Weakly nonlinear wave packets and the nonlinear Schrödinger equation

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Abstract. These lectures describe weakly nonlinear wave packets. The primary model equation is the nonlinear Schrödinger (NLS) equation. Its derivation is presented for two systems : the Korteweg–de Vries equation and the water-wave problem. Analytical as well as numerical results on the NLS equation are reviewed. Several applications are considered, including the study of wave stability. The bifurcation of waves when the phase and the group velocities are nearly equal as well as the effects of forcing on the NLS equation are discussed. Finally, recent results on the effects of dissipation on the NLS equation are also given.

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1 Introduction

The nonlinear Schrödinger (NLS) equation provides a canonical description for the envelope dynamics of a quasi-monochromatic plane wave (the carrier) propagating in a weakly nonlinear dispersive medium when dissipative processes are negligible. The NLS equation assumes weak nonlinearities but a finite dispersion at the scale of the carrier. Cumulative nonlinear interactions result in a modulation of the wave amplitude on large spatial and temporal scales.

2 Physical description

The NLS equation arises in various physical contexts in the description of nonlinear waves such as water waves at the free surface of an ideal fluid or plasma waves. It provides a canonical description of the envelope dynamics of a dispersive wave train

$$\eta = \epsilon A \exp[\mathrm{i}(\mathbf{k} \cdot \mathbf{x} - \omega t)] + \text{ c.c.}, \tag{2.1}$$

with a small ($\epsilon \ll 1$) but finite amplitude, slowly modulated in space and time, propagating in a conservative system. A good physical description is given by Whitham (1974).

Let us consider a nonlinear wave equation written symbolically as

$$\mathcal{L}(\partial_t, \nabla)u + \mathcal{N}(u)u = 0\,, \tag{2.2}$$

where $\mathcal{L}$ is a linear operator with constant coefficients and $\mathcal{N}$ a nonlinear function of u and of its derivatives. For a solution of infinitely small amplitude, the nonlinear effects can be neglected, and the equation admits monochromatic wave solutions of the form

$$u = \epsilon A \exp[\mathrm{i}(\mathbf{k} \cdot \mathbf{x} - \omega t)] + \text{ c.c.}, \tag{2.3}$$

with a constant amplitude ϵA. Here c.c. stands for the complex conjugate. The frequency ω and the wave vector $\mathbf{k}$ are linked through the dispersion relation

$$\mathcal{L}(-\mathrm{i}\omega, \mathrm{i}\mathbf{k}) = 0\,. \tag{2.4}$$

Consider a branch of solution $\omega = \omega(\mathbf{k})$ of (2.4). If we consider a regular perturbation expansion of the solution (2.3), nonlinear effects will accumulate over long times and large propagation distances. Mathematically speaking, resonant terms will be generated in the hierarchy of equations arising at the successive orders, leading to secular terms. One way to deal with these secular terms is the method of multiple scales. The amplitude of the carrier is allowed to vary over slow time and space variables, and its evolution is given by solvability conditions that eliminate the resonances and eventually lead to the NLS equation.

A simple heuristic argument can be given to explain the canonical character of the NLS equation (see for example the monograph by Sulem and Sulem (1999)).

Note that the name “NLS equation” originates from a formal analogy with the Schrödinger equation of quantum mechanics. In this context a nonlinear potential arises in the mean field description of interacting particles.

3 Derivation of the NLS equation

The first derivation of the NLS equation for weakly nonlinear wave packets was apparently given by Benney and Newell (1967). It was re-derived independently by Zakharov (1968). Comprehensive reviews of NLS equations in the context of water waves are given by Peregrine (1983) and by Hammack and Henderson (1993). Craig, Sulem and Sulem (1992) gave a rigorous estimate of the validity of NLS for water waves. But they left open the question of actual convergence of solutions of the water-wave problem to solutions in the form of the modulational ansatz. Some progress was recently made by Schneider (1998), who considered the rigorous derivation of the NLS equation from the Korteweg-de Vries (KdV) equation. A formal derivation of the NLS equation from the KdV equation based on the introduction of multiple scales is presented first. Then two derivations of the NLS equation from the full water-wave equations are given. The first one is based on multiple scale analysis, while the second one is based on Zakharov's integral equation representation of the water-wave problem.

3.1 The method of multiple scales

In order to illustrate the method of multiple scales, we consider the KdV equation, which is the model equation considered in Chapter 1 :

$$u_t + u_{xxx} + 6uu_x = 0\,, \quad (x \in \mathbb{R},\ t \geq 0,\ u(x,t) \in \mathbb{R})\,. \tag{3.1}$$

Its linearization admits monochromatic wave solutions of the form

$$u = \psi \exp[\mathrm{i}(kx - \omega t)] + \text{ c.c.}\,, \quad \text{with } \omega = -k^3\,. \tag{3.2}$$

The common ansatz is that the solution has a uniformly valid asymptotic expansion in terms of a small parameter ϵ (the magnitude of the wave amplitude) and that fast and slow variables are introduced. A regular perturbation expansion would lead to the onset of resonant terms resulting from the cumulative effects of weak nonlinearities on long time or large distances.

We look for a solution in the form

$$u = \epsilon(u_0(x,t,X,T,\cdots) + \epsilon u_1(x,t,X,T,\cdots) + \epsilon^2 u_2 + \cdots),$$

where the slow scales $X = \epsilon x, T = \epsilon t$ have been introduced in addition to the fast scales x and t. The new scaled variables are considered as independent. Therefore the linear operator arising in the KdV equation (3.1) can be rewritten as

$$\mathcal{L} = \mathcal{L}^{(0)} + \epsilon\mathcal{L}^{(1)} + \epsilon^2\mathcal{L}^{(2)} + \cdots\,, \tag{3.3}$$

where the first three contributions are given by

$$\mathcal{L}^{(0)} = \frac{\partial}{\partial t} + \frac{\partial^3}{\partial x^3}\,, \tag{3.4}$$

$$\mathcal{L}^{(1)} = \frac{\partial}{\partial T} + 3\frac{\partial^3}{\partial x^2 \partial X}\,, \tag{3.5}$$

$$\mathcal{L}^{(2)} = \frac{\partial}{\partial \tau} + 3\frac{\partial^3}{\partial x \partial X^2}\,. \tag{3.6}$$

Note the presence of the additional time scale $\tau = \epsilon^2 t$. At order (ϵ), we have

$$\mathcal{L}^{(0)} u_0 = 0\,,$$

and we recover the solution (3.2)

$$u_0(x,t) = \psi_0(X,T)\exp[\mathrm{i}(kx-\omega t)] + \text{ c.c.}$$

of the linear problem (monochromatic wave propagating along the x-axis). The frequency ω is given by the dispersion relation

$$\omega = -k^3\,. \tag{3.7}$$

At this point, we define the *phase velocity*

$$c = \frac{\omega}{k}.$$

Here $c = -k^2$. It depends on k and there is dispersion. We also define the *phase* $\theta = kx - \omega t$ and the *group velocity*

$$c_g = \frac{\mathrm{d}\omega}{\mathrm{d}k}.$$

Here $c_g = -3k^2$.

The next step consists in writing the differential equation at order (ϵ^2) :

$$\mathcal{L}^{(0)} u_1 = -\mathcal{L}^{(1)} u_0 - 3\partial_x(u_0^2)\,. \tag{3.8}$$

The following solvability condition is required at order (ϵ^2) :

$$(\partial_T - 3k^2\partial_X)u_0 = 0 \quad \text{or} \quad (\partial_T + c_g\partial_X)u_0 = 0\,. \tag{3.9}$$

We used the dispersion relation (3.7) to identify $-3k^2$ with the group velocity c_g of the wave packet. On the time scale T, the wave packet is just transported at the group velocity and thus depends only on the variable $\xi = X - c_g T$. We then solve for u_1 :

$$\mathcal{L}^{(0)} u_1 = -3\partial_x(\psi_0^2 \mathrm{e}^{2i(kx-\omega t)} + \text{ c.c.})\,. \tag{3.10}$$

The general solution is

$$u_1 = A_2(X,T)\mathrm{e}^{2i(kx-\omega t)} + \text{ c.c. } + \psi_1 \mathrm{e}^{i(kx-\omega t)} + \text{ c.c. } + A_0(X,T)\,. \tag{3.11}$$

The term $\psi_1 \mathrm{e}^{i(kx-\omega t)}$ can be incorporated into the term $\psi_0 \mathrm{e}^{i(kx-\omega t)}$ by defining $\psi = \psi_0 + \epsilon\psi_1$. Then one has

$$A_2 = \frac{-3(2\mathrm{i}k)\psi^2}{-2\mathrm{i}\omega + (2\mathrm{i}k)^3} = \frac{\psi^2}{k^2}\,.$$

The final step consists in writing the differential equation at order (ϵ^3) :

$$\mathcal{L}^{(0)} u_2 = -\mathcal{L}^{(2)} u_0 - \mathcal{L}^{(1)} u_1 - 6\partial_x(u_0 u_1) - 3\partial_X(u_0^2)\,. \tag{3.12}$$

The following solvability condition that eliminates the terms proportional to $\mathrm{e}^{i(kx-\omega t)}$ and its complex conjugate is required at order (ϵ^3) :

$$-\partial_\tau \psi - 3\mathrm{i}k\partial_{XX}\psi - 6\mathrm{i}k|\psi|^2\psi/k^2 - 6\mathrm{i}kA_0\psi = 0\,. \tag{3.13}$$

Equating the coefficients in front of the oscillation free terms gives

$$-\partial_T A_0 - 6\partial_X |\psi|^2 = 0\,.$$

Taking into account the solvability condition (3.9) yields

$$A_0 = -\frac{2|\psi|^2}{k^2}\,.$$

Finally, noting that $-6k = \mathrm{d}^2\omega/\mathrm{d}k^2$, (3.13) leads to the cubic NLS equation

$$\mathrm{i}\frac{\partial\psi}{\partial\tau} + \frac{1}{2}\frac{\mathrm{d}^2\omega}{\mathrm{d}k^2}\frac{\partial^2\psi}{\partial\xi^2} + \frac{6}{k}|\psi|^2\psi = 0\,. \tag{3.14}$$

Equation (3.14) describes the time evolution of the wave amplitude from an initial modulation $\psi(\xi, \tau = 0)$.

3.2 Rigorous derivation

In the previous subsection, we provided a formal derivation of the NLS equation. Let

$$\tilde{u} = \epsilon\psi(X,T)\exp[\mathrm{i}(kx-\omega t)] + \text{ c.c. } + \epsilon^2 A_2(X,T)\exp[2\mathrm{i}(kx-\omega t)] + \text{ c.c. } + \epsilon^2 A_0(X,T)\,.$$

Then Schneider (1998) proved the following theorem :

Theorem 3.1. *Let $\sigma \geq 3$ and let $\psi \in C([0,T_0], H^{\sigma+5}(\mathbb{R},\mathbb{C}))$ be a solution of the NLS equation (3.14). Then we have $C, \epsilon_0 > 0$ such that for all $\epsilon \in (0,\epsilon_0)$ there are solutions u of the KdV equation (3.1) such that*

$$\sup_{t\in[0,T_0/\epsilon^2]} \|u - \tilde{u}(\psi)\|_{H^\sigma(\mathbb{R},\mathbb{R})} \leq C\epsilon^{3/2}\,.$$

The proof is not easy. There are two main difficulties : we are dealing with a quasi-linear problem in Sobolev spaces, and the eigenvalues of the linearized problem do not satisfy the non-resonance condition which is needed if quadratic terms are present in the nonlinearity.

3.3 The water-wave problem

In order for this subsection to be self-contained, it is necessary to describe first the water-wave problem (its governing equations, its boundary conditions, its linearization).

The governing equations The three-dimensional flow of an ideal and incompressible fluid is governed by the conservation of mass

$$\nabla \cdot \mathbf{u} = 0 \tag{3.15}$$

and by the conservation of momentum

$$\rho \frac{D\mathbf{u}}{Dt} = \rho \mathbf{g} - \nabla p \,, \tag{3.16}$$

where the material derivative is defined as $D(*)/Dt \equiv \partial(*)/\partial t + (\mathbf{u} \cdot \nabla)(*)$. The horizontal coordinates are denoted by x and y, and the vertical coordinate by z. The vector $\mathbf{u}(x, y, z, t) = (u, v, w)$ is the velocity field, $\rho(x, y, z)$ is the fluid density, $\mathbf{g}$ is the acceleration due to gravity and $p(x, y, z, t)$ the pressure.

It is assumed below that the density ρ is constant throughout the fluid domain. This assumption is made for the sake of simplicity. A similar analysis can be made for continuously stratified flows. Another assumption which is commonly made to analyze surface waves is that the flow is irrotational (see however the paper by Colin, Dias and Ghidaglia (1995) for rotational effects). If the flow is irrotational ($\nabla \times \mathbf{u} = \mathbf{0}$), there exists a scalar function $\phi(x, y, z, t)$ (the so-called velocity potential) such that $\mathbf{u} = \nabla\phi$. The continuity equation (3.15) becomes

$$\Delta\phi = 0 \,. \tag{3.17}$$

With all these assumptions, the equation of momentum conservation (3.16) can be integrated into the so-called Bernoulli's equation

$$\frac{\partial\phi}{\partial t} + \frac{1}{2}|\nabla\phi|^2 + gz + \frac{p - p_0}{\rho} = 0 \,, \tag{3.18}$$

which is valid everywhere in the fluid. The constant p_0 is a pressure of reference, for example the atmospheric pressure.

The flow domain The surface wave problem consists in solving the incompressible Euler equations (3.15) and (3.16) in a domain bounded above by a free surface (the interface between air and water) and below by a solid boundary (the bottom). The free surface is represented by $F(x, y, z, t) \equiv \eta(x, y, t) - z = 0$. The bottom can be at any depth and have any shape. The driving force is due to gravity, but the effects of surface tension may be equally important in some physical situations.

Boundary conditions Whenever the potential and its derivatives are evaluated on the free surface, the following notation is used :

$$\varphi(x, y, t) = \phi(x, y, \eta, t) \,, \quad \varphi_{(*)}(x, y, t) = \phi_*(x, y, \eta, t) \,,$$

where the star stands for x, y, z or t. Consequently, φ_* and $\varphi_{(*)}$ have different meanings. They are however related since

$$\varphi_x = \varphi_{(x)} + \varphi_{(z)}\eta_x \,, \quad \varphi_y = \varphi_{(y)} + \varphi_{(z)}\eta_y \,, \quad \varphi_t = \varphi_{(t)} + \varphi_{(z)}\eta_t \,.$$

The free surface must be found as part of the solution. Two boundary conditions are required. The first one is the kinematic condition. It can be stated as $DF/Dt = 0$, which leads to

$$\eta_t + \varphi_{(x)}\eta_x + \varphi_{(y)}\eta_y - \varphi_{(z)} = 0\,. \tag{3.19}$$

In the case of a steady motion, the kinematic condition states that the free surface is a streamline. The second boundary condition is the dynamic condition which states that the forces must be equal on both sides of the free surface. The force normal to the free surface is the difference in pressure and is balanced by the effect of surface tension. If σ denotes the surface tension coefficient and C the curvature of the free surface, $p - p_0 = -\sigma C$. For the air/water interface, $\sigma = 0.074\,\mathrm{N/m}$. The expression for C is

$$C = \left(\frac{\eta_x}{(1+\eta_x^2+\eta_y^2)^{1/2}}\right)_x + \left(\frac{\eta_y}{(1+\eta_x^2+\eta_y^2)^{1/2}}\right)_y\,. \tag{3.20}$$

Bernoulli's equation (3.18) written on the free surface $z = \eta$ gives

$$\varphi_{(t)} + \frac{1}{2}\left(\varphi_{(x)}^2 + \varphi_{(y)}^2 + \varphi_{(z)}^2\right) + g\eta - \frac{\sigma}{\rho}C = 0\,. \tag{3.21}$$

Finally, the bottom $z = -h(x, y)$ is assumed to be flat : $z = -h$. The boundary condition at the bottom simply is

$$\phi_z(x, y, -h, t) = 0\,. \tag{3.22}$$

To summarize, one wants to solve for $\eta(x, y, t)$ and $\phi(x, y, z, t)$ the following set of equations :

$$\begin{aligned}
\Delta\phi \equiv \phi_{xx} + \phi_{yy} + \phi_{zz} &= 0 \quad \text{for} \ (x,y,z) \in \mathbb{R}\times\mathbb{R}\times[-h,\eta],\\
\eta_t + \varphi_{(x)}\eta_x + \varphi_{(y)}\eta_y - \varphi_{(z)} &= 0,\\
\varphi_{(t)} + \frac{1}{2}\left(\varphi_{(x)}^2 + \varphi_{(y)}^2 + \varphi_{(z)}^2\right) + g\eta - \frac{\sigma}{\rho}C &= 0,\\
\phi_z &= 0 \quad \text{on} \ z = -h\,.
\end{aligned}$$

The conservation of momentum equation (3.16) is not stated; it is used to find the pressure p once η and ϕ have been found. In water of infinite depth, the kinematic boundary condition on the bottom (last equation) is replaced by

$$|\nabla\phi| \to 0 \quad \text{as} \ z \to -\infty\,.$$

The surface wave problem has been studied for more than a century. It is a difficult nonlinear problem because of its two nonlinear boundary conditions on the free surface. Assumptions can be made to simplify the problem : linearization of the equations, restriction to periodic solutions or to steady solutions in a moving frame of reference, flat bottom, two spatial dimensions only, model equations.

Linearization of the 2D problem and dispersion relation One looks for solutions which are periodic in x with wave number k and in t with frequency ω. The following dimensionless variables are introduced :

$$(x^*, z^*) = (kx, kz), \quad \eta^* = \frac{\eta}{a}, \quad t^* = \omega t, \quad \phi^* = \frac{k}{\omega a}\phi\,,$$

where a denotes the amplitude of the wave. In terms of the dimensionless variables and after dropping the stars, the surface wave problem linearized around the equilibrium state $\eta = 0$, $\mathbf{u} = \mathbf{0}$ reads :

$$\begin{aligned} \phi_{xx} + \phi_{zz} &= 0 \quad \text{for } (x,z) \in \mathbb{R} \times [-kh, 0], \\ \eta_t - \phi_z &= 0 \quad \text{on } z = 0, \\ \phi_t + \left(\frac{gk}{\omega^2}\right)\eta - \left(\frac{\sigma k^3}{\rho\omega^2}\right)\eta_{xx} &= 0 \quad \text{on } z = 0, \\ \phi_z &= 0 \quad \text{on } z = -kh\,. \end{aligned}$$

Consider solutions in the form of sinusoidal dispersive waves $\eta(x,t) = \cos(x - t + \Theta)$. In the linear theory, the wave number k and the frequency ω are assumed to be constant. Since the equations are linear, the amplitude is arbitrary. On the other hand, for the boundary conditions on the free surface to be satisfied, ω and k must satisfy a *dispersion relation* of the form $D(\omega, k) = 0$. Note that dimensional analysis immediately gives

$$\frac{\omega}{\sqrt{gk}} = \mathcal{G}\left(kh, \frac{\sigma k^2}{\rho g}, ka\right)\,.$$

Let us now obtain the function $\mathcal{G}$. Laplace's equation combined with the kinematic conditions on the bottom and on the free surface gives

$$\phi(x - t, z) = \frac{1}{\tanh(kh)}\sin(x - t + \Theta)\frac{\cosh(z + kh)}{\cosh(kh)}\,.$$

The dynamic condition on the free surface yields

$$D(\omega, k) \equiv \omega^2 - gk\tanh(kh)\left(1 + \frac{\sigma k^2}{\rho g}\right) = 0\,. \tag{3.23}$$

The final expressions for the potential ϕ and for the elevation of the free surface η in the original variables are

$$\phi = b\sin(kx - \omega t + \Theta)\frac{\cosh(kz + kh)}{\cosh(kh)}\,, \quad \eta = a\cos(kx - \omega t + \Theta)\,, \tag{3.24}$$

where

$$b = a\frac{g}{\omega}\left(1 + \frac{\sigma k^2}{\rho g}\right)\,.$$

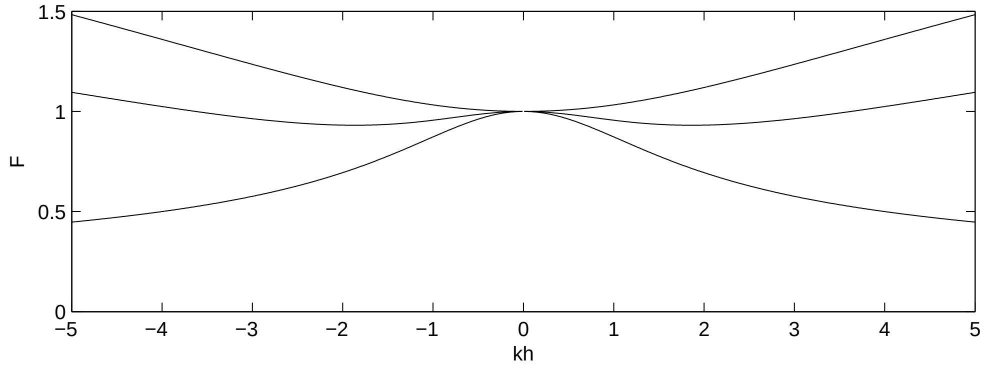

Figure 1. Dispersion relation (3.25) for capillary–gravity waves. The three curves correspond, from top to bottom, to $W = 0.4$, $W = 0.2$ and $W = 0$.

Equation (3.23) is the well-known dispersion relation for linearized 2D capillary–gravity periodic waves in water of finite depth. Written in terms of the phase speed c rather than the frequency ω, the dispersion relation becomes

$$F^2 = \frac{\tanh(kh)}{kh}\left(1 + (kh)^2 W\right) , \tag{3.25}$$

where the Froude number F and the Weber number W are defined as

$$F = \frac{c}{\sqrt{gh}}, \quad W = \frac{\sigma}{\rho g h^2}.$$

Equation (3.25) has been plotted in Figure 1 for three different values of W. When $0 < W < 1/3$, the curve has, in addition to the extremum at the origin, a minimum where phase and group velocities are equal. Note that in infinite depth the dispersion relation becomes

$$D(\omega, k) \equiv \omega^2 - g|k|\left(1 + \frac{\sigma k^2}{\rho g}\right) = 0. \tag{3.26}$$

The presence of the absolute value has subtle consequences.

For the air/water interface, the minimum is obtained (in infinite depth) for a wave length of 1.73 cm and a wave speed of 23.2 cm/s. For short waves (kh large), surface tension has a stronger effect than gravity. For long waves (kh small), it is the opposite. If the Froude number F is between the minimum value and one (and $0 < W < 1/3$), there are two corresponding positive wave numbers, one on the gravity side and one on the capillary side.

In water of 10 centimeter depth, the critical speed $\sqrt{gh}$ is 1m/s. For the air/water interface, the corresponding short wave having the same phase velocity is given by (since kh is large)

$$kh \approx \frac{\sigma}{\rho g h^2}(kh)^2 = 0.00074(kh)^2, \quad \text{or} \quad kh \approx 1350,$$

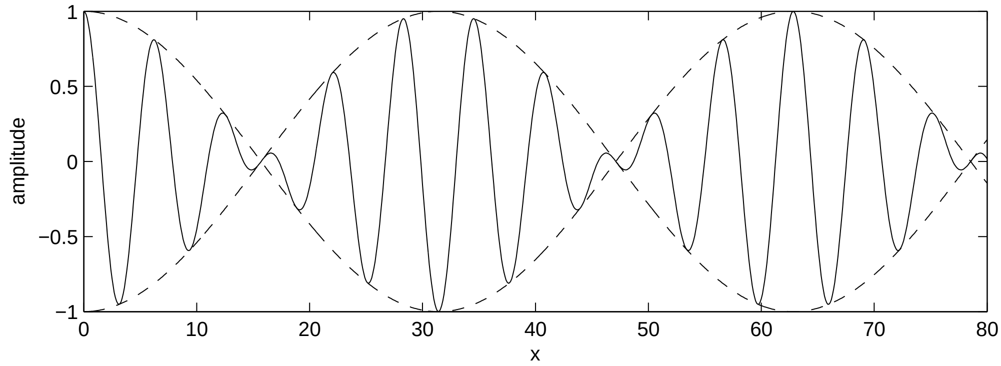

Figure 2. Wave packet (3.27) at time $t = 0$ with $k = 1$, $\delta k = 0.2$ and $A = 1/2$. The envelope is shown in dashed lines.

i.e. its wavelength is half a millimeter. If $h = 5$ cm, the wavelength is 1 mm.

The group velocity can be written as

$$c_g = c\left[\frac{1}{2}\left(\frac{1+3(kh)^2W}{1+(kh)^2W}\right) + \frac{kh}{\sinh 2kh}\right].$$

For long waves ($kh \to 0$), the group and phase velocities are equal to $\sqrt{gh}$. There is no dispersion. For pure capillary waves ($g = 0$), the group velocity is

$$c_g = c\left(\frac{3}{2} + \frac{kh}{\sinh 2kh}\right),$$

while, for pure gravity waves ($\sigma = 0$), the group velocity is

$$c_g = c\left(\frac{1}{2} + \frac{kh}{\sinh 2kh}\right).$$

A first interpretation of the group velocity is given by considering the superposition of two linear waves, with almost equal wave numbers and frequencies :

$$\begin{aligned} \eta_1 &= A\cos(kx - \omega t), \\ \eta_2 &= A\cos[(k+\delta k)x - (\omega + \delta\omega)t]. \end{aligned}$$

The resulting profile is given by

$$\begin{aligned} \eta = \eta_1 + \eta_2 &= 2A\cos\left[\tfrac{1}{2}(\delta k\, x - \delta\omega\, t)\right]\cos\left[kx - \omega t + \tfrac{1}{2}(\delta k\, x - \delta\omega\, t)\right] \\ &\approx 2A\cos\left[\tfrac{1}{2}(\delta k\, x - \delta\omega\, t)\right]\cos(kx - \omega t) \qquad (3.27) \end{aligned}$$

and is shown in Figure 2. The envelope (first cosine term) travels at the group velocity while the carrying wave inside the envelope (second cosine term) travels at the phase velocity.

A second interpretation of the group velocity is given by considering the speed of propagation of energy. We define the kinetic energy density K and the average kinetic energy over a wave length $\overline{K}$

$$K = \int_{-h}^{\eta} \tfrac{1}{2}\rho|\nabla\phi|^2 \,\mathrm{d}z, \quad \overline{K} = \frac{1}{L}\int_{x}^{x+L} K \,\mathrm{d}x\,, \tag{3.28}$$

as well as the potential energy density V and the average potential energy over a wave length $\overline{V}$

$$V = \tfrac{1}{2}\rho g\eta^2 + \sigma\left(\sqrt{1+\eta_x^2}-1\right), \quad \overline{V} = \frac{1}{L}\int_{x}^{x+L} V \,\mathrm{d}x\,, \tag{3.29}$$

where L denotes the wave length. The total energy density is $E = K + V$.

For linearized waves, the expressions for $\overline{K}$ and $\overline{V}$ become

$$\overline{K} = \frac{1}{L}\int_{x}^{x+L}\int_{-h}^{0} \frac{1}{2}\rho|\nabla\phi|^2 \,\mathrm{d}z\,\mathrm{d}x = \frac{1}{4}\rho g a^2\left(1+\frac{\sigma k^2}{\rho g}\right),$$

$$\overline{V} = \frac{1}{L}\int_{x}^{x+L}\left(\frac{1}{2}\rho g\eta^2 + \frac{1}{2}\sigma\eta_x^2\right)\mathrm{d}x = \frac{1}{4}\rho g a^2\left(1+\frac{\sigma k^2}{\rho g}\right).$$

Potential and kinetic energy are equal. This result, however, is no longer true for non-linear waves.

For the sake of simplicity, let us now neglect the effects due to surface tension. The kinetic energy balance for the flow of a perfect fluid can be written as

$$\frac{\partial}{\partial t}\int_{\Omega} \tfrac{1}{2}\rho|\nabla\phi|^2 \,\mathrm{d}\Omega + \int_{\partial\Omega} \tfrac{1}{2}\rho|\nabla\phi|^2 \,\mathbf{u}\cdot\mathbf{n}\,\mathrm{d}a = \int_{\Omega} \rho\,\mathbf{f}\cdot\mathbf{u}\,\mathrm{d}\Omega - \int_{\partial\Omega} p\,\mathbf{u}\cdot\mathbf{n}\,\mathrm{d}a\,,$$

where $\mathbf{f}$ is the force due to gravity, Ω the domain contained in the rectangle between two vertical lines at $x = x_1$ and at $x = x_2$, the bottom and a horizontal line above the free surface, $\partial\Omega$ the boundary of this domain.

Let us transform first the term containing the gravity force, by introducing the potential $\Phi = -gz$. The free surface is denoted by Σ.

$$\begin{aligned}
\int_{\Omega} \rho\,\mathbf{f}\cdot\mathbf{u}\,\mathrm{d}\Omega &= \int_{\Omega} \rho\,\nabla\Phi\cdot\mathbf{u}\,\mathrm{d}\Omega \\
&= \int_{\Omega} \rho\,\nabla\cdot(\Phi\mathbf{u})\,\mathrm{d}\Omega \quad (\text{since } \nabla\cdot\mathbf{u} = 0) \\
&= \int_{\partial\Omega} \rho\,\Phi\,\mathbf{u}\cdot\mathbf{n}\,\mathrm{d}a - \int_{\Sigma} \rho\,[[\Phi\mathbf{u}]]\cdot\mathbf{n}\,\mathrm{d}\Sigma \\
&\quad \text{(divergence theorem in the presence of a free surface)} \\
&= -\int_{\partial\Omega} \rho g z\,\mathbf{u}\cdot\mathbf{n}\,\mathrm{d}a - \int_{\Sigma} \frac{\rho g\eta\eta_t}{\sqrt{1+\eta_x^2}}\,\mathrm{d}\Sigma\,. \\
&\quad \text{(kinematic condition)}
\end{aligned}$$

Therefore the right hand side of the kinetic energy balance can be rewritten as

$$\begin{aligned}\int_\Omega \rho\, \mathbf{f}\cdot\mathbf{u}\, \mathrm{d}\Omega - \int_{\partial\Omega} p\, \mathbf{u}\cdot\mathbf{n}\, \mathrm{d}a &= -\int_{\partial\Omega} (p+\rho g z)\, \mathbf{u}\cdot\mathbf{n}\, \mathrm{d}a - \int_\Sigma \frac{\rho g \eta \eta_t}{\sqrt{1+\eta_x^2}}\, \mathrm{d}\Sigma \\ &= \int_{\partial\Omega} \rho \left(\frac{\partial\phi}{\partial t} + \tfrac{1}{2}|\nabla\phi|^2\right) \mathbf{u}\cdot\mathbf{n}\, \mathrm{d}a - \int_\Sigma \frac{\rho g \eta \eta_t}{\sqrt{1+\eta_x^2}}\, \mathrm{d}\Sigma\,.\end{aligned}$$

It follows that

$$\frac{\partial}{\partial t}\int_\Omega \tfrac{1}{2}\rho\, |\nabla\phi|^2\, \mathrm{d}\Omega + \int_\Sigma \frac{\rho g \eta \eta_t}{\sqrt{1+\eta_x^2}}\, \mathrm{d}\Sigma = \int_{\partial\Omega} \rho\, \frac{\partial\phi}{\partial t}\, \mathbf{u}\cdot\mathbf{n}\, \mathrm{d}a\,,$$

$$\text{or}\quad \frac{\partial}{\partial t}\int_{x_1}^{x_2}\left(\int_{-h}^{\eta} \tfrac{1}{2}\rho\, |\nabla\phi|^2\, \mathrm{d}z + \tfrac{1}{2}\rho g \eta^2\right) \mathrm{d}x = -\mathcal{F}(x_2) + \mathcal{F}(x_1)\,.$$

It yields the equation

$$\frac{\partial E}{\partial t} + \frac{\partial \mathcal{F}}{\partial x} = 0\,,$$

where $\mathcal{F}$ is the energy flux defined by

$$\mathcal{F} = -\int_{-h}^{\eta} \rho \phi_x \phi_t\, \mathrm{d}z\,.$$

A local derivation of that equation can also be given.

For the waves (3.24) of the linearized problem with $\Theta = 0$, one computes easily

$$E = \frac{1}{2}\rho g a^2 \cos^2(kx-\omega t)\left(1+\frac{2kh}{\sinh(2kh)}\right) + \frac{1}{2}\rho g a^2\left(\frac{1}{2} - \frac{kh}{\sinh(2kh)}\right)\,,$$

and

$$\mathcal{F} = \frac{1}{2}\rho g a^2 \frac{\omega}{k}\cos^2(kx-\omega t)\left(1+\frac{2kh}{\sinh(2kh)}\right)\,.$$

By taking the average of these two expressions over a period, one finds

$$\overline{E} = \frac{1}{2}\rho g a^2 \quad\text{and}\quad \overline{\mathcal{F}} = \frac{1}{2}\rho g c_g a^2\,.$$

It follows that $\overline{\mathcal{F}} = c_g \overline{E}$. The energy propagates with the group velocity.

Illustrations of the concept of group velocity There are several experiments which clearly illustrate the concept of group velocity. We have seen that there are two interpretations of the concept of group velocity, one which is purely kinematic and one which is purely dynamical. The link between the two is not that clear (see Stoker (1958)). Stoker gives a preference to the kinematic interpretation. Three well-known illustrations of the concept of group velocity are the wave-maker, the fish-line problem and the ship wake. The main feature of these three examples is that the free surface is perturbed by a surface piercing object. In the problem of the wave maker, a paddle oscillates and disturbs the free surface. In the problem of the fish-line, the object is fixed but the flow has a current. In the problem of the ship wake, the ship is moving on the free surface of a fluid otherwise at rest.

Dispersion relation for 3D waves The extension of the dispersion relation (3.23) to three-dimensional waves is

$$D(\omega,k,l) \equiv \omega^2 - g|\mathbf{k}|\tanh(|\mathbf{k}|h)\left(1+\frac{\sigma|\mathbf{k}|^2}{\rho g}\right) = 0\,, \tag{3.30}$$

where $|\mathbf{k}| = \sqrt{k^2+l^2}$.

A brief account of the method of multiple scales to derive the NLS equation from the full water-wave problem Accounting for nonlinear and dispersive effects correct to third order in the wave steepness, the envelope of a weakly nonlinear gravity–capillary wavepacket in deep water is governed by the NLS equation. A more accurate envelope equation, which includes effects up to fourth order in the wave steepness, was derived by Dysthe (1979) for pure gravity wavepackets. He used the method of multiple scales. Later, Stiassnie (1984) showed that the Dysthe equation is merely a particular case of the more general Zakharov equation that is free of the narrow spectral width assumption. Hogan (1985) extended Stiassnie's results to deep-water gravity–capillary wavepackets. Apart from the leading-order nonlinear and dispersive terms present in the NLS equation, the fourth-order equation of Hogan features certain nonlinear modulation terms and a nonlocal term that describes the coupling of the envelope with the induced mean flow. In addition to playing a significant part in the stability of a uniform wavetrain, this mean flow turns out to be important at the tails of gravity–capillary solitary waves in deep water as shown by Akylas, Dias and Grimshaw (1998).

The common ansatz used in the derivation of the NLS equation (or of the Dysthe equation if one goes to higher order) is that the velocity potential ϕ and the free-surface elevation η have uniformly valid asymptotic expansions in terms of a small parameter ϵ (the dimensionless amplitude of the wave, kA, for example). One writes

$$\eta = \sum_{n=1}^{3} \epsilon^n\, \eta_n(x_0,x_1,x_2,y_1,y_2;t_0,t_1,t_2) + \mathcal{O}(\epsilon^4)\,, \tag{3.31}$$

$$\phi = \sum_{n=1}^{3} \epsilon^n\, \phi_n(x_0,x_1,x_2,y_1,y_2,z;t_0,t_1,t_2) + \mathcal{O}(\epsilon^4)\,, \tag{3.32}$$

where

$$x_0 = x\,, x_1 = \epsilon\, x\,, x_2 = \epsilon^2\, x\,, y_1 = \epsilon\, y\,, y_2 = \epsilon^2\, y\,, t_0 = t\,, t_1 = \epsilon\, t\,, t_2 = \epsilon^2\, t\,. \tag{3.33}$$

The order one component of η is the linearized monochromatic wave

$$\eta_1 = A\,\mathrm{e}^{i(kx-\omega t)} + \text{ c.c.}. \tag{3.34}$$

Applying the method of multiple scales leads to the following equation for the evolution of the complex amplitude A of the wave :

$$2\mathrm{i}\frac{\partial A}{\partial t_2} + p\frac{\partial^2 A}{\partial \xi^2} + q\frac{\partial^2 A}{\partial y_1^2} + \gamma A|A|^2 = -\mathrm{i}\,\epsilon\,(sA_{\xi y_1 y_1} + rA_{\xi\xi\xi} + uA^2A_\xi^* - v|A|^2A_\xi) + \epsilon\, A\,\overline{\phi}_\xi\big|_{z_1=0}\,. \tag{3.35}$$

The evolution equation has been written in dimensionless form, with all lengths nondimensionalized by k, time by ω and potential by $2k^2/\omega$. Here $\xi = x_1 - c_g t_1$ and $z_1 = \epsilon z$ are scaled variables that describe the wavepacket modulations in a frame of reference moving with the group velocity c_g. As expected, to leading order in the wave steepness $\epsilon \ll 1$, equation (3.35) reduces to the familiar NLS equation, while the coupling with the induced mean flow is reflected in the last term of (3.35). Specifically, the mean-flow velocity potential $\epsilon^2\,\overline{\phi}(\xi, z_1, t_2)$ satisfies the boundary-value problem

$$\begin{aligned}
\overline{\phi}_{\xi\xi} + \overline{\phi}_{z_1 z_1} &= 0 && (-\infty < z_1 < 0,\ -\infty < \xi < \infty)\ ,\\
\overline{\phi}_{z_1} &= (|A|^2)_\xi && (z_1 = 0)\ ,\\
\overline{\phi} &\to 0 && (z_1 \to -\infty)\ ,
\end{aligned}$$

from which it follows that

$$\overline{\phi}_\xi\Big|_{z_1=0} = -\int_{-\infty}^{\infty} |s|\, e^{is\xi}\, FT(|A|^2)\, \mathrm{d}s\,, \tag{3.36}$$

where

$$FT(\cdot) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-is\xi}\,(\cdot)\,\mathrm{d}\xi$$

denotes the Fourier transform. Hence, the coupling of the envelope with the induced mean flow enters via a nonlocal term in the fourth-order envelope equation. The coefficients of the rest of the terms in (3.35) are given by the following expressions, where $B = \sigma k^2/\rho g$:

$$p = \frac{k^2}{\omega}\frac{\mathrm{d}^2\omega}{\mathrm{d}k^2} = \frac{3B^2 + 6B - 1}{4(1+B)^2}\,, \tag{3.37}$$

$$q = \frac{k}{\omega}\frac{\mathrm{d}\omega}{\mathrm{d}k} = \frac{1+3B}{2(1+B)}\,, \tag{3.38}$$

$$\gamma = -\frac{2B^2 + B + 8}{8(1-2B)(1+B)}\,, \tag{3.39}$$

$$r = -\frac{(1-B)(B^2 + 6B + 1)}{8(1+B)^3}\,, \tag{3.40}$$

$$s = \frac{3 + 2B + 3B^2}{4(1+B)^2}\,, \tag{3.41}$$

$$u = \frac{(1-B)(2B^2 + B + 8)}{16(1-2B)(1+B)^2}\,, \tag{3.42}$$

$$v = \frac{3(4B^4 + 4B^3 - 9B^2 + B - 8)}{8(1-2B)^2(1+B)^2}\,. \tag{3.43}$$

These expressions are not valid when B is close to $1/2$, which corresponds to the resonance between the first and second harmonics.

Derivation from Zakharov's integral equation This derivation was first considered by Stiassnie (1984) for gravity waves and by Hogan (1985) for capillary–gravity waves.

Instead of working directly with the free-surface elevation $\eta(\mathbf{x},t)$, the new variable $B(\mathbf{k},t)$ is introduced. The link between η and B is :

$$\eta(\mathbf{x},t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\left(\frac{|\mathbf{k}|}{2\omega(\mathbf{k})}\right)^{1/2}\{B(\mathbf{k},t)\exp\{\mathrm{i}[\mathbf{k}\cdot\mathbf{x}-\omega(\mathbf{k})t]\} + \text{ c.c.}\}\,\mathrm{d}\mathbf{k}\,. \tag{3.44}$$

Zakharov's integral equation for $B(\mathbf{k},t)$ is (see also Chapter 5)

$$\begin{aligned}\mathrm{i}\frac{\partial B}{\partial t}(\mathbf{k},t) &= \iiint_{-\infty}^{+\infty} T(\mathbf{k},\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3)B^*(\mathbf{k}_1,t)B(\mathbf{k}_2,t)B(\mathbf{k}_3,t)\delta(\mathbf{k}+\mathbf{k}_1-\mathbf{k}_2-\mathbf{k}_3)\\ &\quad\times\exp\{\mathrm{i}[\omega(\mathbf{k})+\omega(\mathbf{k}_1)-\omega(\mathbf{k}_2)-\omega(\mathbf{k}_3)]t\}\,\mathrm{d}\mathbf{k}_1\,\mathrm{d}\mathbf{k}_2\,\mathrm{d}\mathbf{k}_3\,,\end{aligned} \tag{3.45}$$

where the wave vector $\mathbf{k}$ and the frequency ω are related through the linear dispersion relation (3.30)

$$\omega(\mathbf{k}) = \left(g|\mathbf{k}| + \sigma/\rho\,|\mathbf{k}|^3\right)^{1/2}\,.$$

The function $T(\mathbf{k},\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3)$ is a lengthy scalar function given for example by Krasitskii (1990). Equation (3.45) describes four-wave interaction processes obeying the resonant conditions

$$\mathbf{k}+\mathbf{k}_1 = \mathbf{k}_2+\mathbf{k}_3\,, \tag{3.46}$$

$$\omega(\mathbf{k})+\omega(\mathbf{k}_1) = \omega(\mathbf{k}_2)+\omega(\mathbf{k}_3)\,. \tag{3.47}$$

But, as pointed out by Zakharov (1968), there are difficulties in applying equation (3.45) to capillary–gravity waves. This is because, unlike gravity waves, these waves can satisfy triad resonances. The condition for triad resonance will give a zero denominator in one of the terms of $T(\mathbf{k},\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3)$ corresponding to the second-order interaction. However, if the wave packet is sufficiently narrow, then the resonance condition cannot be satisfied.

Let $\mathbf{k} = \mathbf{k}_0 + \chi$ where $\mathbf{k}_0 = (k,0)$ and $\chi = (p,q)$. Let ϵ denote the order of the spectral width $|\chi|/k$. Let also $\omega(\mathbf{k}_0) = \omega$ and $\chi_i = (p_i,q_i), i = 1,2,3$.

Introducing a new variable $A(\chi,t)$ given by

$$A(\chi,t) = B(\mathbf{k},t)\exp\{-\mathrm{i}[\omega(\mathbf{k})-\omega(\mathbf{k}_0)]t\} \tag{3.48}$$

in equations (3.45) and (3.44), one obtains

$$\begin{aligned}\mathrm{i}\frac{\partial A}{\partial t}(\chi,t) - [\omega(\mathbf{k})-\omega]A(\chi,t) &= \iiint_{-\infty}^{+\infty} T(\mathbf{k}_0+\chi,\mathbf{k}_0+\chi_1,\mathbf{k}_0+\chi_2,\mathbf{k}_0+\chi_3)\\ &\quad \delta(\chi+\chi_1-\chi_2-\chi_3)A^*(\chi_1)A(\chi_2)A(\chi_3)\,\mathrm{d}\chi_1\,\mathrm{d}\chi_2\,\mathrm{d}\chi_3\,,\end{aligned} \tag{3.49}$$

and

$$\eta(\mathbf{x},t) = \exp\{\mathrm{i}(kx-\omega t)\}\frac{1}{2\pi}\int_{-\infty}^{+\infty}\left(\frac{|\mathbf{k}|}{2\omega(\mathbf{k})}\right)^{1/2} A(\chi,t)\,\mathrm{e}^{i\chi\cdot\mathbf{x}}\,\mathrm{d}\chi + \text{ c.c.}\,. \tag{3.50}$$

The Taylor expansion of $|\mathbf{k}_0+\chi|/2\omega(\mathbf{k}_0+\chi)$ in powers of $|\chi|/k$ is

$$\left(\frac{|\mathbf{k}_0+\chi|}{2\omega(\mathbf{k}_0+\chi)}\right)^{1/2} = \left(\frac{\omega}{2g(1+B)}\right)^{1/2}\left(1+\frac{p}{4k}\left(\frac{1-B}{1+B}\right)\right)\,, \tag{3.51}$$

in which terms up to order ϵ have been retained.

Substituting equation (3.51) into (3.50), $\eta(\mathbf{x},t)$ can be expressed as

$$\eta(\mathbf{x},t) = \mathrm{Re}\,\{a(\mathbf{x},t)\mathrm{e}^{i(kx-\omega t)}\}\,, \tag{3.52}$$

where

$$a(\mathbf{x},t) = \frac{1}{2\pi}\left(\frac{2\omega}{g(1+B)}\right)^{1/2}\int_{-\infty}^{+\infty}\left(1+\frac{p}{4k}\left(\frac{1-B}{1+B}\right)\right)A(\chi,t)\mathrm{e}^{i\chi\cdot\mathbf{x}}\,\mathrm{d}\chi\,. \tag{3.53}$$

By Taylor expanding $\omega(\mathbf{k})-\omega$ in powers of $|\chi|/k$ and keeping terms up to order ϵ we get

$$\begin{aligned}\omega(\mathbf{k})-\omega \;=\; & \tfrac{1}{2}\left(\frac{g}{k(1+B)}\right)^{1/2}\left\{p(1+3B)+\frac{p^2}{4k}\left(\frac{-1+6B+3B^2}{1+B}\right)+\frac{q^2}{2k}(1+3B)\right.\\ & \left.+\frac{p^3}{8k^2}\left(\frac{(1-B)(1+6B+B^2)}{(1+B)^2}\right)-\frac{pq^2}{4k^2}\left(\frac{3+2B+3B^2}{1+B}\right)\right\}\,.\end{aligned} \tag{3.54}$$

In (3.49), we substitute the expression (3.54) for $\omega(\mathbf{k})-\omega$. By replacing $A(\chi,t)$ by $a(\mathbf{x},t)$ and taking the inverse Fourier transform, one finds

$$\begin{aligned}&\mathrm{i}\,a_t+\tfrac{1}{2}\left(\frac{g}{k(1+B)}\right)^{1/2}\left\{\mathrm{i}\,(1+3B)a_x+\left(\frac{-1+6B+3B^2}{4k(1+B)}\right)a_{xx}+\frac{(1+3B)}{2k}a_{yy}\right.\\ &\qquad\left.-\mathrm{i}\,\frac{(1-B)(1+6B+B^2)}{8k^2(1+B)^2}a_{xxx}+\mathrm{i}\,\frac{3+2B+3B^2}{4k^2(1+B)}a_{xyy}\right\}\\ &\quad=\frac{1}{2\pi}\left(\frac{2\omega}{g(1+B)}\right)^{1/2}\iiint_{-\infty}^{+\infty}\left[1+\frac{p_2+p_3-p_1}{4k}\left(\frac{1-B}{1+B}\right)\right]\\ &\qquad\times T(\mathbf{k}_0+\chi_2+\chi_3-\chi_1,\mathbf{k}_0+\chi_1,\mathbf{k}_0+\chi_2,\mathbf{k}_0+\chi_3)\\ &\qquad\times A^*(\chi_1)A(\chi_2)A(\chi_3)\exp\{\mathrm{i}(\chi_2+\chi_3-\chi_1)\cdot\mathbf{x}\}\,\mathrm{d}\chi_1\,\mathrm{d}\chi_2\,\mathrm{d}\chi_3\,.\end{aligned} \tag{3.55}$$

Now it can be shown that the Taylor expansion of T keeping terms up to order ϵ becomes

$$\begin{aligned}&T(\mathbf{k}_0+\chi_2+\chi_3-\chi_1,\mathbf{k}_0+\chi_1,\mathbf{k}_0+\chi_2,\mathbf{k}_0+\chi_3)=\frac{k^3}{8\pi^2}\left[\frac{(8+B+2B^2)}{4(1+B)(1-2B)}\right.\\ &\left.+\frac{3(p_2+p_3)}{8k}\frac{(8-B+9B^2-4B^3-4B^4)}{(1+B)^2(1-2B)^2}-\frac{(p_3-p_1)^2}{k|\chi_1-\chi_3|}-\frac{(p_2-p_1)^2}{k|\chi_1-\chi_2|}\right]\,.\end{aligned} \tag{3.56}$$

By using this form of T, we find by using (3.53) that the right-hand side of (3.55) becomes on integration

$$\begin{aligned}&\frac{g}{16\omega}\left[\frac{k^3(8+B+2B^2)}{(1-2B)}a|a|^2-\tfrac{1}{2}\mathrm{i}\,k^2\frac{(1-B)(8+B+2B^2)}{(1+B)(1-2B)}a^2a_x^*\right.\\ &\left.-3\mathrm{i}\,k^2\frac{(8-B+9B^2-4B^3-4B^4)}{(1+B)(1-2B)^2}|a|^2a_x\right]-\frac{k^2}{4\pi^2}aI\,,\end{aligned} \tag{3.57}$$

where

$$I = \iint_{-\infty}^{+\infty} \frac{(p_1 - p_2)^2}{|\chi_1 - \chi_2|} A^*(\chi_1) A(\chi_2) e^{i(\chi_2 - \chi_1)\cdot \mathbf{x}} \, d\chi_1 \, d\chi_2 \,. \tag{3.58}$$

It can be shown that

$$I = \left[\frac{g(1+B)}{2\omega} \right] 2\pi \int_{-\infty}^{+\infty} \frac{\partial}{\partial X}(|a|^2) \frac{x - X}{|\mathbf{x} - \mathbf{X}|^3} \, d\mathbf{X} \,. \tag{3.59}$$

The integral I can be related to the mean-flow velocity potential $\overline{\phi}$ (3.36). Collecting the results from equations (3.55) and (3.59), together with those from the expression (3.57), and making the same scaling transformation as above leads to equation (3.35). More details can be found in Dias and Kharif (1999). See also the interesting paper by Phillips (1981), which provides a history of wave interactions.

4 Mathematical properties

In this section, we consider the NLS equation in the following form :

$$\mathrm{i}\, u_t + \alpha\, u_{xx} + \gamma\, |u|^2 u = 0 \,, \quad u \in \mathbb{C}. \tag{4.1}$$

This form can always been obtained with rescaling. We emphasize here that there is only one NLS equation (focussing if $\alpha\gamma > 0$ and defocussing if $\alpha\gamma < 0$) for all physical systems. Incidentally the NLS equation (3.14) derived from the KdV equation is of the defocussing type. The NLS equation derived from the water-wave problem is of the focussing type if $kh > 1.363$ and of the defocussing type if $kh < 1.363$.

4.1 Hamiltonian structure

The NLS equation (4.1) is a Hamiltonian partial differential equation. Taking real coordinates, the following Hamiltonian system can be derived. Let $u = u_1 + \mathrm{i}u_2$ and write $\mathbf{U} = (u_1, u_2)$. Then the NLS equation takes the form

$$J\mathbf{U}_t + \alpha \mathbf{U}_{xx} + \gamma ||\mathbf{U}||^2 \mathbf{U} = 0 \quad \text{where} \quad J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} , \tag{4.2}$$

or

$$J\mathbf{U}_t = \nabla H(\mathbf{U}) \,, \tag{4.3}$$

with

$$H(\mathbf{U}) = \int_{\mathbb{R}} \left[\tfrac{1}{2}\alpha ||\mathbf{U}_x||^2 - \tfrac{1}{4}\gamma ||\mathbf{U}||^4 \right] \, dx \,. \tag{4.4}$$

The expression for H clearly shows why the case $\alpha\gamma > 0$ is called focussing.

4.2 Conservation laws

There is an infinite number of conservation laws satisfied by solutions to the NLS equation. Only two will be of concern to us. The first one is the conservation law for

$|u(x,t)|^2$, which is given by

$$\frac{\partial}{\partial t}|u|^2 = \mathrm{i}\alpha \frac{\partial}{\partial x}(\overline{u}u_x - u\overline{u}_x)\,.$$

Suppose $|u|^2$ can be integrated over an $x-$interval in such a way that $(\overline{u}u_x - u\overline{u}_x)$ vanishes at the boundary of the interval. Then

$$\frac{d}{dt}\int |u|^2\,\mathrm{d}x = 0\,.$$

The second one is the energy conservation. Define the energy density $E(x,t)$ and the energy flux $\mathcal{F}(x,t)$ by

$$E(x,t) = \tfrac{1}{2}\alpha|u_x|^2 - \tfrac{1}{4}\gamma|u|^4 \tag{4.5}$$

$$\mathcal{F}(x,t) = -\tfrac{1}{2}\alpha(\overline{u_x}u_t + u_x\overline{u_t})\,. \tag{4.6}$$

Note that the energy is indefinite : the integral of $E(x,t)$ over x can be negative or positive, depending on the initial data. A calculation shows that the energy conservation law is

$$\frac{\partial E}{\partial t} + \frac{\partial \mathcal{F}}{\partial x} = 0\,. \tag{4.7}$$

Integrating over x and assuming that the boundary conditions are such that $\int \partial\mathcal{F}/\partial x\,\mathrm{d}x = 0$ yields

$$\frac{d}{dt}\int \left(\tfrac{1}{2}\alpha|u_x|^2 - \tfrac{1}{4}\gamma|u|^4\right)\,\mathrm{d}x = 0\,.$$

5 An NLS model for the Benjamin–Feir instability

The discovery of the Benjamin–Feir (BF) instability of Stokes travelling waves was a milestone in the history of water waves. It is due to the seminal work of Benjamin and Feir (1967) which combined experimental evidence with a weakly nonlinear theory. Mathematically, the BF instability can be characterized as a collision of two pairs of purely imaginary eigenvalues of opposite energy sign (see Figure 3 for a schematic). For the full water-wave problem, there is in addition to the four eigenvalues shown in Figure 3 a countable number of stable purely imaginary eigenvalues. This point of view of the BF instability was a byproduct of the proof of the BF instability provided by Bridges and Mielke (1995). It also appears in the nonlinear Schrödinger model, where it can be shown explicitly. One of the pairs of modes in the collision has negative energy (relative to the basic state) and one pair has positive energy.

As said above, the NLS equation describes the modulations of weakly nonlinear, deep-water gravity waves, with basic wave number k_0 and frequency $\omega_0(k_0)$:

$$\mathrm{i}\,(A_T + c_g A_X) - \frac{\omega_0}{8k_0^2}A_{XX} - \frac{1}{2}\omega_0 k_0^2|A|^2 A = 0, \tag{5.1}$$

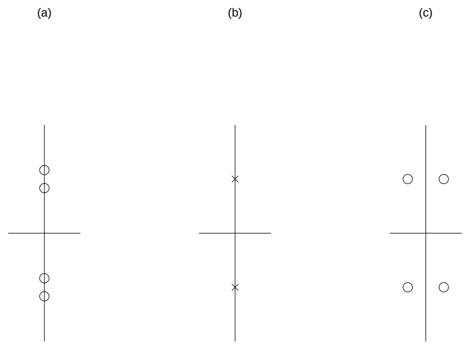

Figure 3. Schematic of the eigenvalue movement associated with the Benjamin–Feir instability. The circles indicate simple eigenvalues, while the crosses indicate double eigenvalues.

where A varies slowly in X and T, and $c_g = \omega_0/(2k_0)$ is the group velocity. At leading order, the free-surface elevation $\eta(x,t)$ is given by

$$\eta(x,t) = A(X,T)\mathrm{e}^{i(k_0 x - \omega_0 t)} + \text{ c.c.}. \tag{5.2}$$

In a frame of reference moving with the group velocity, equation (5.1) becomes

$$\mathrm{i}\, A_T - \frac{\omega_0}{8k_0^2} A_{\xi\xi} - \frac{1}{2}\omega_0 k_0^2 |A|^2 A = 0\,, \quad \text{with} \quad \xi = X - c_g T\,. \tag{5.3}$$

By scaling T, ξ and changing $\overline{A}$ into u, one obtains the normalized equation (4.1), where α and γ are taken to be positive.

The basic travelling wave solution Equation (4.1) admits travelling wave solutions. Let $\theta(x,t) = kx + \omega t + \Theta$, and consider the basic travelling wave solution to (4.2)

$$\widehat{\mathbf{U}}(x,t) = R_{\theta(x,t)}\mathbf{U}_0\,, \quad R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}. \tag{5.4}$$

Then $\mathbf{U}_0, \omega, k$ satisfy

$$\omega + \alpha k^2 = \gamma \|\mathbf{U}_0\|^2\,. \tag{5.5}$$

Below we show that the condition for a BF instability of the basic travelling wave solution of (4.1) is $\alpha\gamma > 0$, a condition satisfied by deep water waves. The BF instability does not occur in the defocussing NLS equation ($\alpha\gamma < 0$).

Formulating the conservative BF stability problem Linearize the partial differential equation (4.2) about the basic travelling wave (5.4). Let $\mathbf{U}(x,t) = R_{\theta(x,t)}(\mathbf{U}_0 + \mathbf{V}(x,t))$, substitute into the governing equation (4.2), linearize about $\mathbf{U}_0$, and simplify using (5.5). One gets

$$J\mathbf{V}_t + 2\alpha k J\mathbf{V}_x + \alpha\mathbf{V}_{xx} + 2\gamma\langle\mathbf{U}_0, \mathbf{V}\rangle\mathbf{U}_0 = \mathbf{0}\,, \tag{5.6}$$

where $\langle\cdot,\cdot\rangle$ is the standard scalar product on $\mathbb{R}^2$.

The class of solutions of interest are solutions which are periodic in x with wavenumber σ. The parameter σ represents the sideband. The BF instability will be associated with the limit $|\sigma| \ll 1$. Therefore let

$$\mathbf{V}(x,t) = \tfrac{1}{2}\mathbf{V}_0(t) + \sum_{n=1}^{\infty}(\mathbf{V}_n(t)\cos n\sigma x + \mathbf{W}_n(t)\sin n\sigma x)\,.$$

The $\sigma-$independent mode satisfies

$$J\partial_t\mathbf{V}_0 + 2\gamma\mathbf{U}_0\mathbf{U}_0^T\mathbf{V}_0 = \mathbf{0}\,. \tag{5.7}$$

Letting $\mathbf{V}_0(t) = \mathrm{e}^{\lambda t}\widetilde{\mathbf{V}}_0$, it is easy to verify that λ satisfies $\lambda^2 = 0$. All solutions of this equation are neutral, and therefore do not contribute to instability. The $\sigma-$independent modes are associated with the *superharmonic* instability which is known to be stable at low amplitude.

The $\sigma-$dependent modes decouple into 4$-$dimensional subspaces for each n, and satisfy

$$\begin{aligned} J\partial_t\mathbf{V}_n + 2\alpha k n\sigma J\mathbf{W}_n - \alpha(n\sigma)^2\mathbf{V}_n + 2\gamma\mathbf{U}_0\mathbf{U}_0^T\mathbf{V}_n &= \mathbf{0} \\ J\partial_t\mathbf{W}_n - 2\alpha k n\sigma J\mathbf{V}_n - \alpha(n\sigma)^2\mathbf{W}_n + 2\gamma\mathbf{U}_0\mathbf{U}_0^T\mathbf{W}_n &= \mathbf{0}\,. \end{aligned}$$

When the amplitude $\|\mathbf{U}_0\| = 0$ it is easy to show that all eigenvalues of the above system (i.e. taking solutions of the form $\mathrm{e}^{\lambda t}$ and computing λ) are purely imaginary. Considering all other parameters fixed, and increasing $\|\mathbf{U}_0\|$, we find that there is a critical amplitude where the $n = 1$ mode becomes unstable through a collision of eigenvalues of opposite signature.

To analyze this instability, take $n = 1$ and study the reduced four dimensional system

$$\begin{aligned} J\partial_t\mathbf{V}_1 + 2\alpha k\sigma J\mathbf{W}_1 - \alpha\sigma^2\mathbf{V}_1 + 2\gamma\mathbf{U}_0\mathbf{U}_0^T\mathbf{V}_1 &= \mathbf{0} \\ J\partial_t\mathbf{W}_1 - 2\alpha k\sigma J\mathbf{V}_1 - \alpha\sigma^2\mathbf{W}_1 + 2\gamma\mathbf{U}_0\mathbf{U}_0^T\mathbf{W}_1 &= \mathbf{0}\,. \end{aligned}$$

To determine the spectrum, let $(\mathbf{V}_1,\mathbf{W}_1) = (\mathbf{Q},\mathbf{P})\mathrm{e}^{\lambda t}$. Then $(\lambda,\mathbf{Q},\mathbf{P})$ satisfy

$$\begin{bmatrix} \lambda J - \alpha\sigma^2 I + 2\gamma\mathbf{U}_0\mathbf{U}_0^T & 2\alpha k\sigma J \\ -2\alpha k\sigma J & \lambda J - \alpha\sigma^2 I + 2\gamma\mathbf{U}_0\mathbf{U}_0^T \end{bmatrix}\begin{pmatrix}\mathbf{Q}\\ \mathbf{P}\end{pmatrix} = \begin{pmatrix}\mathbf{0}\\ \mathbf{0}\end{pmatrix}\,. \tag{5.8}$$

Denote the determinant of the left hand side by $\Delta(\lambda,\sigma)$. Then a calculation shows that

$$\Delta(\lambda,\sigma) = \lambda^4 + 2(p^2 + 4k^2\alpha^2\sigma^2)\lambda^2 + (p^2 - 4k^2\alpha^2\sigma^2)^2\,,$$

where

$$p^2 = \alpha^2\sigma^4 - 2\alpha\gamma\|\mathbf{U}_0\|^2\sigma^2\,.$$

Suppose $p^2 > 0$, then all four roots are purely imaginary (see Figure 3a) and given by

$$\lambda = \mathrm{i}2\alpha k\sigma \pm \mathrm{i}p\,,\ -\mathrm{i}2\alpha k\sigma \pm \mathrm{i}p\,.$$

These modes are purely imaginary as long as $p^2 > 0$ or

$$2\gamma\alpha\|\mathbf{U}_0\|^2 < \alpha^2\sigma^2 .$$

Since $\alpha\gamma > 0$, the instability threshold is achieved when the amplitude reaches

$$\|\mathbf{U}_0\| = \frac{|\alpha\sigma|}{\sqrt{2\alpha\gamma}} . \tag{5.9}$$

At this threshold, a collision of eigenvalues occurs at the points $\lambda = \pm 2\mathrm{i}k\alpha\sigma$ (see Figure 3b).

Signature of the colliding modes Purely imaginary eigenvalues of a Hamiltonian system have a signature associated with them, and this signature is related to the sign of the energy (cf. Cairns (1979), MacKay and Saffman (1986), Bridges (1997)). Collision of eigenvalues of opposite signature is a necessary condition for the collision resulting in instability.

It is straightforward to compute the signature of the modes in the NLS model. Suppose that the amplitude $\|\mathbf{U}_0\|$ of the basic state is smaller than the critical value (5.9) for instability. Then there are two pairs of purely imaginary eigenvalues, and they each have a signature. Let us concentrate on the eigenvalues on the positive imaginary axis

$$\lambda = \mathrm{i}\Omega_\pm \quad \text{with} \quad \Omega_\pm = 2k\sigma\alpha \pm p . \tag{5.10}$$

Then

$$\mathsf{Sign}(\Omega_\pm) = \mathrm{i}\langle\overline{\mathbf{Q}}, J\mathbf{Q}\rangle + \mathrm{i}\langle\overline{\mathbf{P}}, J\mathbf{P}\rangle ,$$

where the inner product is real in order to make the conjugation explicit, and $(\mathbf{Q},\mathbf{P})$ satisfy (5.8). One can also show that this signature has the same sign as the energy perturbation restricted to this mode.

The eigenvalue problem for $\Omega_\pm$ is (5.8) with λ replaced by $\mathrm{i}\Omega_\pm$. From the system (5.8) it follows that

$$2\alpha k\sigma\mathbf{P} = \left[-\mathrm{i}\Omega_\pm I - \alpha\sigma^2 J + 2\gamma J\mathbf{U}_0\mathbf{U}_0^T\right]\mathbf{Q} .$$

First compute the case where $\|\mathbf{U}_0\| = 0$, then

$$2\alpha k\sigma\mathbf{P} = \left[-\mathrm{i}\Omega_\pm I - \alpha\sigma^2 J\right]\mathbf{Q} ,$$

and so

$$(2\alpha k\sigma)^2\mathsf{Sign}(\Omega_\pm) = \mathrm{i}\left(2\Omega_\pm^2 \mp 2\alpha\sigma^2(2\alpha k\sigma)\right)\langle\overline{\mathbf{Q}}, J\mathbf{Q}\rangle - 2\alpha\Omega_\pm\sigma^2\langle\overline{\mathbf{Q}}, \mathbf{Q}\rangle .$$

After some computation it is found that

$$\mathbf{Q}_\pm = \begin{pmatrix} 1 \\ \pm\mathrm{i} \end{pmatrix} , \quad \text{associated with the eigenvalue} \quad \lambda = \mathrm{i}\Omega_\pm ,$$

and so $\langle \overline{\mathbf{Q}}, \mathbf{Q} \rangle = 2$ and $\langle \overline{\mathbf{Q}}, J\mathbf{Q} \rangle = \mp 2\mathrm{i}$. Substitution into the expression for signature yields

$$\mathsf{Sign}(\Omega_\pm) = \pm 4\,.$$

Hence the two modes which ultimately collide have opposite signature when $\mathbf{U}_0 = \mathbf{0}$. Since p^2 decreases as the amplitude increases, the two modes will have opposite signature for all $\|\mathbf{U}_0\|$ between $\|\mathbf{U}_0\| = 0$ and the point of collision.

6 The effect of dissipation on the Benjamin–Feir instability

Having established that there is a collision between modes of opposite signature, we can appeal to the well-known result that dissipation ***always destabilizes*** negative energy modes (Cairns (1979); Craik (1988); MacKay (1991)) to conclude that dissipation enhances the BF instability. But one has to be a bit careful. For example, Segur et al. (2004) recently studied the perturbed NLS equation with weak damping of the form

$$\mathrm{i}\, u_t + \alpha\, u_{xx} + \gamma\, |u|^2 u + \mathrm{i}\delta u = 0\,, \quad u \in \mathbb{C}\,, \tag{6.1}$$

where $\delta \geq 0$.

The perturbed NLS equation takes the form

$$J\mathbf{U}_t + \alpha \mathbf{U}_{xx} + \gamma \|\mathbf{U}\|^2 \mathbf{U} + \delta J\mathbf{U} = \mathbf{0}\,, \tag{6.2}$$

or

$$J\mathbf{U}_t = \nabla H(\mathbf{U}) - \delta J\mathbf{U}\,, \tag{6.3}$$

with H defined by (4.4).

Looking for travelling wave solutions of this perturbed equation leads to

$$\left[(-\omega - \alpha k^2 + \gamma \|\mathbf{U}_0\|^2) I + \delta J\right] \mathbf{U}_0 = \mathbf{0}\,.$$

The main theoretical result of Segur et al. (2004) is that *a uniform train of plane waves of finite amplitude is stable for any physical system modeled by (6.1). Dissipation, no matter how small, stabilizes the Benjamin–Feir instability.*

The cornerstone of their argument is the δ−perturbed conservation law for $|u(x,t)|^2$,

$$\frac{\partial}{\partial t}|u|^2 = \mathrm{i}\alpha \frac{\partial}{\partial x}(\overline{u} u_x - u \overline{u}_x) - 2\delta |u|^2\,.$$

As above suppose $|u|^2$ can be integrated over an x−interval in such a way that $(\overline{u} u_x - u\overline{u}_x)$ vanishes at the boundary of the interval. Then

$$\frac{d}{dt} \int |u|^2 \,\mathrm{d}x = -2\delta \int |u|^2 \,\mathrm{d}x\,.$$

Clearly all solutions are damped no matter how small δ is. This argument is however a global argument. By integrating over space, all x−distribution and modal distribution of $|u|^2$ are smoothed out by the averaging (integration) process. From this global view it is impossible to determine the interplay between the various modes. It is shown below that in fact there is an interplay between the modes. Incidentally, it can be shown easily by using the δ−perturbed conservation law for the energy E (4.5) that the energy can grow with time (at best it is bounded from below).

A model problem Consider the following prototype for a linear ordinary differential equation on $\mathbb{R}^4$ with a collision of eigenvalues of opposite signature, and Rayleigh damping,

$$\mathbf{Q}_{tt} + 2bJ\mathbf{Q}_t + (a^2 - b^2)\mathbf{Q} + 2\delta\mathbf{Q}_t = \mathbf{0}\,, \tag{6.4}$$

where $\delta \geq 0$ represents the Rayleigh damping coefficient, a and b are real parameters and b is the "gyroscopic coefficient". Let

$$\mathbf{Q}(t) = \widehat{\mathbf{Q}}\mathrm{e}^{\lambda t}\,,$$

then substitution into (6.4) leads to the characteristic polynomial

$$\Delta(\lambda) = (\lambda^2 + 2\delta\lambda - b^2 + a^2)^2 + 4b^2\lambda^2 = 0\,,$$

which can be factored into

$$[(\lambda + \mathrm{i}b)^2 + a^2 + 2\delta\lambda] \times [(\lambda - \mathrm{i}b)^2 + a^2 + 2\delta\lambda] = 0\,.$$

When $\delta = 0$ there are four roots $\lambda = \pm\mathrm{i}(b \pm a)$. Suppose a is small and positive (just before the collision) and look at the effect of dissipation on the two modes

$$\lambda_0 = \mathrm{i}b \pm \mathrm{i}a\,.$$

Let

$$\lambda(\delta) = \lambda_0 + \delta\lambda_1 + 0(\delta^2)\,.$$

Then substitution into the characteristic polynomial leads to

$$\lambda_1 = \mp\frac{1}{a}(b \pm a)\,,$$

and so

$$\lambda(\delta) = \mathrm{i}b \pm \mathrm{i}a \mp \frac{\delta}{a}(b \pm a) + \mathcal{O}(\delta^2)\,.$$

In other words, the negative energy mode $\lambda = \mathrm{i}(b - a)$ has positive real part when dissipatively perturbed, and the positive energy mode $\lambda = \mathrm{i}(b + a)$ has negative real part under perturbation. Consequently, when small dissipation is added the mode with negative energy will become unstable before the collision, and after the collision it will have a larger growth rate. This result is generic, and depends only on the property that the system has an instability arising through a collision of eigenvalues of opposite signature.

Effect of dissipation on the NLS model of BF instability Consider the $n = 1$ model reduced system for the BF instability with the type of damping introduced in (6.1) :

$$\begin{aligned} J\partial_t\mathbf{V}_1 + 2\alpha k\sigma J\mathbf{W}_1 - \alpha\sigma^2\mathbf{V}_1 + 2\gamma\mathbf{U}_0\mathbf{U}_0^T\mathbf{V}_1 + \delta J\mathbf{V}_1 &= \mathbf{0} \\ J\partial_t\mathbf{W}_1 - 2\alpha k\sigma J\mathbf{V}_1 - \alpha\sigma^2\mathbf{W}_1 + 2\gamma\mathbf{U}_0\mathbf{U}_0^T\mathbf{W}_1 + \delta J\mathbf{W}_1 &= \mathbf{0}\,. \end{aligned}$$

This form of damping is very special. It can be factored out by the transformation

$$(\mathbf{V}_1, \mathbf{W}_1) = (\widetilde{\mathbf{V}_1}, \widetilde{\mathbf{W}_1})\mathrm{e}^{-\delta t}\,.$$

Let's modify this damping slightly to break the symmetry, and to make it closer in spirit to Rayleigh damping, which is a much more widely used form of damping in mechanics.

Also without loss of generality, set $\|\mathbf{U}_0\| = 0$ since the negative energy and positive energy modes exist already then, and the analysis is exactly the same (but more complicated) for the case $\|\mathbf{U}_0\| > 0$,

$$\begin{aligned} J\partial_t\mathbf{V}_1 + 2\alpha k\sigma J\mathbf{W}_1 - \alpha\sigma^2\mathbf{V}_1 + \delta J\mathbf{V}_1 &= \mathbf{0} \\ J\partial_t\mathbf{W}_1 - 2\alpha k\sigma J\mathbf{V}_1 - \alpha\sigma^2\mathbf{W}_1 + \delta J\mathbf{W}_1 + \frac{\delta^2}{2\alpha k\sigma}J\mathbf{V}_1 - \delta\frac{\sigma}{k}\mathbf{V}_1 &= \mathbf{0}\,. \end{aligned}$$

While the two extra terms in the second equation may look unusual, when the two equations are combined into a single second order equation we find

$$\partial_{tt}\mathbf{V}_1 + 2\alpha\sigma^2 J\partial_t\mathbf{V}_1 + (4\alpha^2k^2\sigma^2 - \alpha^2\sigma^4)\mathbf{V}_1 + 2\delta\partial_t\mathbf{V}_1 = \mathbf{0}\,,$$

which is precisely of the form (6.4), and so we can conclude immediately that dissipation *enhances* the BF instability in the NLS model! More details can be found in Bridges and Dias (2004).

7 Numerical integration of the NLS equation

The experimental and theoretical investigation of Lake et al. (1977) showed that the evolution of a 2D nonlinear wave train on deep water, in the absence of dissipative effects, exhibits the Fermi–Pasta–Ulam (FPU) recurrence phenomenon. This phenomenon is characterized by a series of modulation–demodulation cycles in which initially uniform wave trains become modulated and then demodulated until they are again uniform. As explained in Section 5, modulation is caused by the growth of the two dominant sidebands of the Benjamin–Feir instability at the expense of the carrier. During the demodulation the energy returns to the components of the original wave train (carrier, sidebands, harmonics). The FPU recurrence process is well described by the focusing 1D NLS equation. Lake et al. (1977) successfully solved the NLS equation numerically and found that the instability does not grow unboundedly as expected in linear theory, but that the conservation laws satisfied by the NLS equation tend to inhibit the growth. The result is that the solution returns to the initial condition periodically.

In the numerical study of the NLS equation there are two types of solutions which attract much interest : solitons, in which the solution and its spatial derivatives vanish at infinity (see for example Zakharov and Shabat (1971) for exact soliton solutions), and solutions that describe modulational instability and recurrence. Periodic boundary conditions are then appropriate. We focus on these solutions. Therefore we restrict ourselves to L-periodic solutions defined by

$$u(x+L,t) = u(x,t), \quad -\infty < x < \infty, \;\; t > 0\,. \tag{7.1}$$

Without loss of generality, we consider equation (4.1) with $\alpha = 1$. We have seen in Section 5 that an instability may occur if $\gamma > 0$. If

$$0 < n^2\sigma^2 < 2\gamma\|\mathbf{U}_0\|^2 ,$$

two of the eigenvalues have positive real part and lead to exponential growth.

7.1 The split-step method

For this subsection, we follow closely the paper by Weideman and Herbst (1986). The solution of (4.1) may be advanced from one time level to the next by means of the formula

$$u(x, t+\tau) \approx \exp \mathrm{i}\tau(\mathcal{L} + \mathcal{N}(u)) \cdot u(x,t) , \tag{7.2}$$

where τ denotes the time step. In general (7.2) is first order accurate, but it is exact if $|u|^2$ is time-independent.

The time-splitting procedure consists of replacing the right-hand side of (7.2) by

$$\exp \mathrm{i}\tau(\mathcal{L} + \mathcal{N}(u)) \cdot u(x,t) \approx \exp \mathrm{i}\tau\mathcal{L} \cdot \exp \mathrm{i}\tau\mathcal{N}(u) \cdot u(x,t) .$$

This expression is exact whenever $\mathcal{L}$ and $\mathcal{N}$ commute. Otherwise the splitting is first order accurate. What the splitting does is to solve successively the equations

$$u_t = \mathrm{i}\,\mathcal{N}(u)u , \quad u_t = \mathrm{i}\,\mathcal{L}u ,$$

where the solution of the former equation is used as initial condition for the latter.

From now on, U denotes the approximation to $u(x,t)$. Introducing the quantity

$$V^m = \exp \mathrm{i}\tau\mathcal{N}(U^m) \cdot U^m , \tag{7.3}$$

where U^m denotes the approximation at the time $m\tau$, the split-step scheme can be written as

$$U^{m+1} = \exp \mathrm{i}\tau\mathcal{L} \cdot V^m . \tag{7.4}$$

For the discretization of the space variable, one can use finite differences or a Fourier method. We give details for the Fourier method. The first step is to replace $V^m(x)$ and $U^{m+1}(x)$ in (7.4) by their Fourier series. The Fourier-series of a L-periodic function $w(x)$ is given by

$$w(x) = \sum_{n=-\infty}^{\infty} \hat{w}_n \exp(\mathrm{i}n\sigma x) ,$$

with $\sigma = 2\pi/L$. The Fourier coefficients $\hat{w}_n$ are given by

$$\hat{w}_n = \frac{1}{L}\int_{-L/2}^{L/2} w(x) \exp(-\mathrm{i}n\sigma x)\mathrm{d}x .$$

This yields

$$\hat{u}_n^{m+1} = \exp(-\mathrm{i}n^2\sigma^2\tau)\hat{v}_n^m , \quad n \in \mathbb{Z} , \tag{7.5}$$

where the $\hat{v}_n^m$ and $\hat{u}_n^{m+1}$ are the Fourier coefficients of the continuous functions $V^m(x)$ and $U^{m+1}(x)$, respectively. The discretization of equation (7.5) gives, after replacing $\hat{v}_n^m$ by $\hat{V}_n^m$ and $\hat{u}_n^{m+1}$ by $\hat{U}_n^{m+1}$,

$$\begin{aligned}\hat{V}_n^m &= \frac{1}{N}\sum_j V_j^m \exp(-\mathrm{i}n\sigma x_j)\,, \quad n = -N/2, \cdots, N/2-1 \\ \hat{U}_n^{m+1} &= \exp(-\mathrm{i}n^2\sigma^2\tau)\hat{V}_n^m\,, \quad n = -N/2, \cdots, N/2-1\,.\end{aligned}$$

The intermediate solution V_j^m is given by

$$V_j^m = \exp(\mathrm{i}\tau\gamma|U_j^m|^2)\cdot U_j^m\,.$$

Finally the approximation at the next time level is calculated from the inverse transform

$$U_j^{m+1} = \sum_n \hat{U}_n^{m+1}\exp(\mathrm{i}n\sigma x_j)\,, \quad j = -N/2, \cdots, N/2-1\,.$$

In the numerical computations, Fast Fourier Transforms are used.

Weideman and Herbst (1986) showed that a necessary condition for the solution to be stable again high frequency perturbation in the case $\gamma > 0$ is

$$\tau < \frac{L^2}{\pi N^2}\,.$$

In the case $\gamma < 0$, a necessary condition is

$$\frac{\tau}{2\theta+\pi} < \frac{L^2}{\pi^2 N^2}\,,$$

where

$$\theta = \sin^{-1}\left(\frac{\gamma\|\mathbf{U}_0\|^2\tau}{[1+\gamma^2\|\mathbf{U}_0\|^4\tau^2]^{1/2}}\right)\,.$$

These two conditions prevent any unwanted high frequency instabilities in the numerical solution.

7.2 Numerical results

We show some numerical results, based on the x-independent solution of the NLS equation

$$u(x,t) = \frac{1}{2}\exp\left(\mathrm{i}\frac{\gamma t}{4}\right)\,,$$

with the initial condition $u(x,0) = \frac{1}{2}$. Perturbations of the initial condition will take the following form :

$$u(x,0) = \tfrac{1}{2}\left[1 + 0.1(1 - 2|x|/L)\right]\,.$$

Energy is introduced into all modes.

In Figures 4 and 5, the only difference is the time step τ. The critical value for τ is $\tau^* = 0.0905$. In Figure 4, the numerical stability condition is satisfied ($\tau < \tau^*$). The only

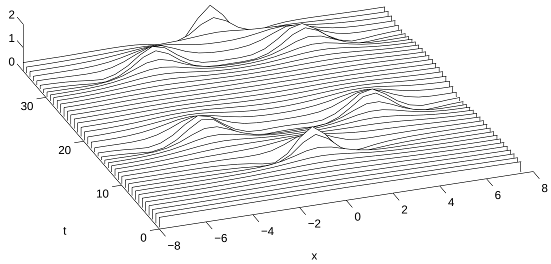

Figure 4. Illustration of the recurrence phenomenon. Split-step Fourier solution of the NLS equation with initial condition $u(x,0) = 0.5[1 + 0.1(1 - |x|/8)]$, $-8 \le x \le 8$. The parameters are $\gamma = 2$, $L = 16$, $\tau = 0.09$. The number of points is $N = 30$.

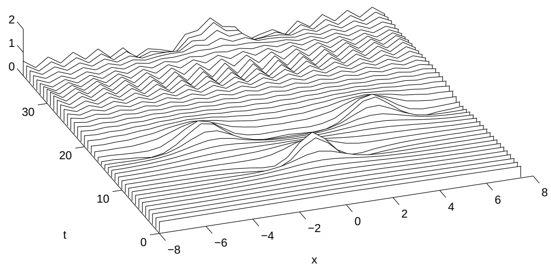

Figure 5. Same as Figure 4 with $\tau = 0.091$.

unstable modes are the two modes of lowest frequency. Most of the energy is transferred between the unstable modes with intermittent returns to the initial condition. In Figure 5 ($\tau > \tau^*$), there are in addition to the two analytically unstable modes corresponding to $|n| = 1$ and $|n| = 2$ a high frequency numerical instability corresponding to $|n| = 15$. After the return to the initial condition, the high frequency component suddenly becomes unstable, resulting in the highly oscillatory appearance of the solution.

In Figure 6, $\gamma < 0$. If the numerical stability condition is satisfied, which is the case with $L = 8$, nothing happens as expected. If we take $L = 7.98$, numerical instability occurs. Figure 6 shows that the mode of highest frequency ($|n| = 10$) grows. But the solution returns to the initial state.

Some other numerical results are shown in Figures 7–9. The initial condition in Figure 7 leads to a homoclinic orbit. The initial conditions in Figures 8 and 9 are on opposite sides of the homoclinic orbit. See Ablowitz and Herbst (1990) for an explicit expression of the homoclinic solution and Osborne et al. (2000) for similar numerical results.

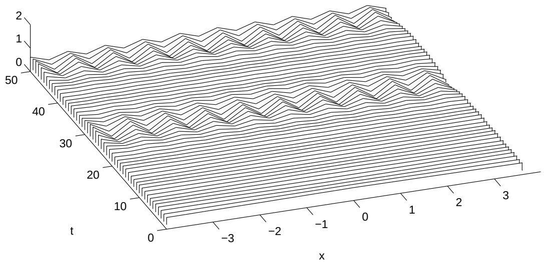

Figure 6. Same as Figure 4 with initial condition $u(x,0) = 0.5[1 + 0.1(1 - |x|/3.99)]$, $-3.99 \leq x \leq 3.99$. The parameters are $\gamma = -2$, $L = 7.98$, $\tau = 0.05$. The number of points is $N = 20$.

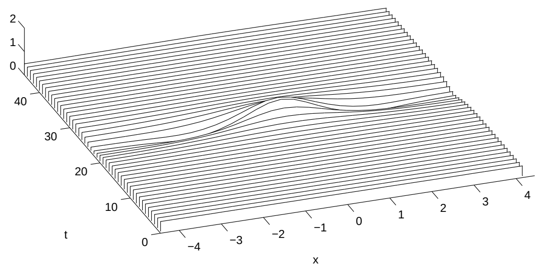

Figure 7. The initial condition is $u(x,0) = \frac{1}{2} + 10^{-5}(1+\mathrm{i})\cos(2\pi x/L)$, with $L = 2\sqrt{2}\pi$. It leads to a homoclinic orbit. The parameters are $\gamma = 2$, $\tau = 0.05$. The number of points is $N = 30$.

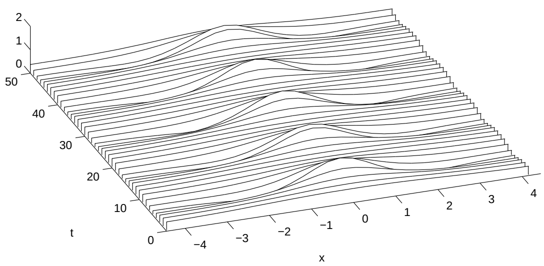

Figure 8. The initial condition is $u(x,0) = \frac{1}{2} + 0.1\cos(\sigma x)$, with $L = 2\sqrt{2}\pi$ and $\sigma = 2\pi/L$. The term $\cos(\sigma x)$ appears periodically. The other parameters are the same as in Figure 7.

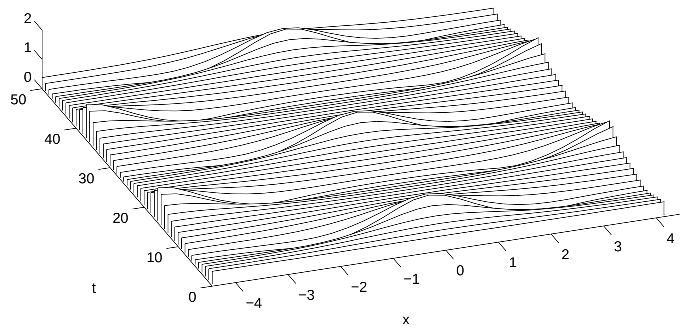

Figure 9. The initial condition is $u(x,0) = \frac{1}{2} + 0.1\ \mathrm{i}\cos(\sigma x)$, with $L = 2\sqrt{2}\pi$ and $\sigma = 2\pi/L$. Interchanges between $\cos(\sigma x)$ and $\cos\sigma(x + \frac{1}{2}L)$ take place.

8 Bifurcation of waves when the phase and group velocities are nearly equal

As shown in Section 2, the dispersion relation for 2D capillary–gravity waves on the surface of a deep layer of water is given by

$$c^2 = g\frac{1}{k} + \frac{\sigma}{\rho}k\,. \tag{8.1}$$

A trivial property of this dispersion relation is that it exhibits a minimum $c_{\min}$, but surprisingly it was only quite recently that some of its consequences were discovered. In the classic textbooks such as Lamb (1932), pp. 462–468; Lighthill (1978), pp. 260–269; Whitham (1974), pp. 407–408, pp. 446–454; Milne-Thomson (1968), pp. 447–449; Stoker (1958), the presence of this minimum is of course mentioned. It represents a real difficulty for linearized versions of the water-wave problem. For example, consider the fishing-rod problem, in which a uniform current is perturbed by an obstacle. Rayleigh (1883) investigated this problem. He assumed a distribution of pressure of small magnitude and linearized the equations around a uniform stream with constant velocity c. He solved the resulting linear equations in closed form. For $c = c_1$ ($> c_{\min}$), the solutions are characterized by trains of waves in the far field of wavenumbers k_2 and $k_1 < k_2$. The waves corresponding to k_1 and k_2 appear behind and ahead of the obstacle, respectively. The asymptotic wave trains are given by

$$\begin{aligned}\eta &\sim -\frac{2\rho P}{(k_2 - k_1)\sigma}\sin(k_1 x)\,, \quad x > 0\,,\\ \eta &\sim -\frac{2\rho P}{(k_2 - k_1)\sigma}\sin(k_2 x)\,, \quad x < 0\,,\end{aligned}$$

where P is the integral of pressure. For $c < c_{\min}$, Rayleigh's solutions do not predict waves in the far field, and the flow approaches a uniform stream with constant velocity

c at infinity. This is consistent with the fact that equation (8.1) does not have real roots for k when $c < c_{\min}$. Rayleigh's solution is accurate for $c \neq c_{\min}$ in the limit as the magnitude of the pressure distribution approaches zero. However it is not uniform as $c \to c_{\min}$. It is clear that the linearized theory fails as one approaches $c_{\min}$: the two wavenumbers k_1 and k_2 merge, the denominators approach zero, and the displacement of the free surface becomes unbounded. Therefore there was a clear need for a better understanding of the limiting process, but for several decades this problem was left untouched. In the late 80's and early 90's, several researchers worked on the nonlinear version of this problem independently. Longuet-Higgins (1989) indirectly touched upon this problem with numerical computations. Iooss and Kirchgässner (1990) tackled the problem mathematically. Vanden-Broeck and Dias (1992) made the link between the numerical computations of Longuet-Higgins (1989) and the mathematical analysis of Iooss and Kirchgässner (1990). One can say that the mathematical results shed some light on the difficulty : there is a difference between a temporal approach and a spatial approach. Roughly speaking, in temporal bifurcation theory the wavenumber k is treated as a given real parameter while in spatial bifurcation theory the wavespeed c is treated as a given real parameter. Again the best way to understand what happens is to consider a model equation. The appropriate model equation is the NLS equation for the amplitude A of a modulated wave train :

$$2\mathrm{i}\frac{\partial A}{\partial t_2} + p\frac{\partial^2 A}{\partial X^2} + q\frac{\partial^2 A}{\partial y_1^2} + \gamma A|A|^2 = 0\,. \tag{8.2}$$

Akylas (1993) and Longuet-Higgins (1993) showed that, for values of c less than $c_{\min}$, equation (8.2) admits particular envelope-soliton solutions, such that the wave crests are stationary in the reference frame of the wave envelope. These solitary waves, which bifurcate from linear periodic waves at the minimum value of the phase speed, have decaying oscillatory tails and are sometimes called "bright" solitary waves. More generally, one can look for stationary solutions of equation (8.2). Using $\sigma/\rho c^2$ as unit length and $\sigma/\rho c^3$ as unit time, allowing for interfacial waves (i.e. waves propagating at the interface between a heavy fluid of density ρ_1 and a lighter fluid of density ρ_2), considering waves without y_1-variations and evaluating the coefficients p and γ at $c = c_{\min}$, one can show that these stationary solutions satisfy the equation

$$-\frac{2r}{1+r}\,\mu A + A_{XX} + \frac{16r^2-5}{2(1+r)^2}\,A|A|^2 = 0\,, \tag{8.3}$$

where the bifurcation parameter μ is defined by $\mu = \alpha - \alpha_{\min}$, with $\alpha = g\sigma/\rho c^4$ and $\alpha_{\min} = g\sigma/\rho c_{\min}^4 = 1/2r(1+r)$. The parameter r is the density ratio $(\rho_1 - \rho_2)/(\rho_1 + \rho_2)$. Note that the coefficient of the cubic term vanishes when $r = r_0 = \sqrt{5}/4$. The corresponding profile for the modulated wave is given by

$$\eta(X) = (1+r)\left[A(X)\exp\left(\mathrm{i}X/(1+r)\right) + \text{ c.c.}\right].$$

Introduce the scaling $|\mu|^{\frac{1}{2}}\tilde{A} = A\,, \tilde{X} = |\mu|^{\frac{1}{2}}\left(2r/(1+r)\right)^{\frac{1}{2}} X$, and the coefficient

$$\tilde{\gamma} = \frac{16r^2-5}{4r(1+r)}\,.$$

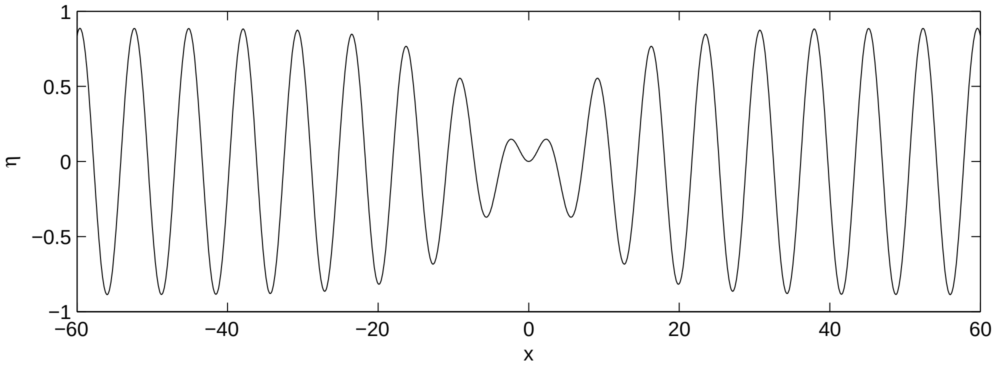

Figure 10. Profile of a dark solitary wave, given by (8.9), with $r = 0.15$ and $\mu = 0.05$.

The resulting equation is

$$\mathsf{Sign}(\mu)\tilde{A} - \tilde{A}_{\tilde{X}\tilde{X}} - \tilde{\gamma}\tilde{A}|\tilde{A}|^2 = 0\,. \tag{8.4}$$

Writing $\tilde{A} = s(\tilde{X})\,\mathrm{e}^{i\theta(\tilde{X})}$ leads to

$$s_{\tilde{X}\tilde{X}} - \mathsf{Sign}(\mu)s + \tilde{\gamma}s^3 - s(\theta_{\tilde{X}})^2 \;=\; 0\,, \tag{8.5}$$

$$2\theta_{\tilde{X}}s_{\tilde{X}} + s\theta_{\tilde{X}\tilde{X}} \;=\; 0\,. \tag{8.6}$$

The system has two first integrals I_1 and I_2, defined as follows :

$$u\theta_{\tilde{X}} \;=\; I_1\,, \tag{8.7}$$

$$\tfrac{1}{4}(u_{\tilde{X}})^2 \;=\; \mathsf{Sign}(\mu)u^2 - \tfrac{1}{2}\tilde{\gamma}u^3 - I_1^2 + I_2 u\,, \tag{8.8}$$

where $u \equiv s^2$. These two integrals are related to the energy flux and flow force, respectively, as shown by Bridges et al. (1995).

For a full description of all the bounded solutions of equation (8.4), one can refer to Iooss and Pérouème (1993), Dias and Iooss (1993) or Dias and Iooss (1996). There are four cases to consider :

- $r < r_0$, $c > c_{\min}$: there are periodic solutions, quasiperiodic solutions and solitary waves, homoclinic to the same periodic wave with a phase shift at $+\infty$ and $-\infty$ (these homoclinic solutions are sometimes called “dark” solitary waves if the amplitude vanishes at the origin and “grey” solitary waves if it does not). A dark solitary wave is shown in Figure 10.
- $r < r_0$, $c < c_{\min}$: there are no bounded solutions.
- $r > r_0$, $c > c_{\min}$: there are periodic solutions and quasiperiodic solutions.
- $r > r_0$, $c < c_{\min}$: there are periodic solutions (of finite amplitude only), quasiperiodic solutions and solitary waves, homoclinic to the rest state (such a solitary wave is shown in Figure 11).

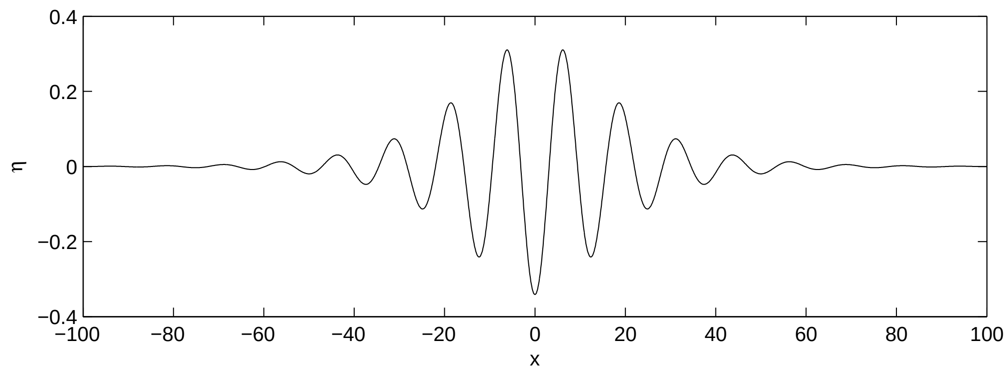

Figure 11. Profile of a bright solitary wave, given by (8.10), with $\mu = 0.005$.

When $r < r_0$ (defocussing case), $c > c_{\min}$ ($\mu < 0$), there is a one-parameter family of grey solitary waves (Dias and Iooss (1996), Laget and Dias (1997)). The "darkest" one, which is such that the amplitude vanishes at the origin, has an envelope given by

$$s = |\tilde{\gamma}|^{-\frac{1}{2}} \tanh\left(\frac{|\tilde{X}|}{\sqrt{2}}\right).$$

The elevation of the dark solitary wave is given by

$$\eta(X) = \pm 4\sqrt{\frac{r(1+r)^3}{5-16r^2}} \tanh\left(\sqrt{\frac{r}{1+r}}\, |\mu|^{\frac{1}{2}}\, X\right) \sin\left(\frac{1}{1+r}\, X\right). \tag{8.9}$$

When $r > r_0$ (focussing case), $c < c_{\min}$ ($\mu > 0$), there are bright solitary waves, the envelope of which is given by

$$\tilde{A} = \pm \frac{\sqrt{2}}{\sqrt{\tilde{\gamma}} \cosh \tilde{X}}.$$

This case includes water waves. The elevation of the solitary wave for water waves is given by

$$\eta(X) = \pm \frac{16\sqrt{\mu}}{\sqrt{11}} \frac{\cos(X/2)}{\cosh \sqrt{\mu} X}. \tag{8.10}$$

The results obtained on the NLS equation also apply to the full water-wave and interfacial wave problems. In particular envelope-soliton solutions have been studied in detail by Longuet-Higgins (1989), Vanden-Broeck and Dias (1992), Dias et al. (1996) and Dias and Iooss (1993).

Supporting the asymptotic and numerical studies cited above, Iooss and Kirchgässner (1990) provided a rigorous proof, based on center-manifold reduction, for the existence of small-amplitude symmetric solitary waves near the minimum phase speed in water of finite depth. The proof could not be extended to the infinite-depth case, however. Later, Iooss and Kirrmann (1996) handled this difficulty by following a different reduction procedure which also brought out the fact that the solitary-wave tails behave differently in water of infinite depth, their decay being slower than exponential, although the precise decay rate could not be determined. By assuming the presence of an algebraic decay, Sun (1997) was able to show that the profiles of interfacial solitary waves in deep fluids must decay like $1/x^2$ at the tails. We also remark that earlier Longuet-Higgins (1989) had inferred such a decay on physical grounds for deep-water solitary waves. Akylas, Dias and Grimshaw (1998) showed that the profile of these gravity–capillary solitary waves actually decays algebraically (like $1/x^2$) at infinity, owing to the induced mean flow that is not accounted for in the NLS equation.

The NLS equation also admits asymmetric solitary waves, obtained by shifting the carrier oscillations relative to the envelope of a symmetric solitary wave. Yang and Akylas (1997) examined the fifth-order Korteweg–de Vries equation, a model equation for gravity–capillary waves on water of finite depth, and showed by using techniques of exponential asymptotics beyond all orders that asymmetric solitary waves are not possible. On the other hand, an infinity of symmetric and asymmetric solitary waves, in the form of two or more NLS solitary wavepackets, exist at finite amplitude.

The implications of the study of the 1:1 resonance in the context of water waves have gone far beyond the field of surface waves. Applications have been given to all sorts of problems in physics, mechanics, thermodynamics and optics since these studies on water waves. See Dias and Iooss (2003) for a mathematical review.

9 The 2D "hyperbolic" NLS equation

As shown in Section 3, the 2D NLS equation that arises in the description of surface gravity waves on deep water is

$$\mathrm{i}\partial_t\psi + \partial_{xx}\psi - \partial_{yy}\psi + |\psi|^2\psi = 0\,. \tag{9.1}$$

This is the two-dimensional "hyperbolic" Schrödinger equation. Sulem and Sulem (1999) write that numerical integrations of (9.1) in spatially periodic domains show no tendency to collapse, and FPU recurrence is observed. However, in contrast with the focusing one-dimensional problem, this recurrence can only be approximate, since a small fraction of the energy is pumped to higher frequencies during each cycle because the region of unstable wave vectors is unbounded. In contrast, for the two-dimensional elliptic NLS equation, numerical evidence of finite-time blowup has been reported.

Osborne et al. (2000) write that the very existence of "coherent structures" in the 2D hyperbolic NLS equation (or "unstable modes" in the sense of the 1D NLS equation) has been left in doubt. They performed more numerical computations and concluded that unstable modes do indeed exist in the 2D hyperbolic NLS equation. They can take the form of large amplitude "rogue" waves. Onorato et al. (2001) extended these results to random initial conditions and concluded that freak waves are more likely to occur for large values of the Phillips parameter and the enhancement coefficient in the Joint North Sea Wave Project (JONSWAP) power spectrum.

10 Forced NLS equation

The NLS equation can still be used as a model when one considers water waves forced by an obstacle or by a pressure disturbance on the free surface. Let us for example consider capillary–gravity waves forced by a pressure distribution acting locally on the free surface and moving at a velocity slightly below the velocity corresponding to the minimum of the dispersion curve. In this regime, the linear analysis of Rayleigh (1883) is no longer valid.

The water wave problem has several formulations. One based on the velocity potential was presented in Section 2. One can also use a formulation based on the conservation of mass and the irrotationality condition. In 2D, the resulting equations are

$$u_x + w_z = 0, \quad u_z - w_x = 0\,. \tag{10.1}$$

In addition to the problem considered in Section 3, there is now a pressure distribution moving along the free surface at the speed c. Introduce nondimensional variables by taking c as unit velocity and $\sigma/\rho c^2$ as unit length. The boundary conditions in a frame of reference moving with the pressure distribution are given on the bottom by

$$w = 0, \quad \text{for } z = -1/b$$

and on the free-surface by

$$u\eta_x - w = 0, \quad \frac{1}{2}(u^2 + w^2 - 1) + b\lambda\eta - \frac{\eta_{xx}}{(1+\eta_x^2)^{3/2}} + \varepsilon p_0 = 0, \quad \text{for } z = \eta(x). \tag{10.2}$$

Here $\lambda = gh/c^2$ and $b = \sigma/\rho h c^2$ are dimensionless numbers. The dimensionless pressure is denoted by $\varepsilon p_0(x)$. The function p_0 is assumed to be of compact support and such that $\langle p_0 \rangle = \int_{-\infty}^{\infty} p_0(x)dx \neq 0$. The dispersion relation for capillary–gravity waves is given by

$$(b\lambda + k^2)\tanh(k/b) - k = 0\,, \tag{10.3}$$

where k is the dimensionless wavenumber. For any $k \in [0, 1/2]$, there exists a unique pair $\lambda(k), b(k)$ such that equation (10.3) admits a double root in k. The corresponding speed is $c_{\min}$.

The analysis is based on reformulating equations (10.1)-(10.2) as a dynamical system in x, on reducing the problem to its center manifold as in Kirchgässner (1988) and on putting the reduced system in normal form in the presence of a reversible 1:1 resonance as in Iooss and Pérouème (1993). Introducing $v_1 = (u^2+w^2-1)/2$, $v_2 = w/u$, v_0 the trace of v_2 on the free surface, $\varphi_0^{\pm}, \varphi_1^{\pm}$ the eigenvectors and generalized eigenvectors corresponding to the double imaginary eigenvectors $\pm ik$, and the parameter μ proportional to $c_{\min} - c$, one can write

$$(v_0, v_1, v_2) = A(x)\varphi_0^+ + B(x)\varphi_1^+ + \bar{A}(x)\varphi_0^- + \bar{B}(x)\varphi_1^- + \Phi(\mu, \varepsilon; x, A, B, \bar{A}, \bar{B})\,, \tag{10.4}$$

with A and B given by the normal form

$$A_x = \mathrm{i}kA + B + \mathrm{i}AP\left(\mu; |A|^2, \tfrac{1}{2}\mathrm{i}(A\bar{B} - \bar{A}B)\right) - \mathrm{i}\mathcal{P}_1\mathcal{P}_0\varepsilon p_0 + \cdots \tag{10.5}$$

$$B_x = \mathrm{i}kB + \mathrm{i}BP\left(\mu; |A|^2, \tfrac{1}{2}\mathrm{i}(A\bar{B} - \bar{A}B)\right) + AQ\left(\mu; |A|^2, \tfrac{1}{2}\mathrm{i}(A\bar{B} - \bar{A}B)\right) + \mathcal{P}_0\varepsilon p_0 + \cdots\,, \tag{10.6}$$

where P and Q are polynomials defined as

$$
\begin{aligned}
P(\mu; U, V) &= p_1\mu + p_2 U + p_3 V + \mathcal{O}(|\mu| + |U| + |V|)^2 \,, && (10.7)\\
Q(\mu; U, V) &= q_1\mu - q_2 U + q_3 V + \mathcal{O}(|\mu| + |U| + |V|)^2 \,. && (10.8)
\end{aligned}
$$

The positive constants $\mathcal{P}_0$ and $\mathcal{P}_1$ have been computed by Dias and Iooss (1993). After rescaling, the system (10.5) can be replaced by

$$
\begin{aligned}
\tilde{A}_{\tilde{x}} &= \tilde{B} + \mathcal{O}\left(|\mu|^{1/2}\right) && (10.9)\\
\tilde{B}_{\tilde{x}} &= \tilde{A}(q_1 \mathsf{Sign}(\mu) - q_2|\tilde{A}|^2) + \tilde{\varepsilon}\mathrm{e}^{-ikx}\frac{P_0}{\langle P_0\rangle} + \mathcal{O}\left(|\mu|^{1/2}\right) \,. && (10.10)
\end{aligned}
$$

The sign of $\tilde{\varepsilon}$ depends on the sign of the pressure distribution. One can replace $P_0/\langle P_0\rangle$ by δ_0, the Dirac delta function, since $P_0/\langle P_0\rangle$ converges towards 0 for all $\tilde{x} \neq 0$ and the average $\langle P_0/\langle P_0\rangle\rangle$ is equal to 1. At leading order, one gets the forced nonlinear Schrödinger equation

$$\tilde{A}_{\tilde{x}\tilde{x}} = q_1 \mathsf{Sign}(\mu)\tilde{A} - q_2\tilde{A}|\tilde{A}|^2 + \tilde{\varepsilon}\delta_0 \,. \qquad (10.11)$$

The mathematical problem of the persistence of the solutions of equation (10.11) when dealing with the full system (10.1)-(10.2) can be tackled as in Iooss and Pérouème (1993) or Iooss and Kirrmann (1996) for the study of persistence without forcing, and Mielke (1986) or Kirchgässner (1988) for the study of persistence with forcing. But there is numerical evidence by Vanden-Broeck and Dias (1992) that the solutions obtained here persist.

We look for continuous and bounded solutions of equation (10.11) on the whole real line, with $\tilde{A}_{\tilde{x}}(0+) - \tilde{A}_{\tilde{x}}(0-) = \tilde{\varepsilon}$, which satisfy (10.11) with $\tilde{\varepsilon} = 0$ for $\tilde{x} \neq 0$. The analysis is restricted to homoclinic solutions. Let $\mu_0 = (q_2/2q_1^2)^{1/2}\varepsilon\mathcal{P}_0\langle p_0\rangle$. When $0 < \mu < \mu_0$, there are no homoclinic solutions. When $\mu = \mu_0$, there are two homoclinic solutions. When $\mu > \mu_0$, there are four homoclinic solutions. Writing $A(x) = s(x)\mathrm{e}^{i(kx+\theta(x))}$, we show in Figure 12 the four possibilities for the envelope $s(x)$ in deep water. The corresponding wave profiles $\eta(x) = 4\mathrm{Re}(s(x)\mathrm{e}^{ikx})$ are shown in Figure 13. The expression of $s(x)$ is given by

$$s(x) = \pm(2q_1\mu/q_2)^{1/2}/\cosh(\sqrt{q_1\mu}|x| \pm \alpha_{1/2}) \,, \qquad (10.12)$$

where $\alpha_{1/2} > 0$ are the roots of $\sinh\alpha/\cosh^2\alpha = \mu_0/2\mu$. Figure 14 shows the amplitude $|\eta(0)|$ as a function of μ for : (i) the solution of the linearized problem with $\varepsilon \neq 0$, (ii) the analytical solution with $\varepsilon = 0$, (iii) the present solutions with $\varepsilon \neq 0$. More details can be found in Părău and Dias (2000).

Forced nonlinear Schrödinger equations have been obtained elsewhere (see for example Akylas (1984) and Barnard et al. (1977)). Even if $q_2 < 0$, $q_1 > 0$, $\mu > 0$, one can obtain homoclinic solutions to zero, although it is impossible without forcing as shown for example by Părău and Dias (2002). Our goal here was to deal with the divergence of the solutions of the linearized problem when the velocity c approaches $c_{\min}$, by studying the influence of the nonlinear terms.

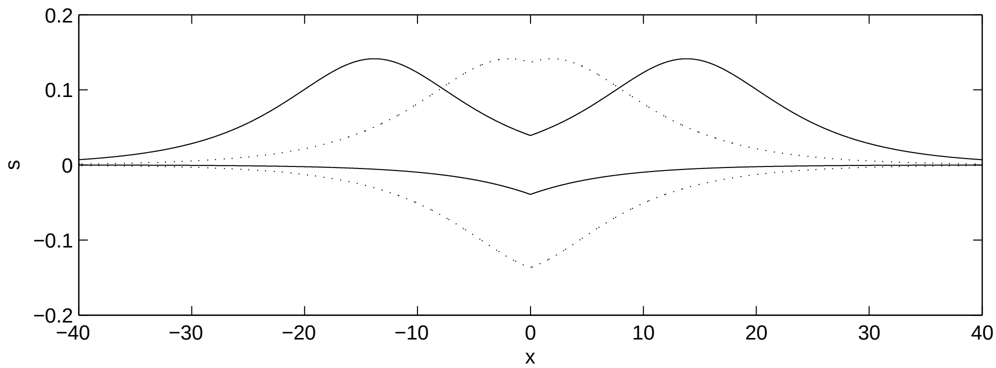

Figure 12. Forced gravity–capillary solitary waves in deep water. The four possibilities for the wave envelope $s(x)$ given by equation (10.12).

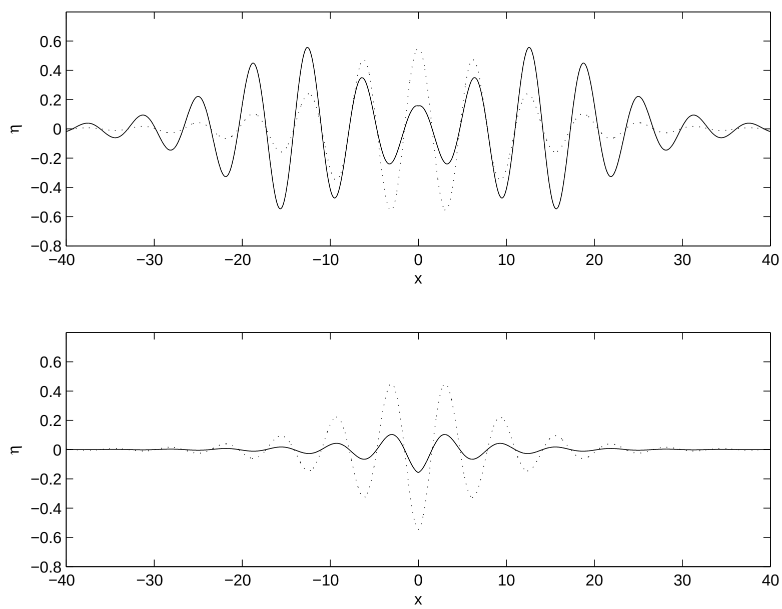

Figure 13. Forced gravity–capillary solitary waves in deep water. The corresponding profiles $\eta(x)$ with $s(x) > 0$ (top) and with $s(x) < 0$ (bottom).

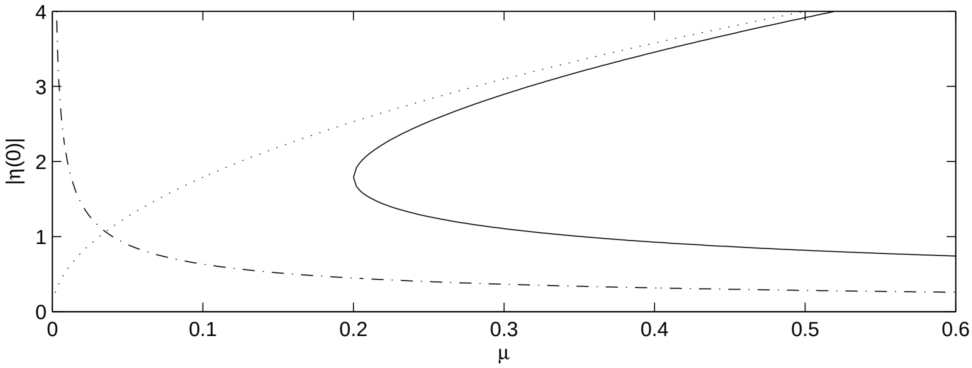

Figure 14. Amplitude at the origin as a function of μ. Comparison between linearized theory with pressure (dash-dotted line), nonlinear theory without pressure (dotted line) and nonlinear theory with pressure (solid line).

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