})}$, $\text{FMatrices}_n^{(\text{V})}$ and $\text{FMatrices}_n^{(\text{VI})}$ are all subsets of $\text{FMatrices}_n^{(\text{II})}$, so Φ_{MA} and Ψ_{MA} are the same bijection when restricted to each of these sets. It follows that

$$\Psi_{MA}(\text{FMatrices}_n^{(\text{R})}) = \Phi_{MA}(\text{FMatrices}_n^{(\text{R})}) = \mathcal{A}_n^{(\text{R})}$$

for $\text{R} \in \{\text{I}, \text{II}, \text{III}, \text{V}, \text{VI}\}$. The only set on which Φ_{MA} and Ψ_{MA} have different images is $\text{FMatrices}_n^{(\text{IV})}$. For this reason we define two new sets

$$\mathcal{A}_n^{(*\text{IV})} := \Psi_{MA}(\text{FMatrices}_n^{(\text{IV})}) \quad \text{and} \quad \mathcal{F}_n^{(*\text{IV})} := \Phi_{AS}(\mathcal{A}_n^{(*\text{IV})}).$$

We will need the following lemma to prove a result about class V, but since the lemma holds in the more general setting of class II we present it here.

Lemma 4.11. If $a \in \mathcal{A}_n^{(\text{II})}$ and $\pi = \Phi_{AS}(a)$, then for all $k \in [n]$ the active site labelled a_{π_k} is somewhere to the left of π_k , and there are no other active sites between π_k and the active site labelled a_{π_k} .

Proof. We prove the claim by induction on n . For the base case $n = 1$, we have $\mathcal{A}_1^{(\text{II})} = \{(0)\}$ and $\Phi_{AS}((0)) = 1$ so the result follows. Let $a' \in \mathcal{A}_{n+1}^{(\text{II})}$ be the result of appending the integer a_{n+1} to $a = (a_1, \dots, a_n)$, and let $\pi' = \Phi_{AS}(a')$. Recall that $\mathcal{A}_n^{(\text{II})}$ is the set of self-modified ascent sequences of length n , so we may consider two cases.

- (i) $a_{n+1} \in [0, a_n]$: In this case $a_{n+1} \leq a_n$ so it follows from Remark 2.13(i) that $n + 1$ is inserted to the left of n in the construction of π' . Hence the site immediately to the right of $n + 1$ in π' is not active. For $k = n + 1$, the claim follows from Remark 2.13(i). Also by Remark 2.13(iii), the status of all other sites as active or not active remains unchanged by the insertion of $n + 1$, and since no new active sites are created, their labelling remains unchanged also. So for $k \in [n]$, the result follows from the induction hypothesis.
- (ii) $a_{n+1} = \text{asc}(a) + 1$: In this case $n + 1$ was inserted into the rightmost site of π by Remark 2.12. Again, the result follows from Remark 2.13(i) for $k = n + 1$. Also by Remark 2.13(iii), the status of all other sites as active or not active remains unchanged, and since they are all to the left of the newly created active site, their labelling remains unchanged also. So for $k \in [n]$, the result follows from the induction hypothesis. □

5. CLASSES III AND IV: $(3 + 1)$ -FREE INTERVAL ORDERS AND N -FREE INTERVAL ORDERS

In this section, we consider the two Catalan-enumerated classes of Fishburn structures, namely those which correspond to N -free interval orders and those which correspond to $(3 + 1)$ -free interval orders. That N -free interval orders are enumerated by the Catalan numbers is shown in [13], and for $(3 + 1)$ -free interval orders it is shown in [35] (see also [33, Exercise 6.19]). Both of these restrictions are well studied, and we will recall some relevant results. We also prove one new result regarding Ψ_{SA} in this context. Lastly, we present a novel interpretation of the correspondence between N -free interval orders, $(3 + 1)$ -free interval orders, and Dyck

paths. This interpretation illustrates the translation of several important statistics and raises some questions.

Definition 5.1. Let $M \in \text{FMatrices}_n$ and suppose that $m_{i,j}$ and $m_{i',j'}$ are non-zero entries of M . If $i < i'$ and $j > j'$, then we say that the entry positions (i, j) and (i', j') are in a *strictly south-west mutual position*. If no such (i, j) and (i', j') exist, then we say that M is *south-west-free*. If $i < i' \leq j < j'$, then we say that the entry positions (i, j) and (i', j') are in a *strictly north-west mutual position*. If no such (i, j) and (i', j') exist, then we say that M is *north-west-free*.

Example 5.2. The following Fishburn matrix from Example 2.3 is south-west free, but not north-west free.

$$\begin{pmatrix} 2 & 1 & 1 & 0 \\ & 0 & 1 & 0 \\ & & 0 & 1 \\ & & & 1 \end{pmatrix}.$$

The entry positions $(1, 3)$ and $(3, 4)$ are in a strictly north-west mutual position.

Lemma 5.3. [22, Lemma 1.2] A Fishburn matrix M of weight n is in $\text{FMatrices}_n^{(\text{IV})}$ iff M is south-west-free. A Fishburn matrix M of weight n is in $\text{FMatrices}_n^{(\text{III})}$ iff M is north-west-free.

Proposition 5.4. [16, Corollary 5.9] We have $\mathcal{A}_n^{(\text{III})} = \mathcal{A}_n(101)$ and $\mathcal{F}_n^{(\text{III})} = S_n(231)$.

It turns out that 101-avoidance in ascent sequences also corresponds to 231-avoidance in Fishburn permutations under the bijection Ψ_{SA} described by Fu et al. [20], a result which we will make use of later.

Proposition 5.5. We have

$$\Psi_{SA}(S_n(231)) = \mathcal{A}_n(101).$$

The following proof (and in fact the definition of Ψ_{SA}) is best understood using the intuition provided by Figure 5. With this in mind, we introduce some terminology. We will say that an entry π_j of a permutation π is *unshaded in $U_i(\pi)$* if $i < j$, and otherwise we will say that π_j is *shaded in $U_i(\pi)$* .

Proof of Proposition 5.5. We prove that for $\pi \in \mathcal{F}_n$, we have $\pi \in S_n(231)$ iff $\Psi_{SA}(\pi) := \beta(b(\pi)) \in \mathcal{A}_n(101)$. First note that $S_n(231)$ is a subset of $\mathcal{F}_n$, the domain of Ψ_{SA} , meaning that $\Psi_{SA}(\pi)$ is in fact defined. Since we already know that $S_n(231)$ and $\mathcal{A}_n(101)$ are in bijection, it suffices to prove that $\pi \in S_n(231)$ implies $\beta(b(\pi)) \in \mathcal{A}_n(101)$. By Remark 2.16 we have that $b \in \mathcal{B}_n$ avoids 101 iff $\beta(b)$ avoids 101. So it is only left to prove that if $\pi = \pi_1 \dots \pi_n \in \mathcal{F}_n$ contains 231, then $b(\pi) = (b_1, \dots, b_n)$ contains 101.

Let $\pi_p \pi_q \pi_r$ be the leftmost occurrence of 231 in π , in the sense that it is lexicographically minimal.

Claim: Let $i \in [n]$ with $p \leq i < q$ and suppose that $U_i(\pi) = (I_1, \ell_1), (I_2, \ell_2), \dots, (I_k, \ell_k)$. Let $L \in [k]$ be such that $\max I_L < \pi_p < \min I_{L-1}$, and let $R \in [k]$ be such that $I_R = I_k$ if $I_k \neq [0, 0]$ and $I_R = I_{k-1}$ otherwise. Then

- (i) $\pi_i < \min I_R$ and $\{\ell_L, \dots, \ell_R\} \subseteq \{b_1, \dots, b_i\}$ or
- (ii) $\pi_i = \max I_R + 1$ and $\{\ell_L, \dots, \ell_{R-1}\} \subseteq \{b_1, \dots, b_i\}$ or
- (iii) $b(\pi)$ contains 101.

Proof of claim: We prove the claim by induction on i . For the base case $i = p$ we consider two cases and show that (ii) is satisfied in either case. In the first case, we suppose that there exists m such that $\pi_m < \pi_p$ and $m < p$ (that is π_m is to the left of π_p in π). Then $\pi_m - 1$ must be to the left of π_p also, since otherwise $\pi_m \pi_p (\pi_m - 1)$ would be a more left-justified occurrence of 231. It follows inductively that there exists some π_m such that $1, 2, \dots, \pi_m$ are all to the left of π_p in π , and $\pi_m + 1, \pi_m + 2, \dots, \pi_p - 1$ are all to the right of π_p in π . This means that all elements of the interval $[1, \pi_m]$ are shaded in $U_p(\pi)$ and all elements of the interval

$[\pi_m + 1, \pi_p - 1]$ are unshaded in $U_p(\pi)$. Note that necessarily $\pi_m < \pi_r$, since π_r is unshaded in $U_p(\pi)$, so $[\pi_m + 1, \pi_p - 1] \neq \emptyset$. It follows that $I_k = [0, 0]$ and $I_{k-1} = [\pi_m + 1, \pi_p - 1]$, so $I_{k-1} = I_R$ and $\pi_p = \max I_R + 1$. Also, we have $L = k - 1 = R$ so $\{\ell_L, \dots, \ell_{R-1}\}$ is empty. In the second case, we suppose that there does not exist m such that $\pi_m < \pi_p$ and $m < p$. This means that every element of the interval $[1, \pi_p - 1]$ is unshaded in $U_p(\pi)$. Note also that $\pi_p - 1 \neq 0$ since $\pi_r < \pi_p$. It follows that $I_k = [0, \pi_p - 1]$, so $I_k = I_R$ and $\pi_p = \max I_R + 1$. Also, we have $L = k = R$ so $\{\ell_L, \dots, \ell_{R-1}\}$ is empty.

For the induction step, suppose that $U_{i+1}(\pi) = (I'_1, \ell'_1), (I'_2, \ell'_2), \dots, (I'_{k'}, \ell'_{k'})$. Let $L' \in [k']$ be such that π_p is between $I'_{L'}$ and $I'_{L'+1}$, and let $R' \in [k']$ be such that $I'_{R'} = I'_{k'}$ if $I'_{k'} \neq [0, 0]$ and $I'_{R'} = I'_{k'-1}$ otherwise.

We first suppose that (i) holds for i . Note that since $\pi_i < \min I_R$, we cannot have $R = k$, so $R = k - 1$. Let $\pi_{i+1} \in I_v$ and suppose that $v \neq R$, so we must have $\ell_v < \ell_R$. Since $\pi_{i+1} \notin I_R$ and $R \neq k$, it can be seen from the construction of $U_{i+1}(\pi)$ that ℓ_R will still be the label of some interval of $U_{i+1}(\pi)$ (i.e. $\ell_R \in \{\ell'_1, \ell'_2, \dots, \ell'_{k'}\}$). It can further be seen that ℓ_R cannot cease to be the value of some label before we encounter some $i' > i + 1$ such that $\pi_{i'}$ is an element of the interval of $U_{i'}(\pi)$ labelled ℓ_R , so we may conclude that $\ell_R \in \{b_{i+2}, \dots, b_n\}$. But we also have $\ell_R \in \{b_1, \dots, b_i\}$ and $\ell_v = b_{i+1}$, and so (ℓ_R, ℓ_v, ℓ_R) is an occurrence of 101 in $b(\pi)$. Otherwise if we suppose that $v = R$, then there are two cases. If π_{i+1} invokes **Slice1** or **Slice2** then clearly $\pi_{i+1} = \max I'_{R'} + 1$. It can straightforwardly be verified that $\{\ell'_{L'}, \dots, \ell'_{R'-1}\} \subseteq \{\ell_L, \dots, \ell_R\}$. Combining this with the induction hypothesis we get $\{\ell'_{L'}, \dots, \ell'_{R'-1}\} \subseteq \{b_1, \dots, b_i\} \subseteq \{b_1, \dots, b_{i+1}\}$, so (ii) holds for $i + 1$. Otherwise, if π_{i+1} invokes **Slice3** or **Slice4** then $\pi_{i+1} < \min I'_{R'}$. It can straightforwardly be verified that $b_{i+1} = \ell_R$ and $\{\ell'_{L'}, \dots, \ell'_{R'}\} \subseteq \{\ell_L, \dots, \ell_R\}$. Combining this with the induction hypothesis we get $\{\ell'_{L'}, \dots, \ell'_{R'}\} \subseteq \{b_1, \dots, b_{i+1}\}$, so (i) holds for $i + 1$.

Next we suppose that (ii) holds for i . It follows that $\pi_i - 1$ is unshaded in $U_i(\pi)$, meaning that $\pi_i - 1$ is to the right of π_i in π . Now we must have $\pi_{i+1} < \pi_i$ since otherwise $\pi_i \pi_{i+1} (\pi_i - 1)$ would form an occurrence of $2|3\bar{1}$ in π . Hence $\pi_{i+1} \in I_R$. As before there are two cases. If π_{i+1} invokes **Slice1** or **Slice2** then it can be shown in a similar fashion as before that (ii) holds for $i + 1$. Otherwise, if π_{i+1} invokes **Slice3** or **Slice4** then it can be shown in a similar fashion as before that (i) holds for $i + 1$.

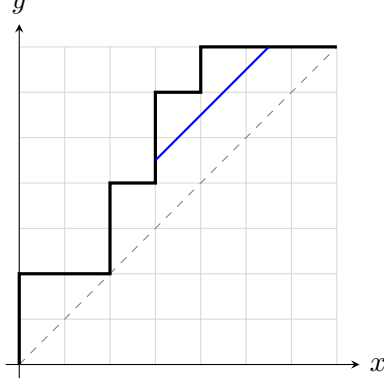
Finally, (iii) is independent of i , so if (iii) holds for i , it will also hold for $i + 1$. This proves the claim.

We now apply our claim to the case $i = q - 1$. Note that $\pi_q > \max I_R$, since π_r is unshaded and π_p is shaded in $U_{q-1}(\pi)$, and $\pi_r < \pi_p < \pi_q$. If (iii) holds then we are done. Suppose for a contradiction that (ii) holds for $i = q - 1$. Since $\pi_{q-1} = \max I_R + 1$ and $\max I_R < \pi_q$, we have $\pi_{q-1} < \pi_q$. But recall that in the induction step we showed that if (ii) holds for i then we must have $\pi_{i+1} < \pi_i$.

Hence (i) must hold for $i = q - 1$. If we let $\pi_{i+1} = \pi_q \in I_v$, then we must have $v \neq R$ since $\pi_q > \max I_R$. Now we may follow the exact same reasoning as in the induction step to conclude that $b(\pi)$ contains 101. $\square$

5.1. Dyck path correspondences. We define a *Dyck path* of semi-length n to be a lattice path from $(0, 0)$ to (n, n) which consists of n up steps and n right steps, and never falls below the line $y = x$. We denote by $\mathcal{D}_n$ the set of Dyck paths of semi-length n . A *tunnel* of $D \in \mathcal{D}_n$ is a line segment joining the midpoint of an up step and the midpoint of a down step of D , which doesn't intersect D at any other points and is parallel to the line $y = x$. A Dyck path of semi-length n has n tunnels. We will say a *run of up steps* (resp. *run of right steps*) to mean a maximal consecutive sequence of up (resp. right) steps. We denote by $\text{uprun}_i(D)$ (resp. $\text{rightrun}_i(D)$) the number of up steps (resp. right steps) in the i th run of up steps (resp. right steps) of D .

Example 5.6. A Dyck path $D \in \mathcal{D}_7$ and an example of a tunnel (in blue) which begins at the third run of up steps and ends at the fourth run of right steps of D .



The following map is equivalent to the one between Dyck paths and series-parallel interval orders described in [13]. Given a Dyck path D , we associate with D an integer matrix $\phi_1(D)$ whose (i, j) entry is the number of tunnels which begin at the i th run of up steps and end at the j th run of right steps. It is easily seen that $\phi_1(D)$ is a Fishburn matrix.

Example 5.7. The Fishburn matrix corresponding to the Dyck path from Example 5.6 via ϕ_1 is

$$\phi_1(D) = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

For example, the tunnel marked in blue contributes +1 to the entry at $(3, 4)$.

Proposition 5.8. If $D \in \mathcal{D}_n$ then $\phi_1(D) \in \text{FMatrices}_n^{(\text{III})}$.

Proof. By Lemma 5.3 we may show that $\phi_1(D)$ is north-west-free. Suppose that both the (i, j) and (i', j') entries of $\phi_1(D)$ are non-zero, and that $i < i'$ and $j < j'$. We wish to show that (i, j) and (i', j') are not in a strictly north-west mutual position. By definition of ϕ_1 , there is a tunnel t (resp. t') from the i th (resp. i' th) up step to the j th (resp. j' th) right step of D . Since the start of t precedes start of t' in D , and the end of t precedes the end of t' in D , it is easily seen that the end of t must precede the start of t' in D also. That is, $j < i'$. $\square$

Remark 5.9. It can be seen from the definition of ϕ_1 that

$$\text{rowsum}_i(\phi_1(D)) = \text{uprun}_i(D) \quad \text{and} \quad \text{colsum}_j(\phi_1(D)) = \text{rightrun}_j(D).$$

Hence the map $\psi : \text{FMatrices}_n \rightarrow \mathcal{D}_n$ defined by

$$\text{uprun}_i(\psi(M)) = \text{rowsum}_i(M) \quad \text{and} \quad \text{rightrun}_j(\psi(M)) = \text{colsum}_j(M),$$

for $i, j \in [\dim(M)]$, provides a way to recover D from $\phi_1(D)$, and so ϕ_1 is a bijection $\mathcal{D}_n \rightarrow \text{FMatrices}_n^{(\text{III})}$ whose inverse is the map ψ restricted to $\text{FMatrices}_n^{(\text{III})}$.

The following map is equivalent to the one between Dyck paths and semiorders described in [33, Exercise 6.19], which is also equivalent to the one in [22]. We associate with D another Fishburn matrix $\phi_2(D)$ whose (i, j) entry is the number of indices $k \in [n]$ such that the k th up step is part of the i th run of up steps, and the k th right step is part of the j th run of right steps.

Example 5.10. The Fishburn matrix corresponding to the Dyck path from Example 5.6 via ϕ_2 is

$$\phi_2(D) = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

For example take $k = 4$. The 4th up step is part of the second run of up steps, and the 4th right step is part of the third run of rights steps, so the index $k = 4$ contributes $+1$ to the entry at $(2, 3)$.

We leave it as an exercise to verify the following remark whose proof is very similar to the proof of Proposition 5.8..

Remark 5.11. If $D \in \mathcal{D}_n$ then $\phi_2(D) \in \text{FMatrices}_n^{(\text{IV})}$.

The unexpected connection between ϕ_1 and ϕ_2 is that Remark 5.9 can also be seen to hold if we replace ϕ_1 by ϕ_2 , so ϕ_2 is also a bijection and its inverse is the restriction of the map ψ to $\text{FMatrices}_n^{(\text{IV})}$. In the language of category theory, ϕ_1 and ϕ_2 are both sections of ψ .

Question 5.12. What can be said about Fishburn matrices which map to the same Dyck path under ψ ? How many Fishburn matrices map to a given Dyck path under ψ ? Any section of ψ defines a Catalan-enumerated subset of FMatrices_n : are there any other combinatorially interesting ones? If so, could it be used to define a new type of Catalan pair as Jelínek did with semiorders in [22]?

A nice property of ψ is that reflecting a Fishburn matrix M of weight n along its anti-diagonal, so that its i th row becomes its i th last column (which amounts to taking the dual of the interval order represented by M), corresponds to reflecting the Dyck path $\psi(M)$ along the line $y = n - x$, so that its start point and end point switch positions. Another noteworthy implication of our discussion is that an element of $\text{FMatrices}_n^{(\text{III})}$ (resp. $\text{FMatrices}_n^{(\text{IV})}$) is uniquely determined by its row and column sums. Or in terms of posets, a series-parallel interval order (resp. semiorder) is uniquely determined by the size of each of its level sets and the size of each of the level sets of its dual.

We proceed to introduce a number statistics and demonstrate the translation of these statistics under ψ (and thus under ϕ_1 and ϕ_2). A *peak* in a Dyck path $D \in \mathcal{D}_n$ is an up step which is immediately followed by a right step. The number of peaks in D is denoted $\text{pea}(D)$. A *return* in D is a right step whose right endpoint touches the line $y = x$. The number of returns in D is denoted $\text{ret}(D)$. Given Dyck paths D_1 and D_2 , we denote their concatenation by $D_1 \cdot D_2$. Any Dyck path D allows for a decomposition $D = D_1 \cdot D_2 \cdot \dots \cdot D_{\text{ret}(D)}$ where each D_i is indecomposable in the sense that it is not the concatenation of two smaller Dyck paths. Define $\text{semilen}_i(D)$ to be the semi-length of the Dyck path D_i from this decomposition.

Given upper triangular matrices M_1 and M_2 , their direct sum is the upper triangular matrix

$$M_1 \oplus M_2 = \begin{pmatrix} M_1 & 0 \\ 0 & M_2 \end{pmatrix}.$$

Again it is clear that any upper triangular integer matrix M allows for a decomposition $M = M_1 \oplus M_2 \oplus \dots \oplus M_k$, and if each M_i is indecomposable in the sense that it cannot be written as the direct sum of two smaller matrices, then we define $\text{blockweight}_i(M)$ to be sum of the entries in M_i , and $\text{blocks}(M) := k$.

Before proving the translation of statistics under ψ , we show that ψ is a monoid homomorphism from the set of all Fishburn matrices under direct sum to the set of all Dyck paths under concatenation.

Lemma 5.13. If M_1 and M_2 are Fishburn matrices then

$$\psi(M_1 \oplus M_2) = \psi(M_1) \cdot \psi(M_2).$$

Proof. Let M_1, M_2 be Fishburn matrices with $\dim(M_1) = d_1$ and $\dim(M_2) = d_2$. Let us construct the Dyck path $\psi(M_1 \oplus M_2)$ in two stages. For $i \in [d_1]$ we have

$$\text{uprun}_i(\psi(M_1 \oplus M_2)) = \text{rowsum}_i(M_1 \oplus M_2) = \text{rowsum}_i(M_1) = \text{uprun}_i(\psi(M_1))$$

and

$$\text{rightrun}_i(\psi(M_1 \oplus M_2)) = \text{colsum}_i(M_1 \oplus M_2) = \text{colsum}_i(M_1) = \text{rightrun}_i(\psi(M_1)).$$

So the first d_1 runs of up steps and runs of right steps form the Dyck path $\psi(M_1)$, and naturally end on the diagonal $y = x$. From here, we have for $i \in [d_1 + 1, d_2]$ that

$$\text{uprun}_i(\psi(M_1 \oplus M_2)) = \text{rowsum}_i(M_1 \oplus M_2) = \text{rowsum}_i(M_2) = \text{uprun}_i(\psi(M_2))$$

and

$$\text{rightrun}_i(\psi(M_1 \oplus M_2)) = \text{colsum}_i(M_1 \oplus M_2) = \text{colsum}_i(M_2) = \text{rightrun}_i(\psi(M_2)).$$

So the final d_2 runs of up steps and runs of right steps form the Dyck path $\psi(M_1)$, completing the Dyck path $\psi(M_1 \oplus M_2)$. $\square$

Theorem 5.14. Let $M \in \text{FMatrices}_n$ and $D = \psi(M) \in \mathcal{D}_n$. Then

- (i) $\dim(M) = \text{pea}(D)$,
- (ii) $\text{rowsum}_i(M) = \text{uprun}_i(D)$ for $i \in [\dim(M)]$,
- (iii) $\text{colsum}_i(M) = \text{rightrun}_i(D)$ for $i \in [\dim(M)]$,
- (iv) $\text{blocks}(M) = \text{ret}(D)$, and
- (v) $\text{blockweight}_i(M) = \text{semilen}_i(D)$ for $i \in [\text{blocks}(M)]$.

In particular we have $\text{rowsum}_1(M) = \text{uprun}_1(D)$ and $\text{colsum}_{\dim(M)}(M) = \text{rightrun}_{\text{pea}(D)}(D)$.

Proof. Note that $\text{pea}(D)$ is exactly the number of runs of up steps (resp. right steps) of D , which is equal to $\dim(M)$ by the definition of ψ . Parts (ii) and (iii) are exactly the definition of ψ . If $M = M_1 \oplus \dots \oplus M_{\text{blocks}(M)}$ as described previously, then $\psi(M) = \psi(M_1) \cdot \dots \cdot \psi(M_{\text{blocks}(M)})$ by Lemma 5.13. Since each M_i is not the direct sum of smaller matrices, it also follows from Lemma 5.13 that each $\psi(M_i)$ is not the concatenation of smaller Dyck paths. Now if $D = D_1 \cdot D_2 \cdot \dots \cdot D_{\text{ret}(D)}$ as described previously, then we have $\psi(M_1) \cdot \dots \cdot \psi(M_{\text{blocks}(M)}) = D_1 \cdot D_2 \cdot \dots \cdot D_{\text{ret}(D)}$. Since each D_i is also not the concatenation of smaller Dyck paths, we must have $\text{blocks}(M) = \text{ret}(D)$ and $\psi(M_i) = D_i$ for all $i \in [\text{blocks}(M)]$. It follows that $\text{blockweight}_i(M) = \text{semilen}_i(D)$. $\square$

Although Theorem 5.14 is stated for the general case of Fishburn matrices under the surjection ψ , it produces an immediate equidistribution result when restricted to $\text{FMatrices}_n^{(\text{III})}$ (resp. $\text{FMatrices}_n^{(\text{IV})}$).

Remark 5.15. Since the bijection $\phi_1 : \mathcal{D}_n \rightarrow \text{FMatrices}_n^{(\text{III})}$ (resp. $\phi_2 : \mathcal{D}_n \rightarrow \text{FMatrices}_n^{(\text{IV})}$) is a right inverse of the surjection ψ , Theorem 5.14(i)-(v) also holds for $D \in \mathcal{D}_n$ and $M = \phi_1(D)$ (resp. $M = \phi_2(D)$). It follows that each pair of statistics given in Theorem 5.14 is in fact equidistributed on the sets $\mathcal{D}_n$ and $\text{FMatrices}_n^{(\text{III})}$ (resp. $\text{FMatrices}_n^{(\text{IV})}$).

Each of the given statistics also has a significance for the interval order P represented by M , meaning that the equidistribution result naturally extends to the sets $\text{FPosets}_n^{(\text{III})}$ (resp. $\text{FPosets}_n^{(\text{IV})}$). We have already seen that $\dim(M) = \text{mag}(P)$, the number of unique strict down-sets of P . The statistics $\text{rowsum}_i(M)$ and $\text{colsum}_i(M)$ are the cardinalities of $L_{i-1}(P)$ and $L_{i-1}(P^d)$ respectively, where P^d denotes the dual of P . In particular, $\text{rowsum}_1(M)$ and $\text{colsum}_{\dim(M)}(M)$ are the number of minimal and maximal elements in P respectively. The fact that the number of minimal elements of a series-parallel interval order (resp. semiorder) is equal to the length of the first run of up steps of the corresponding Dyck path, has already been observed in [22], along with the dual statement for maximal elements. The ordinal sum of posets, denoted by $\oplus$, also defines a decomposition $P = P_1 \oplus P_2 \cdot \dots \cdot \oplus P_k$ of P into interval orders which are not themselves the ordinal sum of smaller interval orders. If the direct sum decomposition of M is $M = M_1 \oplus M_2 \oplus \dots \oplus M_{\text{blocks}(M)}$, then M_i is the Fishburn matrix which represents P_i . It follows that $\text{blocks}(M) = k$, the number of posets in the ordinal sum decomposition of P , and that $\text{blockweight}_i(M)$ is the cardinality of P_i . Most of the statistics discussed have also been shown in [7, Theorem 12] to have combinatorial significance when translated to ascent sequences and Fishburn permutations under the classical bijections.

It is well known (see for example [12]) that the distribution of pea on $\mathcal{D}_n$ is given by the Narayana numbers [29, A001263]

$$N(n, k) = \frac{1}{n} \binom{n}{k} \binom{n}{k-1}.$$

It is also shown in [12] that the distributions of the statistics ret and uprun_1 (and $\text{rightrun}_{\text{pea}(D)}$ by symmetry) on $\mathcal{D}_n$ are given by

$$\frac{k}{2n-k} \binom{2n-k}{n},$$

which is the triangle [29, A033184]. It follows from the first return decomposition of Dyck paths that the distribution of semilen_1 on $\mathcal{D}_n$ is given by

$$C_{k-1} C_{n-k}$$

where C_i is the i th Catalan number [29, A078391].

6. CLASS V: $(3 + 1, \text{COVER-}N)$ -FREE INTERVAL ORDERS

In this section, we consider the more restricted class of Fishburn structures which correspond to interval orders which are $(3 + 1)$ -free and whose covering relation is N -free. We present characterizations of each of the associated Fishburn structures, and show that they are counted by the Fibonacci numbers F_{2n-1} [29, A001519].

Along with the condition (inherited from $\text{FMatrices}_n^{(\text{II})}$) that all diagonal entries are non-zero, elements of $\text{FMatrices}_n^{(\text{V})}$ are characterized by having no non-zero entries above the super diagonal.

Proposition 6.1. $M \in \text{FMatrices}_n^{(\text{V})}$ iff M is a non-negative integer matrix of weight n and dimension d satisfying

- (i) $m_{i,i} \neq 0$ for all $i \in [d]$ (i.e M is invertible) and
- (ii) $m_{i,j} = 0$ whenever $j \notin \{i, i+1\}$ (i.e. M is bidiagonal).

Proof. Recall that class V is the intersection of classes II and IV. Hence, by Proposition 4.6, it suffices to show that a Fishburn matrix $M \in \text{FMatrices}_n^{(\text{II})}$ is also an element of $\text{FMatrices}_n^{(\text{IV})}$ iff $m_{i,j} = 0$ whenever $j \notin \{i, i+1\}$. Or equivalently by Lemma 5.3, that $M \in \text{FMatrices}_n^{(\text{II})}$ is south-west-free iff $m_{i,j} = 0$ whenever $j \notin \{i, i+1\}$.

$\Rightarrow$: The matrix M is upper triangular so $m_{i,j} = 0$ for all $j < i$. Suppose for a contradiction that $m_{i,j}$ is non-zero for some $j > i+1$. By Proposition 4.6, we have that $m_{i+1,i+1}$ is non-zero, and so it is clear that (i, j) and $(i+1, i+1)$ are in a strictly south-west mutual position.

$\Leftarrow$: All non-zero entries are at most one row/column removed from the diagonal, so M cannot have two entry positions which are in a strictly south-west mutual position. $\square$

Example 6.2. The following matrix is an element of $\text{FMatrices}_{19}^{(\text{V})}$.

$$\begin{pmatrix} 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ & 1 & 3 & 0 & 0 & 0 & 0 \\ & & 1 & 2 & 0 & 0 & 0 \\ & & & 2 & 0 & 0 & 0 \\ & & & & 1 & 1 & 0 \\ & & & & & 4 & 0 \\ & & & & & & 1 \end{pmatrix}$$

Lemma 6.3. Let $M \in \text{FMatrices}_n^{(\text{V})}$ with $\dim(M) = d$. Then

$$\Phi_{MA}(M) = (0^{m_{1,1}}, 1^{m_{2,2}}, 0^{m_{1,2}}, 2^{m_{3,3}}, 1^{m_{2,3}}, 3^{m_{4,4}}, 2^{m_{3,4}}, \dots, (d-1)^{m_{d,d}}, (d-2)^{m_{d-1,d}}).$$

It follows that $a \in \mathcal{A}_n^{(\text{V})}$ iff a is of the form

$$a = (0^{m_{1,1}}, 1^{m_{2,2}}, 0^{m_{1,2}}, 2^{m_{3,3}}, 1^{m_{2,3}}, 3^{m_{4,4}}, 2^{m_{3,4}}, \dots, (d-1)^{m_{d,d}}, (d-2)^{m_{d-1,d}}),$$

for some $d \in [n]$, where the $m_{i,j}$ are non-negative integers and $m_{i,i} \neq 0$ for all $i \in [d]$.

Proof. Recall that class V is contained in class II. The result follows from Proposition 6.1 and Lemma 4.4. $\square$

The ascent sequences in this class allow for a characterization in terms of pattern avoidance which intuitively displays their containment in $\mathcal{A}_n^{(\text{II})} = \mathcal{A}_n(10|1)$. That is to say that they simply avoid one additional pattern.

Proposition 6.4. We have

$$\mathcal{A}_n^{(\text{V})} = \mathcal{A}_n(10|1, 120).$$

Proof. $\subseteq$: Let $a \in \mathcal{A}_n^{(\text{V})} \subseteq \mathcal{A}_n^{(\text{II})}$. Then a avoids $10|1$ by Proposition 4.8. Consulting the characterization of a in Lemma 6.3, is clear by inspection that a avoids 120.

$\supseteq$: We prove the claim by induction on n . The base case $n = 1$ is immediate. Let $a' \in \mathcal{A}_{n+1}(10|1, 120)$ and let a be the result of deleting the final entry a_{n+1} from a' . Clearly $a \in \mathcal{A}_n(10|1, 120)$, so by the induction hypothesis we have that $a \in \mathcal{A}_n^{(\text{V})}$, and hence by Proposition 6.3,

$$a = (0^{m_{1,1}}, 1^{m_{2,2}}, 0^{m_{1,2}}, 2^{m_{3,3}}, 1^{m_{2,3}}, \dots, (d-1)^{m_{d,d}}, (d-2)^{m_{d-1,d}})$$

for some $d \in [n]$, where $m_{i,i} \neq 0$ for all $i \in [d]$. We wish to prove that a' satisfies the same characterization. By counting ascents, we find that $\text{asc}(a) = d-1$, regardless of which $m_{i,j}$ are zero or non-zero, so $a_{n+1} \leq d$ since a' is an ascent sequence. If $d = 1$ then $a = (0^{m_{1,1}})$ so either $a' = (0^{m_{1,1}+1})$ or $a' = (0^{m_{1,1}}, 1)$ and we are done. If $d \geq 2$ then $d-2$ and $d-1$ occur in a in that order, so we must have $a_{n+1} \geq d-2$ in order to avoid 120. We consider three cases.

(i) $a_{n+1} = d$: we have

$$a' = (0^{m_{1,1}}, 1^{m_{2,2}}, 0^{m_{1,2}}, 2^{m_{3,3}}, 1^{m_{2,3}}, \dots, (d-1)^{m_{d,d}}, (d-2)^{m_{d-1,d}}, d)$$

and we are done.

(ii) $a_{n+1} = d-1$: we must have $m_{d-1,d} = 0$ in order to avoid $10|1$, so we have

$$a' = (0^{m_{1,1}}, 1^{m_{2,2}}, 0^{m_{1,2}}, 2^{m_{3,3}}, 1^{m_{2,3}}, \dots, (d-1)^{m_{d,d}+1})$$

and we are done.

(iii) $a_{n+1} = d-2$: we have

$$a' = (0^{m_{1,1}}, 1^{m_{2,2}}, 0^{m_{1,2}}, 2^{m_{3,3}}, 1^{m_{2,3}}, \dots, (d-1)^{m_{d,d}}, (d-2)^{m_{d-1,d}+1}).$$

and we are done. $\square$

The Fishburn permutations in this class may also be characterized by the avoidance of the same patterns as in class II, along with one additional pattern.

Proposition 6.5. We have

$$\mathcal{F}_n^{(\text{V})} = S_n(2|31, \bar{2}31, 4123).$$

Proof. By Proposition 6.4, the statement is the same as $\Phi_{AS}(\mathcal{A}_n(10|1, 120)) = S_n(2|31, \bar{2}31, 4123)$. Recall from Section 4 that $\Phi_{AS}(\mathcal{A}_n(10|1)) = S_n(2|31, \bar{2}31)$ and that $\mathcal{A}_n(10|1)$ is the set of self-modified ascent sequences of length n , so it suffices to prove that if $a = (a_1, \dots, a_n)$ is a self-modified ascent sequences of length n , then a contains 120 iff $\Phi_{AS}(a) := \pi = \pi_1 \dots \pi_n$ contains 4123.

$\implies$: Suppose that we have an occurrence of 120 in a , that is $i_1 < i_2 < i_3$ with $a_{i_3} < a_{i_1} < a_{i_2}$. Recall the characterization of self-modified ascent sequences given in Lemma 4.4. It is clear that, due to their relative sizes and positions, a_{i_1} and a_{i_2} must be part of two different weakly descending runs of a . Let a_{j_1} and a_{j_2} be the respective maximum (i.e. leftmost) terms of each of these weakly descending runs, and note that $j_1 < j_2$. Since $a_{i_1} > 0$, there must also exist another weakly descending run immediately to the left of a_{j_1} . Let a_{j_0} be the maximum term of this weakly descending run. Consulting again the characterization of a given

in Lemma 4.4, we find that $a_{j_1} = a_{j_0} + 1$. We also find that $a_{j_1} = \text{asc}((a_1, \dots, a_{j_1-1})) + 1$ and $a_{j_2} = \text{asc}((a_1, \dots, a_{j_2-1})) + 1$. Let us now consider the recursive process by which we construct π from a . Note that $j_0 < j_1 < j_2 < i_3$. Out of the four terms, j_0 is inserted first since it is the smallest. Eventually, j_1 is inserted into the active site labelled a_{j_1} , which is to the right of j_0 by Lemma 2.12. Similarly j_2 is inserted to the right of j_1 . Lastly, we eventually insert i_3 into the active site labelled a_{i_3} . But $a_{i_3} < a_{i_1} \leq a_{j_1} = a_{j_0} + 1$ so $a_{i_3} \leq a_{j_0}$. By Lemma 4.11, the active site labelled a_{j_0} is somewhere to the left of j_0 , and thus i_3 must be inserted to the left of j_0 . Now $i_3 j_0 j_1 j_2$ forms an occurrence of 4123 in π .

$\Leftarrow$: Suppose that we have an occurrence $\pi_{i_1} \pi_{i_2} \pi_{i_3} \pi_{i_4}$ of 4123 in π , that is, indices $i_1 < i_2 < i_3 < i_4$ such that $\pi_{i_2} < \pi_{i_3} < \pi_{i_4} < \pi_{i_1}$. Let us again consider the recursive process by which π is constructed from a . Out of the four terms, we insert π_{i_2} first, since it is the smallest. Eventually π_{i_3} is inserted somewhere to the right of π_{i_2} since $i_2 < i_3$. But the active site labelled $a_{\pi_{i_2}}$ is somewhere to the left of π_{i_2} by Lemma 4.11, so we must have that $a_{\pi_{i_2}} < a_{\pi_{i_3}}$. Similarly we have that $a_{\pi_{i_3}} < a_{\pi_{i_4}}$. When we insert π_{i_1} , it is somewhere to the left of π_{i_2} , so it follows from Lemma 4.11 that $a_{\pi_{i_1}} \leq a_{\pi_{i_2}}$. Now $a_{\pi_{i_1}} < a_{\pi_{i_3}} < a_{\pi_{i_4}}$ so $(a_{\pi_{i_1}}, a_{\pi_{i_3}}, a_{\pi_{i_4}})$ forms an occurrence of 120 in a . $\square$

Proposition 6.6. We have $|\mathcal{A}_n^{(V)}| = \text{Fib}_{2n-1}$, where Fib_n is the n th Fibonacci number.

Proof. From [26, Theorem 12.4] we have the classic identity

$$\sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n-k}{k} = \text{Fib}_{n+1}.$$

By changing variables $n \mapsto 2n - 2$ followed by $k \mapsto n - 1 - k$ we obtain

$$\text{Fib}_{2n-1} = \sum_{k=0}^n \binom{n-1+k}{n-1-k} = \sum_{k=0}^n \binom{n-1+k}{2k}.$$

Since, by Proposition 8.9, $\binom{n-1+k}{2k}$ is the number of elements of $\mathcal{A}_n^{(V)}$ with k ascents the result follows. Note that this is the sequence [29, A001519]. $\square$

7. CLASS VI: $(3 + 1, N)$ -FREE INTERVAL ORDERS

In this section, we further restrict our attention to the class of Fishburn structures which correspond to interval orders which are N -free and $(3 + 1)$ -free. We present characterizations of each of the associated Fishburn structures, and give an enumeration of this class.

Let us first consider the Fishburn matrices. Along with the conditions inherited from $\text{FMatrices}_n^{(V)}$, elements of $\text{FMatrices}_n^{(VI)}$ are characterized by the property that no two entries which are “adjacent” on the superdiagonal are both positive.

Proposition 7.1. $M \in \text{FMatrices}_n^{(VI)}$ iff M is a non-negative integer matrix of weight n and dimension d satisfying

- (i) $m_{i,i} \neq 0$ for all $i \in [d]$ (i.e M is invertible),
- (ii) $m_{i,j} = 0$ whenever $j \notin \{i, i + 1\}$ (i.e. M is bidiagonal) and
- (iii) $m_{i,i+1} m_{i+1,i+2} = 0$ for all $i \in [d - 2]$.

Proof. Recall that class VI is the intersection of classes III and V. Hence, by Proposition 6.1, it suffices to show that a Fishburn matrix $M \in \text{FMatrices}_n^{(V)}$ is also an element of $\text{FMatrices}_n^{(III)}$ iff $m_{i,i+1} m_{i+1,i+2} = 0$ for all $i \in [d - 2]$. Or equivalently by Lemma 5.3, that $M \in \text{FMatrices}_n^{(V)}$ is north-west-free iff $m_{i,i+1} m_{i+1,i+2} = 0$ for all $i \in [d - 2]$.

$\Rightarrow$ by contrapositive: If $m_{i,i+1}$ and $m_{i+1,i+2}$ are both non-zero for some $i \in [d - 2]$, then $(i, i + 1)$ and $(i + 1, i + 2)$ are in a strictly north-west mutual position.

$\Leftarrow$: Suppose that $m_{i,j}$ and $m_{i',j'}$ are non-zero entries with $i < i'$ and $j < j'$. We wish to show that we cannot have $i' \leq j$, so that (i, j) and (i', j') are not in a strictly north-west mutual position. If either entry is on the diagonal, that is $i = j$ or $i' = j'$, then $j < i'$ and

we are done. Otherwise, both entries must be on the superdiagonal by Proposition 6.1, that is $j = i + 1$ and $j' = i' + 1$. Now we have $m_{i,i+1}m_{i',i'+1} \neq 0$ so we cannot have $i' = i + 1$. Hence $j = i + 1 < i'$. $\square$

Example 7.2. The following matrix is an element of $\text{FMatrices}_{16}^{(\text{VI})}$.

$$\begin{pmatrix} 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ & 1 & 0 & 0 & 0 & 0 & 0 \\ & & 1 & 2 & 0 & 0 & 0 \\ & & & 2 & 0 & 0 & 0 \\ & & & & 1 & 0 & 0 \\ & & & & & 4 & 1 \\ & & & & & & 1 \end{pmatrix}$$

The ascent sequences of this class are characterized by the avoidance of two patterns, as are the Fishburn permutations.

Proposition 7.3. We have

$$\mathcal{A}_n^{(\text{VI})} = \mathcal{A}_n(101, 120).$$

Proof. Recall that class VI is the intersection of classes III and V. So

$$\begin{aligned} \mathcal{A}_n^{(\text{VI})} &= \mathcal{A}_n^{(\text{III})} \cap \mathcal{A}_n^{(\text{V})} \\ &= \mathcal{A}_n(101) \cap \mathcal{A}_n(10|1, 120), && \text{by Propositions 5.4 and 6.4} \\ &= \mathcal{A}_n(101, 120), && \text{by pattern containment.} \end{aligned} \quad \square$$

This settles the open Problem 6.3 of [16], which asks for a description of the ascent sequences which correspond to series-parallel semiorders.

In [1, Proposition 14] Baxter and Pudwell showed that $\mathcal{A}_n(101, 120)$ and $S_n(231, 4123)$ are in bijection because they have the same enumeration. It turns out that Φ_{AS} serves as an explicit bijection between these restricted sets.

Proposition 7.4. We have

$$\mathcal{F}_n^{(\text{VI})} = S_n(231, 4123).$$

Proof. Recall that class VI is the intersection of classes III and V. So

$$\begin{aligned} \mathcal{F}_n^{(\text{VI})} &= \mathcal{F}_n^{(\text{III})} \cap \mathcal{F}_n^{(\text{V})} \\ &= S_n(231) \cap S_n(2|31, \bar{2}31, 4123), && \text{by Propositions 5.4 and 6.5} \\ &= S_n(231, 4123), && \text{by pattern containment.} \end{aligned} \quad \square$$

The ordinary generating function for this class is

$$\sum_{n \geq 0} |S_n(231, 4123)|z^n = \frac{(z-1)^3}{(3z^3 - 5z^2 + 4z - 1)},$$

which is derived in [1, Prop. 14]. The corresponding counting sequence is [29, A116703].

8. ASCENT POLYNOMIALS AND EULERIAN DISTRIBUTIONS

In this section we consider the distribution of the ascent statistic on the subsets of ascent sequences that correspond to classes I through VI. We also consider the distribution of the descent statistic (also known as the Eulerian distribution) on the six subsets of Fishburn permutations that correspond to classes I through VI. It turns out that the two are closely related to one another in just over half of the cases and these are summarized in Table 3. Another aim of this section is to consider the number sequences from the resulting distributions with respect to log-concavity and unimodality, and this is summarized in Table 4.

As we have seen in Proposition 2.10, the dimension of a given Fishburn matrix (or equivalently the magnitude of the interval order which it represents) is one more than the number of ascents of the corresponding ascent sequence. Thus the generating function of the dimension statistic on Fishburn matrices for each of the six classes equals that of the corresponding class of ascent sequences.

Given $R \in \{I, II, III, IV, *IV, V, VI\}$, define

$$A_n^{(R)}(q) := \sum_{a \in \mathcal{A}_n^{(R)}} q^{\text{asc}(a)} \quad \text{and} \quad F_n^{(R)}(q) := \sum_{\pi \in \mathcal{F}_n^{(R)}} q^{\text{des}(\pi)}.$$

In addition to this, let us define two generating functions for the sequences of these polynomials:

$$A^{(R)}(q, z) := \sum_{n \geq 0} A_n^{(R)}(q) z^n \quad \text{and} \quad F^{(R)}(q, z) := \sum_{n \geq 0} F_n^{(R)}(q) z^n.$$

The tables for $A_n^{(R)}(q)$ and $F_n^{(R)}(q)$ for some small values of n are illustrated in Table 2.

Proposition 8.1. [7, Prop. 15] We have

$$A^{(I)}(q, z) = \sum_{k \geq 1} \frac{(1-q)q^{k-1}(1-z)^k}{(q-(q-1)(1-z)^k) \prod_{i=1}^k (q-(q-1)(1-z)^i)}.$$

The coefficients of the first few polynomials $A_n^{(I)}(q)$ and $F_n^{(I)}(q)$ are given in Table 2 and are sequence [29, A137251].

Proposition 8.2. For all $n \geq 1$ we have $A_n^{(I)}(q) = F_n^{(I)}(q)$.

Proof. Proposition 2.18 (due to [20, Theorem 12]) states that the descents of a Fishburn permutation π are exactly the ascents of the corresponding ascent sequence $\Psi_{SA}(\pi)$, so des and asc are equidistributed on the respective sets $\mathcal{F}_n^{(I)} = \mathcal{F}_n$ and $\mathcal{A}_n^{(I)} = \mathcal{A}_n$. $\square$

In other words, the ascent statistic on the set of ascent sequences of length n and the descent statistic on the set of Fishburn permutations of length n are equidistributed. In fact it seems empirically that a stronger result holds, namely that $\text{asc}(\cdot)$ and $\text{des}(\Phi_{AS}(\cdot))$ have a joint symmetric distribution on $\mathcal{A}_n$. Table 5 illustrates this for $n = 8$. This leads us to the following conjecture.

Conjecture 8.3. For all $n \geq 1$,

$$\sum_{a \in \mathcal{A}_n} q^{\text{asc}(a)} w^{\text{des}(\Phi_{AS}(a))} = \sum_{a \in \mathcal{A}_n} w^{\text{asc}(a)} q^{\text{des}(\Phi_{AS}(a))}$$

It is not true that the sequence of coefficients of $A_n^{(I)}(q)$ are log-concave. The smallest counterexample is for $n = 24$. The polynomial $A_{24}^{(I)}(q) = 1 + 276q + 76406q^2 + \dots$ and $276^2 - 76406 < 0$. However it does seem that the sequence of coefficients of $A_n^{(I)}(q)$ is unimodal, and we have verified this up to $n = 300$.

Conjecture 8.4. The sequence of coefficients of $A_n^{(I)}(q)$ is unimodal.

For the next class, a closed form for the coefficients was already derived in [7].

Proposition 8.5. For $n \geq 1$,

$$[q^k] A_n^{(II)}(q) = \binom{n-1 + \binom{k+1}{2}}{n-1-k}.$$

Proof. The claim is equivalent to [7, Proposition 11], recalling that $\Phi_{AS}(\mathcal{A}_n^{(II)}) = S_n(3\bar{1}52\bar{4})$ and that Φ_{AS} preserves the ascent statistic in this class by Proposition 4.9. $\square$

$A_1^{(I)}(q) = 1$ $A_2^{(I)}(q) = 1 + q$ $A_3^{(I)}(q) = 1 + 3q + q^2$ $A_4^{(I)}(q) = 1 + 6q + 7q^2 + q^3$ $A_5^{(I)}(q) = 1 + 10q + 26q^2 + 15q^3 + q^4$ $A_6^{(I)}(q) = 1 + 15q + 71q^2 + 98q^3 + 31q^4 + q^5$	$F_1^{(I)}(q) = 1$ $F_2^{(I)}(q) = 1 + q$ $F_3^{(I)}(q) = 1 + 3q + q^2$ $F_4^{(I)}(q) = 1 + 6q + 7q^2 + q^3$ $F_5^{(I)}(q) = 1 + 10q + 26q^2 + 15q^3 + q^4$ $F_6^{(I)}(q) = 1 + 15q + 71q^2 + 98q^3 + 31q^4 + q^5$
$A_1^{(II)}(q) = 1$ $A_2^{(II)}(q) = 1 + q$ $A_3^{(II)}(q) = 1 + 3q + q^2$ $A_4^{(II)}(q) = 1 + 6q + 6q^2 + q^3$ $A_5^{(II)}(q) = 1 + 10q + 21q^2 + 10q^3 + q^4$ $A_6^{(II)}(q) = 1 + 15q + 56q^2 + 55q^3 + 15q^4 + q^5$	$F_1^{(II)}(q) = 1$ $F_2^{(II)}(q) = 1 + q$ $F_3^{(II)}(q) = 1 + 3q + q^2$ $F_4^{(II)}(q) = 1 + 6q + 6q^2 + q^3$ $F_5^{(II)}(q) = 1 + 10q + 21q^2 + 10q^3 + q^4$ $F_6^{(II)}(q) = 1 + 15q + 55q^2 + 56q^3 + 15q^4 + q^5$
$A_1^{(III)}(q) = 1$ $A_2^{(III)}(q) = 1 + q$ $A_3^{(III)}(q) = 1 + 3q + q^2$ $A_4^{(III)}(q) = 1 + 6q + 6q^2 + q^3$ $A_5^{(III)}(q) = 1 + 10q + 20q^2 + 10q^3 + q^4$ $A_6^{(III)}(q) = 1 + 15q + 50q^2 + 50q^3 + 15q^4 + q^5$	$F_1^{(III)}(q) = 1$ $F_2^{(III)}(q) = 1 + q$ $F_3^{(III)}(q) = 1 + 3q + q^2$ $F_4^{(III)}(q) = 1 + 6q + 6q^2 + q^3$ $F_5^{(III)}(q) = 1 + 10q + 20q^2 + 10q^3 + q^4$ $F_6^{(III)}(q) = 1 + 15q + 50q^2 + 50q^3 + 15q^4 + q^5$
$A_1^{(IV)}(q) = 1$ $A_2^{(IV)}(q) = 1 + q$ $A_3^{(IV)}(q) = 1 + 3q + q^2$ $A_4^{(IV)}(q) = 1 + 6q + 6q^2 + q^3$ $A_5^{(IV)}(q) = 1 + 10q + 20q^2 + 10q^3 + q^4$ $A_6^{(IV)}(q) = 1 + 15q + 50q^2 + 50q^3 + 15q^4 + q^5$	$F_1^{(IV)}(q) = 1$ $F_2^{(IV)}(q) = 1 + q$ $F_3^{(IV)}(q) = 1 + 3q + q^2$ $F_4^{(IV)}(q) = 1 + 5q + 7q^2 + q^3$ $F_5^{(IV)}(q) = 1 + 7q + 18q^2 + 15q^3 + q^4$ $F_6^{(IV)}(q) = 1 + 9q + 35q^2 + 55q^3 + 31q^4 + q^5$
$A_1^{(*IV)}(q) = 1$ $A_2^{(*IV)}(q) = 1 + q$ $A_3^{(*IV)}(q) = 1 + 3q + q^2$ $A_4^{(*IV)}(q) = 1 + 6q + 6q^2 + q^3$ $A_5^{(*IV)}(q) = 1 + 10q + 20q^2 + 10q^3 + q^4$ $A_6^{(*IV)}(q) = 1 + 15q + 50q^2 + 50q^3 + 15q^4 + q^5$	$F_1^{(*IV)}(q) = 1$ $F_2^{(*IV)}(q) = 1 + q$ $F_3^{(*IV)}(q) = 1 + 3q + q^2$ $F_4^{(*IV)}(q) = 1 + 5q + 7q^2 + q^3$ $F_5^{(*IV)}(q) = 1 + 7q + 18q^2 + 15q^3 + q^4$ $F_6^{(*IV)}(q) = 1 + 9q + 35q^2 + 55q^3 + 31q^4 + q^5$
$A_1^{(V)}(q) = 1$ $A_2^{(V)}(q) = 1 + q$ $A_3^{(V)}(q) = 1 + 3q + q^2$ $A_4^{(V)}(q) = 1 + 6q + 5q^2 + q^3$ $A_5^{(V)}(q) = 1 + 10q + 15q^2 + 7q^3 + q^4$ $A_6^{(V)}(q) = 1 + 15q + 35q^2 + 28q^3 + 9q^4 + q^5$	$F_1^{(V)}(q) = 1$ $F_2^{(V)}(q) = 1 + q$ $F_3^{(V)}(q) = 1 + 3q + q^2$ $F_4^{(V)}(q) = 1 + 5q + 6q^2 + q^3$ $F_5^{(V)}(q) = 1 + 7q + 15q^2 + 10q^3 + q^4$ $F_6^{(V)}(q) = 1 + 9q + 28q^2 + 35q^3 + 15q^4 + q^5$
$A_1^{(VI)}(q) = 1$ $A_2^{(VI)}(q) = 1 + q$ $A_3^{(VI)}(q) = 1 + 3q + q^2$ $A_4^{(VI)}(q) = 1 + 6q + 5q^2 + q^3$ $A_5^{(VI)}(q) = 1 + 10q + 14q^2 + 7q^3 + q^4$ $A_6^{(VI)}(q) = 1 + 15q + 30q^2 + 26q^3 + 9q^4 + q^5$	$F_1^{(VI)}(q) = 1$ $F_2^{(VI)}(q) = 1 + q$ $F_3^{(VI)}(q) = 1 + 3q + q^2$ $F_4^{(VI)}(q) = 1 + 5q + 6q^2 + q^3$ $F_5^{(VI)}(q) = 1 + 7q + 14q^2 + 10q^3 + q^4$ $F_6^{(VI)}(q) = 1 + 9q + 26q^2 + 30q^3 + 15q^4 + q^5$

TABLE 2. $A_n^{(R)}(q)$ and $F_n^{(R)}(q)$ for all $n \in [1, 6]$.

R	$A_n^{(R)}$ coefficients	$F_n^{(R)}$ coefficients
I	A137251 (Prop. 8.1)	A137251 (Prop. 8.2)
II	A098568 (Prop. 8.5)	Reverse A098568 (Prop. 8.6)
III	A001263 Narayana (Prop. 8.8)	A001263 Narayana (Prop. 8.8)
IV	A001263 Narayana (Prop. 8.8)	Unknown
*IV	A001263 Narayana (Prop. 8.8)	Unknown
V	A085478 (Prop. 8.9)	Reverse A085478 (Prop. 8.6)
VI	New (Prop. 8.14)	New (Prop. 8.11) Reverse of $A_n^{(VI)}$ (Prop. 8.6)

TABLE 3. Coefficient sequences of the polynomials considered in Section 8.

R	Coefficients of $A_n^{(R)}$	Coefficients of $F_n^{(R)}$
I	Conjectured to be unimodal in Conj. 8.4	Conjectured to be unimodal in Conj. 8.4
II	Not log-concave, Conjectured to be unimodal in Conj. 8.7	Not log-concave, Conjectured to be unimodal in Conj. 8.7
III	Log-concave (since Narayana)	Log-concave (since Narayana)
IV	Log-concave (since Narayana)	Unknown
*IV	Log-concave (since Narayana)	Unknown
V	Log-concave (Prop. 8.10)	Log-concave (Prop. 8.10)
VI	Not log-concave Conjectured to be unimodal in Conj. 8.13 Partial unimodality proven in Prop. 8.15	Not log-concave Conjectured to be unimodal in Conj. 8.13 Partial unimodality follows from Prop. 8.15

TABLE 4. Unimodal/log-concave properties of the polynomial coefficient sequences. We use the phrase ‘since Narayana’ to mean they are so by virtue of the coefficients being the Narayana numbers.

$k \setminus \ell$	0	1	2	3	4	5	6	7
0	0	0	0	0	0	0	0	1
1	0	0	0	0	0	0	28	0
2	0	0	0	0	0	252	70	0
3	0	0	0	0	760	645	28	0
4	0	0	0	760	1291	233	1	0
5	0	0	252	645	233	8	0	0
6	0	28	70	28	1	0	0	0
7	1	0	0	0	0	0	0	0

TABLE 5. The distribution of $|\{a \in \mathcal{A}_n : \text{asc}(a) = k, \text{des}(\Phi_{AS}(a)) = \ell\}|$ for $n = 8$.

The coefficients of $A_n^{(II)}(q)$ are the sequence [29, A098568], which were also found by Khamis [24] to count the number of interval orders whose covering relation is N -free and whose *height* is k . The *height* of a poset P is the maximum cardinality of a chain in P . Recall from Section 4 that an interval order whose covering relation is N -free may equivalently be characterized by the existence of a chain whose cardinality is equal to the magnitude of the interval order. Clearly the height of such a poset is equal to its magnitude. But the magnitude of an interval order is one less than the ascents of the corresponding ascent sequence, which explains this connection.

Proposition 8.6. For $R \in \{\text{II}, \text{III}, \text{V}, \text{VI}\}$ we have

$$A_n^{(R)}(q) = q^{n-1} F_n^{(R)}(1/q).$$

That is, the sequence of coefficients of $F_n^{(R)}(q)$ is the reverse of that of $A_n^{(R)}(q)$.

Proof. Recall that $A_n^{(\text{III})}(q)$, $A_n^{(\text{V})}(q)$, and $A_n^{(\text{VI})}(q)$ are all subsets of $A_n^{(\text{II})}(q)$. Now it follows from Proposition 4.9 that $\text{asc}(a) = \text{asc}(\Phi_{AS}(a))$ for all $a \in \mathcal{A}_n^{(R)}$ when $R \in \{\text{II}, \text{III}, \text{V}, \text{VI}\}$. But $\text{asc}(\pi) = n - 1 - \text{des}(\pi)$ for any $\pi \in S_n$. It follows that the distribution of asc on $\mathcal{A}_n^{(R)}$ and the distribution of des on $\mathcal{F}_n^{(R)}$ are the reverse of one another. $\square$

The first few polynomials $A_n^{(\text{II})}(q)$ and $F_n^{(\text{II})}(q)$ are given in Table 2. The coefficient sequences fail to be log-concave for $n \geq 29$. Once again, the coefficients appear to be unimodal, and we have verified this up to $n = 1000$.

Conjecture 8.7. The sequence of coefficients of $A_n^{(\text{II})}(q)$ is unimodal.

The n th Narayana polynomial is

$$N_n(q) = \sum_{k=1}^n N(n, k) q^{k-1} \quad \text{where} \quad N(n, k) = \frac{1}{n} \binom{n}{k} \binom{n}{k-1}.$$

Note the choice of exponent $k - 1$ rather than k .

Proposition 8.8. For all $n \geq 1$ we have $A_n^{(\text{III})}(q) = F_n^{(\text{III})}(q) = A_n^{(\text{IV})}(q) = A_n^{(*\text{IV})}(q) = N_n(q)$.

Proof. We present one of the many ways to prove each of the equalities in this proposition.

- (i) $A_n^{(\text{III})}(q) = N_n(q)$: Let $M \in \text{FMatrices}_n^{(\text{III})}$ and $a = \Phi_{MA}(M)$. Recall that $\text{asc}(a) = \dim(M) + 1$. By Remark 5.15, the statistic $\dim$ on $\text{FMatrices}_n^{(\text{III})}$ and the statistic pea on $\mathcal{D}_n$ have the same distribution. As we have already seen, the Dyck path peak distribution is given by the Narayana numbers. The result follows.
- (ii) $A_n^{(\text{IV})}(q) = N_n(q)$: Analogous to (i).
- (iii) $F_n^{(\text{III})}(q) = N_n(q)$: We have $N(n, k) = N(n, n + 1 - k)$, meaning that $N_n(q)$ is palindromic. So the result follows from (i) and Proposition 8.6.
- (iv) $A_n^{(*\text{IV})}(q) = A_n^{(\text{IV})}(q)$: Follows immediately from Proposition 2.10. $\square$

In the recent preprint [28, Theorem 1.4], it is shown that the distributions of mag on $\text{FPosets}_n^{(\text{III})}$ and on $\text{FPosets}_n^{(\text{IV})}$ (as well as several other related distributions) are given by the Narayana numbers, which is analogous to part of the above result. The Narayana polynomials are known to be real-rooted [9, 8], and hence log-concave.

From Table 2 it appears that $F_n^{(\text{IV})}(q) = F_n^{(*\text{IV})}(q)$, but this is not the case and they begin to differ starting at $n = 8$.

Proposition 8.9. For all $n \geq 1$,

$$[q^k] A_n^{(\text{V})}(q) = \binom{n-1+k}{2k}.$$

Proof. $[q^k] A_n^{(\text{V})}(q)$ counts the number of elements of $\mathcal{A}_n^{(\text{V})}$ with k ascents, or equivalently by Proposition 2.10, the number of elements of $\text{FMatrices}_n^{(\text{V})}$ of dimension $k + 1$.

Recall the characterization of $\text{FMatrices}_n^{(\text{V})}$ given in Proposition 6.1. Of the total weight n of our matrix, $k + 1$ is accounted for by necessarily non-zero diagonal entries. To construct an element of $\text{FMatrices}_n^{(\text{V})}$ of dimension $k + 1$, we are left to distribute the remaining $n - k - 1$ weight amongst the $2k + 1$ diagonal and bidiagonal entries as we please. The number of ways of doing this is

$$\binom{(n-k-1) + (2k+1) - 1}{(2k+1) - 1} = \binom{n-1+k}{2k}.$$

□

The first few polynomials $A_n^{(V)}(q)$ and $F_n^{(V)}(q)$ are given in Table 2. The coefficients are the sequence [29, A085478].

Proposition 8.10. The sequences of coefficients of $A_n^{(V)}(q)$ and $F_n^{(V)}(q)$ are log-concave.

Proof. Define $a_k := \binom{n-1+k}{2k}$. It follows that

$$\begin{aligned} \frac{a_k^2}{a_{k-1}a_{k+1}} &= \frac{n-1+k}{n+k} \cdot \frac{n-k}{n-1-k} \cdot \frac{(2k+1)(2k+2)}{(2k-1)(2k)} \\ &= \frac{(n+k)(n-k) - n+k}{(n+k)(n-k) - n-k} \cdot \frac{(2k+1)(2k+2)}{(2k-1)(2k)}. \end{aligned}$$

For $k \in [n]$, both factors in the product are greater than or equal to 1. It is now easy to see that the sequence of coefficients of $F_n^{(V)}(q)$ is also log-concave, since it is simply the reverse of the sequence of coefficients of $A_n^{(V)}(q)$ by Proposition 8.6. □

Proposition 8.11. We have

$$F^{(VI)}(q, z) = \frac{q(1-qz)^3}{(1-qz)^3 - z + 2qz^2 - (q+q^2)z^3}$$

Lemma 8.12. For all $n \geq 1$, let

$$H_n(q) := \sum_{\pi \in S_n(231, 123)} q^{1+\text{des}(\pi)},$$

set $H_0(q) := 1$ and define $H(q, z) := \sum_{n \geq 0} H_n(q)z^n$. Then

$$H(q, z) = \frac{(1-qz)^2 + z^2q}{(1-qz)^3}.$$

Proof. It is shown in [32, Proposition 11] that the number of (231, 123)-avoiding permutations of length n is $1 + \binom{n}{2}$. Taking the reverse of the patterns (132, 321) gives the patterns (231, 123), meaning that this is also the number of (231, 123)-avoiding permutations of length n .

We claim that an element $\pi \in S_n(231, 123)$ cannot have more than one ascent. Suppose for a contradiction that we have $i_1 < i_2 \leq i_3 < i_4$ such that $\pi_{i_1} < \pi_{i_2}$ and $\pi_{i_3} < \pi_{i_4}$. Since π avoids 123, we must have $\pi_{i_1} > \pi_{i_3}$ and $\pi_{i_2} > \pi_{i_3}$. But now $\pi_{i_1}\pi_{i_2}\pi_{i_3}$ is an occurrence of 231. There is exactly one permutation with no ascents, and the rest have one ascent, so we have that

$$H_n(q) = q^n + \binom{n}{2}q^{n-1}$$

for $n \geq 1$. Multiplying by z^n and summing over $n \geq 1$:

$$\begin{aligned} H(q, z) &= 1 + \sum_{n \geq 1} H_n(q)z^n = 1 + \sum_{n \geq 1} q^n z^n + qz^2 \sum_{n \geq 2} \binom{n}{2} q^{n-2} z^{n-2} \\ &= \frac{1}{1-qz} + qz^2 \sum_{n \geq 0} \binom{n+2}{2} q^n z^n. \end{aligned}$$

We apply [34, Equation (2.5.7)] with $k = 2$ to find that

$$H(q, z) = \frac{1}{1-qz} + \frac{qz^2}{(1-qz)^3},$$

from which the result follows. □

Proof of Proposition 8.11. Let $\pi \in \mathcal{F}_n^{(VI)} = S_n(231, 4123)$ and write $\pi = LnR$. In the case that R is empty, we have $L \in S_{n-1}(231, 4123)$ and $\text{des}(L) = \text{des}(\pi)$. Otherwise, every term of L must be less than every term of R , because if not, π would have an occurrence of 231 consisting of some term of L , n and some term of R . Hence $L \in S_k(231, 4123)$ for some $k \in [0, n-2]$ and

$R' := (R_1 - k, R_2 - k, \dots, R_{n-1-k} - k) \in S_{n-1-k}(231, 4123)$. In fact, $R' \in S_{n-1-k}(231, 123)$, since an occurrence of 123 in R would complete an occurrence of 4123 in π , the first term being n . Also note that we have $\text{des}(\pi) = \text{des}(L) + \text{des}(R) + 1$, and $\text{des}(R) = \text{des}(R')$. It follows that

$$\begin{aligned}
F_n^{(\text{VI})}(q) &= \sum_{k=0}^{n-1} \sum_{L \in S_k(231, 4123)} \sum_{R' \in S_{n-1-k}(231, 123)} q^{\text{des}(LnR)} \\
&= \sum_{k=0}^{n-1} \sum_{L \in S_k(231, 4123)} \sum_{R' \in S_{n-1-k}(231, 123)} q^{\text{des}(L) + \text{des}(R) + 1} \\
&= \sum_{k=0}^{n-1} \left(\sum_{L \in S_k(231, 4123)} q^{\text{des}(L)} \right) \left(\sum_{R' \in S_{n-1-k}(231, 123)} q^{\text{des}(R') + 1} \right) \\
&= \sum_{k=0}^{n-1} F_k^{(\text{VI})}(q) H_{n-1-k}(q).
\end{aligned}$$

Multiplying by z^n and summing over $n \geq 1$ we obtain

$$\begin{aligned}
F^{(\text{VI})}(q, z) - 1 &= \sum_{n \geq 1} F_n^{(\text{VI})}(q) z^n = \sum_{n \geq 1} \left(\sum_{k=0}^{n-1} F_k^{(\text{VI})}(q) H_{n-1-k}(q) \right) z^n \\
&= z \sum_{n \geq 0} \left(\sum_{k=0}^n F_k^{(\text{VI})}(q) H_{n-k}(q) \right) z^n \\
&= z F^{(\text{VI})}(q, z) H(q, z).
\end{aligned}$$

This yields

$$F^{(\text{VI})}(q, z) = \frac{1}{1 - zH(q, z)}.$$

The result follows after using Lemma 8.12 to substitute for $H(q, z)$ and simplifying. $\square$

The first few polynomials $A_n^{(\text{VI})}(q)$ and $F_n^{(\text{VI})}(q)$ are given in Table 2. The coefficients do not appear in the OEIS, and they fail to be log-concave for $n \geq 20$. As before, they still seem to be unimodal, and we have verified this up to $n = 1000$.

Conjecture 8.13. The sequence of coefficients of $A_n^{(\text{VI})}(q)$ is unimodal.

We proceed to present an explicit formula for the number of elements of $\mathcal{A}_n^{(\text{VI})}$ with k ascents, which we then use to prove a partial unimodality result to support our conjecture.

Proposition 8.14. For all $n \geq 1$,

$$[q^k]A_n^{(\text{VI})}(q) = \sum_i \binom{k+1-i}{i} \binom{n-1}{k+i}.$$

Proof. $[q^k]A_n^{(\text{VI})}(q)$ counts the number of elements of $\mathcal{A}_n^{(\text{VI})}$ with k ascents, or equivalently by Proposition 2.10, the number of elements of $\text{FMatrices}_n^{(\text{VI})}$ of dimension $k+1$.

Recall the characterization of $\text{FMatrices}_n^{(\text{VI})}$ given in Proposition 7.1. To construct an element of $\text{FMatrices}_n^{(\text{VI})}$ of dimension $k+1$, we first choose i of the k superdiagonal entries to be non-zero in such a way that no two chosen entries are adjacent. This is equivalent to choosing a non-adjacent i -subset of $[k]$, and the number of ways of doing this is

$$\binom{k+1-i}{i}.$$

Now, of the total weight n of our matrix, $k+1$ is accounted for by necessarily non-zero diagonal entries, and i is accounted for by necessarily non-zero superdiagonal entries, so we are

left to distribute the remaining $n - k - 1 - i$ weight amount the $k + 1 + i$ non-zero entries as we please. The number of ways of doing this is

$$\binom{(n - k - 1 - i) + (k + 1 + i) - 1}{(k + 1 + i) - 1} = \binom{n - 1}{k + i}.$$

We multiply together the two terms and sum over all i to obtain the result. $\square$

Proposition 8.15. For $k \leq \frac{n}{3}$ we have

$$[q^{k-1}]A_n^{(\text{VI})}(q) \leq [q^k]A_n^{(\text{VI})}(q),$$

and for $k \geq \frac{2n}{3}$ we have

$$[q^k]A_n^{(\text{VI})}(q) \geq [q^{k+1}]A_n^{(\text{VI})}(q).$$

Proof. We prove both parts of this proposition by considering the sum given in Proposition 8.14 termwise.

Let $k \leq \frac{n}{3}$. We wish to show that

$$\binom{k-i}{i} \binom{n-1}{k-1+i} \leq \binom{k+1-i}{i} \binom{n-1}{k+i}$$

for all i . The fact that $\binom{k-i}{i} \leq \binom{k+1-i}{i}$ is immediate. Also, $\binom{k-i}{i}$ is only non-zero for $0 \leq i \leq \lfloor \frac{k}{2} \rfloor$, so we need to show that $\binom{n-1}{k-1+i} \leq \binom{n-1}{k+i}$ only in this range. This now follows from the fact that $k+i \leq k + \lfloor \frac{k}{2} \rfloor \leq \frac{3k}{2} \leq \frac{n}{2}$.

Next, let $k \geq \frac{2n}{3}$. We wish to show that

$$\binom{k+2-i}{i} \binom{n-1}{k+1+i} \leq \binom{k+1-i}{i} \binom{n-1}{k+i}$$

for all i . The left side of the inequality is only non-zero for $0 \leq i \leq n - k - 2$, so we may restrict our attention to this range. Consider

$$\begin{aligned} & \binom{k+1-i}{i} \binom{n-1}{k+i} - \binom{k+2-i}{i} \binom{n-1}{k+1+i} \\ &= \binom{n-1}{k+1+i} \left(\binom{k+1-i}{i} \frac{k+1+i}{n-1-k-i} - \left(\binom{k+1-i}{i-1} + \binom{k+1-i}{i} \right) \right) \\ &= \binom{n-1}{k+1+i} \binom{k+1-i}{i} \left(\frac{k+1+i}{n-1-k-i} - \left(\frac{i}{k+2-2i} + 1 \right) \right) \end{aligned}$$

It is easily checked that the denominators of both fractions in this expression are positive as a consequence of the constraints $i \leq n - k - 2$ and $k \geq \frac{2n}{3}$. After simplifying, we find that the expression is non-negative if and only if

$$f(i) := -3i^2 + (n - k + 1)i + (2k^2 + 6k - kn - 2n + 4)$$

is non-negative. Since f is a concave quadratic in i , it suffices to prove that $f(0)$ and $f(n - k - 1)$ are both non-negative ($n - k - 1$ is easier to work with in this case than $n - k - 2$). We have

$$f(0) \geq 2k^2 + 6k - \frac{3k^2}{2} - 3k + 4 > 0$$

and, after simplifying,

$$f(n - k - 1) = -2n^2 + 3nk + 4n \geq -2n^2 + 2n^2 + 4n \geq 0. \quad \square$$

9. ASCENT-MAJOR INDEX AND MAJOR INDEX STATISTICS

The ascent major index and major index statistics are natural refinements of the ascent and descent statistics. We will extend two of the results from the previous sections to these refined statistics. First let us define the relevant polynomials. Given $R \in \{I, II, III, IV, *VI, V, VI\}$, define

$$M_n^{(R)}(q) := \sum_{a \in \mathcal{A}_n^{(R)}} q^{\text{amaj}(a)} \quad \text{and} \quad G_n^{(R)}(q) := \sum_{\pi \in \mathcal{F}_n^{(R)}} q^{\text{maj}(\pi)}.$$

Proposition 9.1. $M_n^{(I)}(q) = G_n^{(I)}(q)$.

Proof. Proposition 2.18 (due to [20, Theorem 12]) states that the descents of a Fishburn permutation π are exactly the ascents of the corresponding ascent sequence $\Psi_{SA}(\pi)$, so maj and amaj are equidistributed on the respective sets $\mathcal{F}_n$ and $\mathcal{A}_n$. $\square$

The first few polynomials $M_n^{(I)}(q)$ and $G_n^{(I)}(q)$ are listed in Table 6.

$M_1^{(I)}(q)$	$=$	$G_1^{(I)}(q)$	$=$	1
$M_2^{(I)}(q)$	$=$	$G_2^{(I)}(q)$	$=$	$1 + q$
$M_3^{(I)}(q)$	$=$	$G_3^{(I)}(q)$	$=$	$1 + 2q + q^2 + q^3$
$M_4^{(I)}(q)$	$=$	$G_4^{(I)}(q)$	$=$	$1 + 3q + 2q^2 + 4q^3 + 3q^4 + q^5 + q^6$
$M_5^{(I)}(q)$	$=$	$G_5^{(I)}(q)$	$=$	$1 + 4q + 3q^2 + 8q^3 + 9q^4 + 8q^5 + 7q^6 + 7q^7 + 4q^8 + q^9 + q^{10}$
$M_6^{(I)}(q)$	$=$	$G_6^{(I)}(q)$	$=$	$1 + 5q + 4q^2 + 13q^3 + 17q^4 + 20q^5 + 25q^6 + 28q^7 + 32q^8 + 22q^9 + 18q^{10} + 14q^{11} + 11q^{12} + 5q^{13} + q^{14} + q^{15}$
$M_1^{(III)}(q)$	$=$	$G_1^{(III)}(q)$	$=$	1
$M_2^{(III)}(q)$	$=$	$G_2^{(III)}(q)$	$=$	$1 + q$
$M_3^{(III)}(q)$	$=$	$G_3^{(III)}(q)$	$=$	$1 + 2q + q^2 + q^3$
$M_4^{(III)}(q)$	$=$	$G_4^{(III)}(q)$	$=$	$1 + 3q + 2q^2 + 4q^3 + 2q^4 + q^5 + q^6$
$M_5^{(III)}(q)$	$=$	$G_5^{(III)}(q)$	$=$	$1 + 4q + 3q^2 + 8q^3 + 6q^4 + 6q^5 + 6q^6 + 4q^7 + 2q^8 + q^9 + q^{10}$
$M_6^{(III)}(q)$	$=$	$G_6^{(III)}(q)$	$=$	$1 + 5q + 4q^2 + 13q^3 + 11q^4 + 14q^5 + 19q^6 + 15q^7 + 15q^8 + 10q^9 + 11q^{10} + 6q^{11} + 4q^{12} + 2q^{13} + q^{14} + q^{15}$

TABLE 6. Polynomials $M_n^{(I)}(q)$, $G_n^{(I)}(q)$, $M_n^{(III)}(q)$, and $G_n^{(III)}(q)$ for small values of n

Proposition 9.2. $M_n^{(III)}(q) = G_n^{(III)}(q)$.

Proof. By Proposition 5.4 and Proposition 5.5 we have that $\Psi_{SA}(\mathcal{F}_n^{(III)}) = \mathcal{A}_n^{(III)}$. Proposition 2.18 states that the descents of a Fishburn permutation π are exactly the ascents of the corresponding ascent sequence $\Psi_{SA}(\pi)$, so maj and amaj are equidistributed on the respective sets $\mathcal{A}_n^{(III)}$ and $\mathcal{F}_n^{(III)}$. $\square$

See Table 6 for the first few polynomials $M_n^{(III)}(q)$ and $G_n^{(III)}(q)$.

10. THE CHEN-YAN-ZHOU BIJECTION FOR SEMIORDERS

The problem of characterizing the set of ascent sequences which correspond to semiorders under the classical bijection Φ_{MA} appears to be quite difficult. It was considered in [23], where the authors characterize a certain subset of the semiorders which they call *hereditary semiorders*. Looking at Table 1, it certainly seems that 120-avoidance is in some way relevant to this class, but we have found it to be neither a necessary nor sufficient condition for belonging. Adding to the intrigue of the story is that the 120-avoiding ascent sequences have evaded enumeration at the time of writing. In an attempt to shed some light on this class, we consider instead the ascent sequences which correspond to semiorders under the bijection Ψ_{MA} of Chen et al [11].

We demonstrate that semiorders interact far more nicely with this bijection, and we settle an open problem of [23, Page 25]. We also prove a characterization of these ascent sequences.

First, let us describe the elements of $\text{FMatrices}_n^{(\text{IV})}$ in a way that makes them easier to visualize.

Remark 10.1. Let $M \in \text{FMatrices}_n^{(\text{IV})}$ and suppose that $m_{i,j}$ and $m_{i',j'}$ are non-zero entries of M . Lemma 5.3 guarantees that (without loss of generality) $i \leq i'$ and $j \leq j'$. This means that we can list the entry positions of the non-zero entries of M from $(1,1)$ to $(\dim(M), \dim(M))$ so that each entry position is “weakly south-east” of its predecessor in the list. Moreover, two consecutive entry positions in this list can differ in each of their coordinates by at most one, since otherwise a row or column would be left empty. We refer to this kind of list as an *uninterrupted staircase* from $(1,1)$ to $(\dim(M), \dim(M))$, and for a square matrix of non-negative integers, it is a necessary and sufficient condition for belonging to $\text{FMatrices}_n^{(\text{IV})}$.

Note that the uninterrupted staircase is just the list of the entry positions of the non-zero entries of M given in row-major order, or equivalently column-major order. In fact, the equivalence of row-major and column major-order in this context can be seen to be another characterization of the matrices in this set.

Example 10.2. Below are three elements of $\text{FMatrices}_n^{(\text{IV})}$.

$$A = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{pmatrix} \quad B = \begin{pmatrix} 3 & 2 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad C = \begin{pmatrix} 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 & 2 \end{pmatrix}$$

Let us investigate the interaction of each of the above matrices with the bijection Ψ_{MA} by finding $g(A)$, $g(B)$ and $g(C)$.

Rem1: We have $\text{mindex}(A) = 3$ and $\text{rowsum}_3(M) > 1$, so **Rem1** is invoked and

$$g(A) = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}.$$

It is not difficult to see in general that if $M \in \text{FMatrices}_n^{(\text{IV})}$ invokes **Rem1**, then (this will be relevant to prove Proposition 10.3) we must have

$$\text{mindex}(g(M)) = \begin{cases} \text{mindex}(M) + 1 & \text{if } m_{\text{mindex}(M), \dim(M)} = 1, \\ \text{mindex}(M) & \text{otherwise.} \end{cases}$$

Furthermore, the non-zero entries of $g(M)$ will still form an uninterrupted staircase from $(1,1)$ to $(\dim(M), \dim(M))$, and hence $g(M) \in \text{FMatrices}_{n-1}^{(\text{IV})}$.

Rem2: Next, $\text{mindex}(B) = \dim(B) = 4$ and $\text{rowsum}_4(B) = 1$, so **Rem2** is invoked and

$$g(B) = \begin{pmatrix} 3 & 2 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 3 \end{pmatrix}.$$

Again is not difficult to see in general that if $M \in \text{FMatrices}_n^{(\text{IV})}$ invokes **Rem2**, then the non-zero entries of $g(M)$ still form an uninterrupted staircase from $(1,1)$ to $(\dim(M) - 1, \dim(M) - 1)$, and hence $g(M) \in \text{FMatrices}_{n-1}^{(\text{IV})}$. This time we note the $\text{mindex}(g(M)) < \text{mindex}(M)$.

Rem3: Lastly, $\text{mindex}(C) = 4 < \dim(C)$ and $\text{rowsum}_4(B) = 1$, so **Rem2** is invoked and the result is

$$g(C) = \begin{pmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 3 & 2 \\ 0 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix}.$$

In general if $M \in \text{FMatrices}_n^{(\text{IV})}$ invokes **Rem3**, then the uninterrupted staircase from $(1, 1)$ to $(\dim(M), \dim(M))$ may be split into two parts: first the uninterrupted staircase from $(1, 1)$ to $(\text{mindex}(M)-1, \dim(M)-1)$, followed by the uninterrupted staircase from $(\text{mindex}(M), \dim(M))$ to $(\dim(M), \dim(M))$ (which is confined entirely to column $\dim(M)$). The rule **Rem3** moves all of these non-zero entries in column $\dim(M)$ to the left by one, into column $\dim(M)-1$. Then row $\text{mindex}(M)$ and column $\dim(M)$ are simultaneously deleted, and the result is an uninterrupted staircase of non-zero entries from $(1, 1)$ to $(\dim(M)-1, \dim(M)-1)$, so $g(M) \in \text{FMatrices}_{n-1}^{(\text{IV})}$. Note that like last time, we must have $\text{mindex}(g(M)) < \text{mindex}(M)$.

We come to the conclusion that if $M \in \text{FMatrices}_n^{(\text{IV})}$ then $g(M) \in \text{FMatrices}_{n-1}^{(\text{IV})}$. Now consider the definition

$$\Psi_{MA}(M) := (\text{mindex}(g^{n-1}(M))-1, \text{mindex}(g^{n-2}(M))-1, \dots, \text{mindex}(g(M))-1, \text{mindex}(M)-1).$$

Inductively, we have that $g^k(M) \in \text{FMatrices}_{n-k}^{(\text{IV})}$ for all $k \in [n-1]$. It follows that if $a' = \Psi_{MA}(M')$ is the leading initial subsequence of length k of the ascent sequence $\Psi_{MA}(M)$, then $M' \in \text{FMatrices}_k^{(\text{IV})}$. This settles an open problem of [23, Page 25], which asks for a bijection between ascent sequences and interval orders with the property that every initial subsequence of an ascent sequence corresponding to a semiorder is one that also corresponds to a semiorder itself.

We make use of the discussion in this section to characterize the elements of the set $\mathcal{A}_n^{(*\text{IV})}$ of ascent sequences which correspond to semiorders under the bijection Ψ_{MA} . We note that this characterization coincides with that of the *restricted ascent sequences* introduced in [25], where these sequences were shown to be enumerated by the Catalan numbers.

Proposition 10.3. Let $a = (a_1, \dots, a_n) \in \mathcal{A}_n$. Then $a \in \mathcal{A}_n^{(*\text{IV})}$ iff

$$a_i \geq \max\{a_1, \dots, a_{i-1}\} - 1$$

for all $i \in [n]$.

Proof. We argue inductively on n in both directions. The base case $n = 1$ holds since $\mathcal{A}_1 = \mathcal{A}_1^{(*\text{IV})} = \{(0)\}$, and the sequence (0) has no descents.

$\implies$: Let $a = (a_1, \dots, a_{n+1}) \in \mathcal{A}_{n+1}^{(*\text{IV})}$ and let $M \in \text{FMatrices}_{n+1}$ such that $\Psi_{MA}(M) = a$. Then $(a_1, \dots, a_n) \in \mathcal{A}_n^{(*\text{IV})}$ since $(a_1, \dots, a_n)$ is a leading initial subsequence of a , so by the induction hypothesis all that is left to show is that

$$a_{n+1} \geq \max\{a_1, \dots, a_n\} - 1.$$

If $a_{n+1} \geq a_n$ then we have $a_{n+1} \geq a_n \geq \max\{a_1, \dots, a_{n-1}\} - 1$ by the induction hypothesis, and we are done. So suppose instead that $a_{n+1} < a_n$, or equivalently $\text{mindex}(M) < \text{mindex}(g(M))$. By the discussion following Example 10.2, the only possibility is that M invokes **Rem1**, and in particular that $\text{mindex}(g(M)) = \text{mindex}(M) + 1$ and $[M]_{\text{mindex}(M), \dim(M)} = 1$. The fact that **Rem1** is invoked also implies that $\dim(g(M)) = \dim(M)$. Besides $[M]_{\text{mindex}(M)+1, \dim(M)}$, there are no other non-zero entries in row $\text{mindex}(M) + 1$ of M , since otherwise such an entry position would be in a strictly south-west mutual position with $(\text{mindex}(M), \dim(M))$. When **Rem1** is applied to M , the entry $[M]_{\text{mindex}(M), \dim(M)}$ is reduced from 1 to 0, but no other

entries are changed. So $[g(M)]_{\text{mindex}(M)+1, \text{dim}(M)}$ must still be the only non-zero entry in row $\text{mindex}(M) + 1 = \text{mindex}(g(M))$ of $g(M)$. Denote the value of this entry by k . Then we have

$$\text{mindex}(M) + 1 = \text{mindex}(g(M)) = \text{mindex}(g^2(M)) = \dots = \text{mindex}(g^k(M))$$

by repeated application of **Rem1**, and we end up with $[g^k(M)]_{\text{mindex}(g^k(M)), \text{dim}(g^k(M))} = 1$, which is still the only non-zero entry of row $\text{mindex}(g^k(M))$ of $g^k(M)$. Hence $\text{rowsum}_{\text{mindex}(g^k(M))} = 1$, so $g^k(M)$ does not invoke **Rem1**. Therefore $g^k(M)$ invokes **Rem2** or **Rem3**, and in either case we have seen that $\text{mindex}(g^{k+1}(M)) < \text{mindex}(g^k(M))$. So we have

$$a_{n+1} + 1 = a_n = a_{n-1} = \dots = a_{n-k+1} > a_{n-k}.$$

Now $a_{n+1} \geq a_{n-k} \geq \max\{a_1, \dots, a_{n-k-1}\} - 1$ by the induction hypothesis, and we also have $a_{n+1} = a_n - 1 = a_{n-1} - 1 = \dots = a_{n-k+1} - 1$, so the result follows.

$\Leftarrow$: Let $a = (a_1, \dots, a_{n+1}) \in \mathcal{A}_{n+1}$ such that

$$a_i \geq \max\{a_1, \dots, a_{i-1}\} - 1$$

for all $i \in [n+1]$. Also let $M \in \text{FMatrices}_{n+1}$ such that $\Psi_{MA}(M) = a$. It is not difficult to see that $\Psi_{MA}(g(M)) = (a_1, \dots, a_n)$. Note that $(a_1, \dots, a_n)$ must certainly also satisfy the aforementioned property of a , and so it follows from the induction hypothesis that $(a_1, \dots, a_n) \in \mathcal{A}_n^{(*IV)}$, and hence $g(M) \in \text{FMatrices}_n^{(IV)}$. We will recover M from $g(M)$ depending on the value of a_{n+1} , and show that in any case we have that $M \in \text{FMatrices}_{n+1}^{(IV)}$, from which it follows that $a \in \mathcal{A}_{n+1}^{(*IV)}$. Define $i := \text{mindex}(g(M))$ and $d := \text{dim}(g(M))$, then $g(M)$ is of the form

$$g(M) = \begin{pmatrix} & & & & & & 0 \\ & & & & & & 0 \\ & & & & & & \vdots \\ & & & & & & 0 \\ & & X & & & & x_{i,d} \\ 0 & 0 & 0 & 0 & \dots & 0 & x_{i+1,d} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 & x_{i+1,d} \end{pmatrix}.$$

In this $i \times d - 1$ submatrix X has an uninterrupted staircase from $(1, 1)$ to either $(i-1, d)$ or (i, d) .

(i) If $a_{n+1} = a_n$ then $\text{mindex}(M) = \text{mindex}(g(M))$, so M invokes **Rem1** and we have

$$M = \begin{pmatrix} & & & & & & 0 \\ & & & & & & 0 \\ & & & & & & \vdots \\ & & & & & & 0 \\ & & X & & & & x_{i,d} + 1 \\ 0 & 0 & 0 & 0 & \dots & 0 & x_{i+1,d} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 & x_{i+1,d} \end{pmatrix}.$$

The zero-non-zero pattern of M is the same as that of $g(M)$, so M still has an uninterrupted staircase from $(1, 1)$ to $(\text{dim}(M), \text{dim}(M))$. Hence $M \in \text{FMatrices}_{n+1}^{(IV)}$.

- (ii) If $a_{n+1} > a_n$ then $\text{mindex}(M) > \text{mindex}(g(M))$, so M invokes either Rem2 or Rem3 and we have

$$M = \begin{pmatrix} & & & & & & 0 & 0 \\ & & & & & & 0 & 0 \\ & & & & & & \vdots & \\ & & & X & & & 0 & 0 \\ & & & & & & x_{i,d} & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & x_{i+1,d} & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 & x_{j-1,d} & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & 1 \end{pmatrix} \quad \text{or} \quad M = \begin{pmatrix} & & & & & & 0 & 0 \\ & & & & & & 0 & 0 \\ & & & & & & \vdots & \\ & & & X & & & 0 & 0 \\ & & & & & & x_{i,d} & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & x_{i+1,d} & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 & x_{j-1,d} & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & x_{j,d} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & x_{d,d} \end{pmatrix},$$

where $j = \text{mindex}(M) = a_{n+1} + 1$. It is apparent that since X had an uninterrupted staircase, so too does M , and so $M \in \text{FMatrices}_{n+1}^{(\text{IV})}$.

- (iii) If $a_{n+1} < a_n$ then it follows from $a_{n+1} \geq \max\{a_1, \dots, a_n\} - 1$ that

$$a_{n+1} + 1 = a_n = a_{n-1} = \dots = a_{n-k+1} > a_{n-k},$$

or, equivalently

$$\text{mindex}(M) + 1 = \text{mindex}(g(M)) = \dots = \text{mindex}(g^k(M)) > \text{mindex}(g^{k+1}(M))$$

for some $k \in [n-1]$. We have $\text{mindex}(g(M)) = \text{mindex}(g^2(M)) = \dots = \text{mindex}(g^k(M))$, so $g^k(M)$ is obtained from $g(M)$ by $k-1$ repeated applications of Rem1. We also have $\text{mindex}(g^k(M)) \neq \text{mindex}(g^{k+1}(M))$, so we must have $[g^k(M)]_{i,d} = 1$ and it follows that $[g(M)]_{i,d} = k$. We can also conclude that $[g(M)]_{i,d-1} = [g^k(M)]_{i,d-1} = 0$, since otherwise $g^k(M)$ would invoke Rem1 and we would have $\text{mindex}(g^{k+1}(M)) = \text{mindex}(g^k(M)) + 1$, contradicting $\text{mindex}(g^k(M)) > \text{mindex}(g^{k+1}(M))$. Now

$$g(M) = \begin{pmatrix} & & & & & & 0 \\ & & & & & & 0 \\ & & & & & & \vdots \\ & & & X & & & 0 \\ & & & & & & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & x_{i,d} = k \\ 0 & 0 & 0 & 0 & \dots & 0 & x_{i+1,d} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 & x_{i+1,d} \end{pmatrix}$$

where the $i-1 \times d-1$ submatrix X has an uninterrupted staircase from $(1,1)$ to $(i-1, d-1)$. Finally $\text{mindex}(M) = \text{mindex}(g(M)) - 1$ means that

$$M = \begin{pmatrix} & & & & & & 0 \\ & & & & & & 0 \\ & & & & & & \vdots \\ & & & X & & & 0 \\ & & & & & & 1 \\ 0 & 0 & 0 & 0 & \dots & 0 & x_{i,d} = k \\ 0 & 0 & 0 & 0 & \dots & 0 & x_{i+1,d} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 & x_{i+1,d} \end{pmatrix}.$$

Again it is apparent that since X had an uninterrupted staircase, so too does M , and so $M \in \text{FMatrices}_{n+1}^{(\text{IV})}$. □

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