Physics Reports 507 (2011) 43–105

Contents lists available at SciVerse ScienceDirect

Physics Reports

journal homepage: www.elsevier.com/locate/physrep

Bifurcations of solitons and their stability

E.A. Kuznetsov[a,b,c,*], F. Dias[d,e]

[a] P.N. Lebedev Physical Institute, 119991 Moscow, Russia
[b] L.D. Landau Institute for Theoretical Physics, 119334 Moscow, Russia
[c] Novosibirsk State University, 630090 Novosibirsk, Russia
[d] School of Mathematical Sciences, University College Dublin, Ireland
[e] CMLA, Ecole Normale Supérieure de Cachan, France

ARTICLE INFO

Article history:
Accepted 20 June 2011
Available online 1 July 2011
editor: I. Procaccia

Keywords:
Soliton
Supercritical & subcritical bifurcations
Lyapunov stability
Integral majorized inequalities

ABSTRACT

In spite of the huge progress in studies on solitary waves in the seventies and eighties of the XX century as well as their practical importance, the theory of solitons is far from being complete. Only in 1989, Longuet-Higgins in his numerical experiments discovered one-dimensional solitons for gravity–capillary waves in deep water. These solitons essentially differed from those in shallow water where the KDV equation could be used. Being localized, these solitons, unlike the KDV solitons, contain many oscillations in their shape. The number of oscillations was found to increase while approaching the maximal phase velocity for linear gravity–capillary waves and simultaneously the soliton amplitude was demonstrated to vanish. In fact, it was the first time ever that the bifurcation of solitons was observed.

This review discusses bifurcations of solitons, both supercritical and subcritical, with applications to fluids and nonlinear optics as well. The main attention is paid to the universality of soliton behavior and stability of solitons while approaching supercritical bifurcations. For all physical models considered in this review, solitons are stationary points of the corresponding Hamiltonian for the fixed integrals of motion, i.e., the total momentum, number of quasi-particles, etc. Two approaches are used for the soliton stability analysis. The first method is based on the Lyapunov theory and another one is connected with the linear stability criterion of the Vakhitov–Kolokolov type. The Lyapunov stability proof is maintained by means of application of the integral majorized inequalities being sequences of the Sobolev embedding theorem. This allows one to demonstrate the boundedness of the Hamiltonians and show that solitons, as stationary points, which realize the minimum (or maximum) of the Hamiltonian, are stable in the Lyapunov sense. In the case of unstable solitons, the nonlinear stage of their instability near the bifurcation point results in the distraction of the solitons due to the wave collapse.

Contents

* Corresponding address: P.N. Lebedev Physical Institute, 53 Leninsky ave., 119991 Moscow, Russia. Tel.: +7 499 1326819; fax: +7 499 1326819.
E-mail address: kuznetso@itp.ac.ru (E.A. Kuznetsov).

0370-1573/$ – see front matter
doi:10.1016/j.physrep.2011.06.002

1. Introduction

After their discovery in the nineteenth century on the surface of fluids (see [1]) solitary waves (or solitons) remained for a long time of interest only to a small number of specialists in hydrodynamics and mathematics who tried to prove their existence. In the late 1950s the soliton concept penetrated into plasma physics. Here, due to the work by Sagdeev [2], Gardner and Morikawa [3] and others, solitons were successfully used to construct the theory of a fine structure of shock waves under the conditions of rare collisions. Then solitons started to be used widely in all branches of physics. In the late sixties the interest to solitons grew tremendously with the discovery in 1967 of the Inverse Scattering Transform (IST) suggested by Gardner et al. [4]. They applied this method to integrate the Korteweg–de Vries (KDV) equation. This equation describes the propagation of one-dimensional (1D) acoustic waves in nonlinear media with weak dispersion; in particular, it can be applied to shallow water gravity waves. In this paper it was first demonstrated that solitons in the KDV equation occur to be structurally stable entities: they collide elastically between each other as well as with the non-soliton part of the spectrum so that the asymptotic states are defined by solitons only. The next equation of great physical importance to which the IST can be applied turned out to be the 1D nonlinear Schrödinger (NLS) equation which was integrated in 1971 by Zakharov and Shabat [5]. Like the KDV equation, the NLS equation is a universal model. It describes the propagation of wave envelopes in nonlinear media with cubic nonlinearities; in particular, it can be used for the description of pulse propagation in nonlinear fiber optics when the main nonlinearity is connected with the Kerr effect. At the beginning of the seventies, when solitons in the NLS equation were shown to be structurally stable [5] and when, a bit later, Hasegawa and Tappet [6] suggested to use optical solitons as the information bit in fiber communications, solitons became a very popular object for study in optical fibers. Interest in optical solitons increased enormously in the last decades, stimulated by practical applications of the use of solitons in modern communications systems [7,8] (see the book [9] for recent progress in this area), coupled with the availability if advanced techniques for characterizing their evolution [10]. Soliton dynamics have also been shown to play a central role in the nonlinear propagation dynamics of supercontinuum generation and other phenomena in new generation optical fibers [11].

In spite of the huge progress connected with the development of IST (this is now a whole branch of mathematical physics, often called "mathematical theory of solitons", see, e.g., [12–14]) as well as the practical importance of solitons, their theory is far from being complete. For instance, in 1989, Longuet-Higgins [15] in his numerical experiments discovered 1D solitons for gravity–capillary waves in deep water. These solitons essentially differed from those in shallow water when the KDV equation can be used. Being localized, these solitons, unlike the KDV solitons, contain many oscillations in their shape. The number of oscillations was found to increase while approaching the maximal phase velocity for linear gravity–capillary

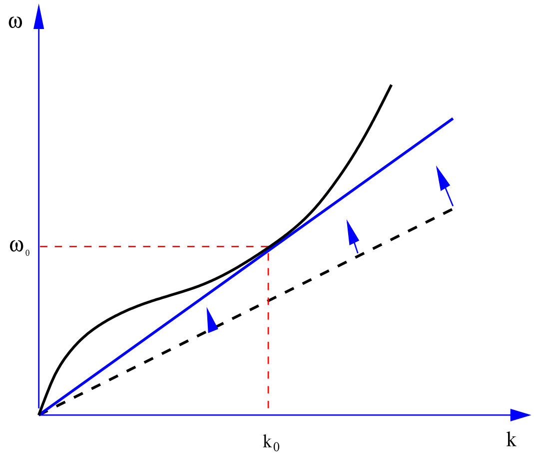

Fig. 1. The solid line corresponds to the dispersion relation $\omega = \sqrt{gk + \sigma k^3}$ for gravity–capillary waves. The dashed (straight) line corresponds to the soliton velocity. The arrows show the direction of increase of the soliton velocity. At the point ω_0, k_0, the straight line $\omega = kV$ touches the dispersive curve.

waves and simultaneously the soliton amplitude was demonstrated to vanish. In fact it was the first time ever that the bifurcation of solitons was observed. This bifurcation was first explained by Iooss and Kirchgässner [16] and independently by Akylas [17].

The physical reason for such a bifurcation can be easily understood. According to the usual definition, solitons are nonlinear localized objects propagating uniformly with a constant velocity (see, for example, [12,13]). Thus, the soliton velocity **V** represents the main soliton characteristics which often defines the soliton shape, in particular its amplitude and width.

It is well known that if the velocity **V** of a moving object is such that the equation

$$\omega_k = \mathbf{k} \cdot \mathbf{V}, \tag{1.1}$$

where $\omega = \omega_k$ is the dispersion law for linear waves and **k** is the wave vector, has a nontrivial solution, then such an object will lose energy due to Cherenkov radiation. This also pertains, to a large extent, to solitons as localized stationary entities. They cannot exist if the resonance condition (1.1) is satisfied. Hence follows the first, and simplest, selection rule for solitons: the soliton velocity must be either less than the minimum phase velocity of linear waves or greater than the maximum phase velocity. Mathematically it can be formulated also as the condition of positiveness for the function $L_k = \omega_k - \mathbf{k} \cdot \mathbf{V} > 0$ if the touching of the plane $\omega = \mathbf{k} \cdot \mathbf{V}$ with the dispersive surface $\omega = \omega_k$ happens from below and respectively negativeness for L_k for the touching from above. The boundary separating the region of existence of solitons from the resonance region (1.1) determines the critical soliton velocity V_{cr}.

As it is easily seen (Fig. 1), this velocity coincides with the group velocity of linear waves at the touching point where the straight line $\omega = kV$ (in 1D) is tangent to the dispersion curve $\omega = \omega_k$ (in the multi-dimensional case—the point of tangency of the plane $\omega = \mathbf{k} \cdot \mathbf{V}$ to the dispersion surface). If touching occurs from below, then the critical velocity V_{cr} determines the maximum soliton velocity for this parameter range and, conversely, for touching from above V_{cr} coincides with the minimum phase velocity. Two regimes are possible in crossing this boundary: they correspond to supercritical or subcritical bifurcations (soft or rigid excitation regimes).

While approaching the supercritical bifurcation point from below or above, the soliton amplitude vanishes smoothly according to the same-Landau-law ($\propto |V - V_{cr}|^{1/2}$) as for phase transitions of the second kind (see, for instance, Ref. [18]). The behavior of solitons in this case is completely universal, both for their amplitudes and their shapes. As $V \to V_{cr}$ solitons transform into oscillating wave trains with the carrying frequency corresponding to the extremal phase velocity of linear waves V_{cr}. The shape of the wave train envelope coincides with that for the soliton of the standard cubic NLS equation. The soliton width happens to be proportional to $|V - V_{cr}|^{-1/2}$. Thus, the pulse monochromaticity improves as $V \to V_{cr}$, becoming complete at the bifurcation point.

As already mentioned above, bifurcations of solitons were first observed for gravity–capillary waves in numerical simulations by Longuet-Higgins [15] and explained later in [16,17]. Then a bifurcation – a transition from periodic solutions to a soliton solution – was studied in Refs. [16,19] using normal forms (see also the review paper [20]). The stationary NLS equation for gravity–capillary wave solitons was derived in Ref. [21]. In Ref. [22] it was shown that this mechanism can be extended to optical solitons. In fact this paper provided the first demonstration of the universality of soliton behavior near a supercritical bifurcation for waves of arbitrary nature. It is worth noting that the universal character of solitons allows not only to find their shapes but also to investigate their stability. This analysis, as shown in Ref. [23], demonstrates that near supercritical bifurcation solitons are stable only in the 1D case. This means that in two (2D) and three (3D) dimensions the soliton may be stable for velocities smaller or larger than the critical velocity depending on whether the touching occurs from below or above. For instance, such a situation arises for 3D solitons in three-wave systems [24,25] where, following the paper by Kanashev & Rubenchik [26], it is possible to estimate the region of stable solitons using the Lyapunov approach. This region turns out to be separated from the surface in the soliton parameter space where supercritical bifurcation happens [27].

The question of whether the bifurcation is supercritical or subcritical depends on the character of the nonlinear interaction. In the 1D case, the supercritical bifurcation occurs for a focusing nonlinearity when the product $\omega''T < 0$, where $\omega'' = \partial^2\omega/\partial k^2$ is the second derivative of the frequency with respect to the wave number, evaluated at the touching point $k = k_0$, and T is the value of the matrix element $\widetilde{T}_{k_1k_2k_3k_4}$ of the four-wave interaction for $k_i = k_0, i = 1, 2, 3, 4$. Here the tilde means that the matrix element is renormalized due to three-wave interactions; in the present case this is the interaction with the zeroth and second harmonics. If $\omega''T > 0$, which corresponds to a defocusing nonlinearity, then there are no solitons – localized solutions – with amplitude vanishing gradually as $V \to V_{cr}$. In the theory of phase transitions this corresponds to a first-order phase transition, and in the theory of turbulence, using Landau's terminology [28], it corresponds to a rigid regime of excitation. The transition through the critical velocity is accompanied by a jump in the soliton amplitude. For Hamiltonian systems such as those considered in the present paper, the magnitude of the jump is determined by the next higher-order nonlinear terms in the expansion of the Hamiltonian. Like for first-order phase transitions, the universality of soliton behavior is no longer guaranteed in this situation. When the amplitude jump at this transition is small, it is enough to keep a finite number of next order terms in the Hamiltonian expansion to describe such a bifurcation. In phase transitions this corresponds to a first-order phase transition close to a second-order transition, which occurs, for example, near a tricritical point. As shown in Ref. [29], this situation arises for 1D internal wave solitons propagating along the interface between two ideal fluids with different densities in the presence of both gravity and capillarity. According to Ref. [29] the matrix element T in this case vanishes for a critical value θ_{cr} of the density ratio ρ_1/ρ_2 equal to $(21 - 8\sqrt{5})/11 \approx 0.2828$, where $\rho_{1,2}$ are mass densities for upper and lower fluids, respectively. In particular, it follows that the bifurcation for gravity–capillary waves in the deep water case is supercritical (when $\rho_1/\rho_2 = 0$); this case corresponds to the first numerical experiments by Longuet-Higgins [15], followed by the numerical experiments of Vanden-Broeck and Dias [30]. Subcritical bifurcations can also be met for gravity water waves with finite depth when the matrix element $T = 0$ at $k_0h \approx 1.363$. In nonlinear optics, as shown in [23], a decrease of T (Kerr constant) can be provided by the interaction of light pulses with acoustic waves (Mandelstamm–Brillouin scattering). If the jump in the soliton amplitude is of order one then we need to keep all the remainder terms in the Hamiltonian expansion. The situation in nonlinear optics, however, is different from that for internal waves propagating along the interface between two fluids. First of all, the difference is connected with the nonlocal character of the Hamiltonian expansion beyond the classical cubic NLS nonlinearity for the fluid case [31,32]. In both cases, however, in order to find the Hamiltonian expansion the most simple way is to use the Hamiltonian formalism [33]. In this review we will keep mainly the Hamiltonian description as the most adequate for this problem; the alternative method of normal forms will be used for comparison. It is clear that the method of normal forms demonstrates its efficiency for the analysis of bifurcations for ordinary differential equations. The method of normal forms has some advantages and weak points as well. For instance, unlike the Hamiltonian formalism, the introduction of envelopes by means of the normal form method is not unique. This is due to the fact that the original Hamiltonian equations lose their initial Hamiltonian structure after they are averaged. In the Hamiltonian description, the introduction of the envelope is natural: it is constructed from the inverse Fourier transform from normal wave amplitudes and can be used for the description of any nonlinear waves. The difference in such a case will be in different constants, first of all in the nonlinear coupling coefficients.

The above approach based on the Hamiltonian perturbation technique assumes renormalization of the four-wave matrix element due to three-wave interactions. As mentioned above, for the case of a wave train with carrying frequency ω_0 and carrying wave vector k_0, it accounts for the non-resonant interaction of a wave packet with its zeroth and second harmonics. If the interaction between the fundamental harmonics and the second harmonics becomes resonant,

$$2\omega(k_0) \approx \omega(2k_0),$$

the renormalization of the four-wave matrix element breaks down. As a result the one-envelope approximation can no longer be applied. In this situation one needs to consider two equations for the two independent envelope amplitudes related to the first and second harmonics. In the simplest case, the corresponding system can be obtained as the reduction of the three-wave system [24] when two amplitudes are identified. It is well known that three wave packets with carrier frequencies satisfying the triad resonant condition can form bound states – solitons – due to their mutual nonlinear interaction [25]. In nonlinear optics the three-wave system describes spatial solitons as well as spatial–temporal solitons in χ^2 media [25,34]. This system couples amplitudes of three quasi-monochromatic waves due to quadratic nonlinearity. The familiar results about the existence of bound states – solitons – due to their mutual nonlinear interaction is also valid for the interaction of the first and second harmonics. The soliton family is characterized by two independent parameters, a soliton potential and a soliton velocity. It can be shown that this system, in the general situation, is not Galilean invariant. As a result, the family of movable solitons cannot be obtained from the rest soliton solution by applying the corresponding Galilean transformation. In Ref. [27] the region of soliton parameters was found analytically and confirmed by numerical integration of the steady equations. On the boundary of the region the solitons bifurcate. For this system there exist two kinds of bifurcations: supercritical and subcritical. In the first case, the soliton amplitudes vanish smoothly as the boundary is approached. Near the bifurcation point the soliton form is universal, determined from the NLS equation. For the second type of bifurcations the wave amplitudes remain finite at the boundary. In this case the Manley–Rowe integral increases indefinitely as the boundary is approached, and therefore according to the Vakhitov–Kolokolov type stability criterion, the solitons are unstable [27].

In this review we will discuss all the above problems dealing with the bifurcations of solitons and their stability. All systems which are considered in this review belong to the Hamiltonian type and soliton solutions are stationary points of the Hamiltonian, for fixed other integrals of motion, such as the momentum, the number of particles, the Manley–Rowe integrals. In all the systems under consideration solitons are possible as a result of a balance between nonlinear interaction and dispersive effects. In this review we follow two approaches for the analysis of soliton stability. The first approach to soliton stability is based on the Lyapunov theorem. In the conservative case, if some integral, say the Hamiltonian, is bounded from below (or above), the soliton realizing this extremum will be stable in the Lyapunov sense. Because soliton solutions represent stationary points of the Hamiltonian for certain fixed other integrals of motion, they correspond to a conditional variational problem, and so to prove stability one needs to demonstrate the boundedness of the Hamiltonian for these fixed integrals. One should note, however, that without these fixed integrals, the Hamiltonians of such systems are usually unbounded due to the nonlinear contribution; in other words, one can say that the Hamiltonians of these systems do not possess a vacuum. This is an essential part of Derrick's arguments [35]. But fixing other integrals of motion causes significant changes. It provides the Hamiltonian boundedness that establishes stability for solitons realizing the corresponding extremum. First, this approach was applied to KDV solitons in 1972 by Benjamin [36] and two years later to three-dimensional solitons for ion-acoustic waves in magnetized plasma with low pressure [37]. Then this method was applied to the NLS equation and its generalizations (for more details see the review [38]). Now it is one of the most powerful tools in soliton stability analysis. In this paper we would like to pay a special attention to the use of the embedding theorems, and to demonstrate how with their help it is possible to construct an estimate for the Hamiltonian for a lot of models considered here.

Another method used in this paper is the linear stability analysis which for the NLS equation gives the so-called Vakhitov–Kolokolov (VK) criterion [39]. This criterion says that if $\partial N_s/\partial\lambda^2 > 0$ then the soliton is stable and respectively unstable if this derivative is negative, where N_s is the total number of waves for the soliton. This criterion has a simple physical meaning. The value $-\lambda^2$ for the NLS soliton can be interpreted as the energy of the bound state. If we add one particle to the system and the energy of this bound state decreases then one has a stable situation. If by adding one particle the level $-\lambda^2$ is pushed toward the continuous spectrum, then such a soliton is unstable.

It is important to point out that establishing the Lyapunov stability for solitons is often a problem which is solved more easily than that for linear stability. The linear stability analysis assumes linearization of the equations of motion on the background of the soliton solution and leads to an eigenvalue problem for some differential operators. The proof of linear stability includes the establishment of completeness of the eigenfunctions. This in itself is a hard problem, let alone determining the linear stability as a whole. However, while being effective for the stability study of ground-state solitons, the Lyapunov method is hardly applicable to the stability study of local stationary points. In this case linear analysis should be used to draw a conclusion about their stability.

The main attention in this review will be paid to the universality of behavior of solitons while approaching the supercritical bifurcation point. The first three sections are devoted to this topic. We consider first the simplest model where all effects concerning supercritical bifurcation can be analyzed easily. This is the KDV equation with fifth-order dispersion, i.e. it also has, besides the third-order spatial derivative, a fifth-order derivative relative to x. This model can be derived for shallow water waves in the presence of surface tension when the Bond number $\sigma/(\rho g h^2)$ is close to 1/3. Here σ is the surface tension coefficient, ρ the water density, g the acceleration due to gravity and h the water depth. When the Bond number is close to 1/3, the coefficient of the third-order dispersion term is small, and one needs to keep the next (fifth) order dispersion term. In Section 2, we demonstrate how soliton solutions transform into NLS envelope solitons while approaching the critical velocity for solitons. We show also that for the fifth-order KDV equation the Hamiltonian is bounded from below for fixed momentum. If there exists a solitary wave solution which realizes this minimum, then the soliton is stable with respect to not only small perturbations but also finite ones. The proof is based on both the Lyapunov theorem and an integral estimate of the Sobolev–Gagliardo–Nirenberg inequalities [40,41]. These inequalities follow from the general embedding theorems first proved by Sobolev. In this section we also demonstrate that the supercritical bifurcation takes place for movable solitons for the 1D NLS equation.

In Section 3, following Ref. [22], optical solitons and quasisolitons are examined relative to the Cherenkov radiation. Both solitons and quasisolitons are shown to exist if the linear operator defining their asymptotics at infinity is sign definite. In particular, applying this criterion to the stationary optical solitons yields the soliton carrying frequency where the first derivative of the dielectric permittivity vanishes. At this point the phase and group velocities coincide. Both solitons and quasisolitons are absent if the third-order dispersion is taken into account. By means of the sign definiteness of the operator and by use of integral estimates of Sobolev type the soliton stability is established for the fourth-order dispersion for all dimensions. This proof again is based on the boundedness of the Hamiltonian in the case of fixed pulse power. Besides, in this section we develop the Hamiltonian expansion for nonlinear optics beyond the classical cubic NLS equation. As is well known in optics (see, e.g. [42]), the spatial dispersion effects are small in comparison with the temporal dispersion ones (their ratio is a small relativistic factor, $\sim v/c$ where v is the characteristic electron velocity in atoms and c the light velocity, thus, this ratio is of order $\alpha = 1/137$). Therefore the expansion of the electric induction $D(t, r)$ in terms of the electric field $E(t, r)$ represents an infinite set with respect to powers of the electric field, evaluated at the same point as the electric induction. Each term of this set contains only time convolutions. This is why in nonlinear optics the NLS equation, for example, is usually written for the electric field envelope, where the spatial coordinate z plays the role of time in the usual NLS equation and t represents the analog of coordinate (see, e.g., [9]). Note that the nonlinear optics NLS equation is also used in hydrodynamics for the wavemaker problem [43,44].

Section 4 is devoted to stationary solitons for arbitrary nonlinear wave media and their properties near the supercritical bifurcation. The stability of solitons is based on the Lyapunov theorem and the Hamiltonian approach. It is shown by means of integral estimates of Sobolev type in their multiplicative variant (Gagliardo–Nirenberg inequalities [40]) that only 1D solitons are Lyapunov-stable. It is worth noting that, in contrast to the method of normal forms, which is extensively used in Refs. [16,30,29,45] to study bifurcations of solitons, the Hamiltonian approach is fundamental for investigating soliton stability. It is necessary to add also that in the method of normal forms, the introduction of envelopes is not unique. Consequently the Hamiltonian equations of motion lose their initial Hamiltonian structure after averaging. In the 3D geometry solitons near the supercritical bifurcation undergo modulation instability that follows directly from the VK criterion [39]. In 3D, the derivative $\partial N_s/\partial\lambda^2$ is negative (here the subscript s means that N is evaluated on the soliton solution) and therefore such solitons are unstable. From the Hamiltonian point of view such soliton solutions viewed as stationary points of the Hamiltonian for fixed N represent saddle points and this is why they are unstable. Moreover, it is possible to establish that the Hamiltonian in this case will be unbounded from below and therefore the nonlinear stage of this instability will be collapse of the soliton when the field intensity blows up and its size shrinks. This compression will happen at least up to the scale of the wavelength of order k_0^{-1}. The final stage of this instability depends on whether the primitive Hamiltonian is bounded from below (or above).

In Section 5, we consider which nonlinear effects must be taken into account near the transition from supercritical to subcritical bifurcations and how they change the shape of solitons and their stability. As examples of such transition we consider internal waves, surface gravity waves at finite depth near $k_0h \approx 1.363$ and short optical pulses when the Kerr constant becomes small enough. In order to describe the behavior of solitons and their bifurcations, a generalized NLS equation describing the behavior of solitons and their bifurcations is derived. In comparison with the classical NLS equation this equation takes into account three additional nonlinear terms: the so-called Lifshitz term responsible for pulse steepening, a nonlocal term analogous to that first found by Dysthe [46] for gravity waves and the six-wave interaction term. Near the transition point, the soliton family from the supercritical branch, which is defined from the solution of the generalized NLS equation, changes noticeably, but at $V = V_{cr}$ these solitons vanish smoothly. All 1D solitons corresponding to the family of supercritical bifurcations are shown to be stable in the Lyapunov sense. Above the transition point, solitons from the subcritical branch undergo a jump at $V = V_{cr}$ (for interfacial waves, this jump is proportional to $\sqrt{\theta - \theta_{cr}}$ where $\theta = \rho_1/\rho_2$). At large distances their amplitude decays algebraically. Secondly, the soliton family of this branch turns out to be unstable. The development of this instability results in the collapse of solitons. Near the time of collapse, the pulse amplitude and its width exhibit a self-similar behavior with a small asymmetry in the pulse tails due to self-steepening.

Section 6 deals with solitons involving the interaction between the fundamental and second harmonics. The soliton family for this system is characterized by two independent parameters, a soliton chemical potential and a soliton velocity. It is shown that this system, in the general situation, is not Galilean invariant. As a result, the family of movable solitons cannot be obtained from the rest soliton solution by applying the corresponding Galilean transformation. The region of soliton parameters is found analytically and confirmed by numerical integration of the steady equations. On the boundary of the region the solitons bifurcate. For this system there exist two kinds of bifurcations: supercritical and subcritical. In the first case the soliton amplitudes vanish smoothly as the boundary is approached. Near the bifurcation point the soliton form is universal. It is determined from the NLS equation. For the second type of bifurcations the wave amplitudes remain finite at the boundary. In this case the Manley–Rowe integral increases indefinitely as the boundary is approached, and therefore according to the VK type stability criterion, the solitons are unstable.

In the last section, Section 7, we obtain the VK type criterion for NLS type models. The crucial point in its derivation for the scalar NLS equation is based on the oscillation theorem for the stationary Schrödinger operator. This theorem establishes the one-to-one correspondence between a level number and a number of nodes of the eigenfunction. As is well known, this theorem is valid only for scalar (one-component) Schrödinger operators and cannot be extended, for example, to the analogous matrix operators. This means that the Vakhitov–Kolokolov type of criteria, as a rule, defines only sufficient conditions for soliton instability and cannot necessarily determine the stability of solitons. The three-wave system can be used as such an example. For this system the linearized operator represents a product of two (3×3)-matrix Schrödinger operators for which the oscillation theorem cannot be applied. We discuss this situation in detail for solitons describing a bound state of the fundamental frequency and its second harmonics [47].

2. Bifurcations of gravity–capillary solitary waves in shallow water

We start from the 2D capillary–gravity waves for shallow water. For arbitrary water depth h the linear dispersion for gravity–capillary waves is given by the expression

$$\omega = \sqrt{\left(gk + \frac{\sigma}{\rho}k^3\right)\tanh kh},$$

where σ is the surface tension coefficient, ρ is the water density, and g is the acceleration due to gravity. In the long wave limit the expansion of this dispersion has the form

$$\omega = kc_s\left[1+\frac{1}{2}(kh)^2\left(B-\frac{1}{3}\right)-(kh)^4\left(\frac{1}{6}B-\frac{1}{15}+\frac{1}{8}\left(B-\frac{1}{3}\right)^2\right)+\cdots\right],$$

where $c_s=\sqrt{gh}$ is the velocity of long gravity waves and $B=\sigma/(\rho gh^2)$ is the Bond number. The cubic dispersion vanishes at $B\equiv B_{cr}=1/3$. In a small vicinity of the critical Bond number this expansion is simplified:

$$\omega = kc_s\left[1+\frac{\Delta B}{2}(kh)^2+\frac{1}{90}(kh)^4\right]$$

where $\Delta B = B-B_{cr}$. Hence one can see that the sign of the cubic dispersion coincides with the sign of ΔB. If $\Delta B<0$ this dispersion curve has a saddle point, it is similar to that for the deep water case (compare with Fig. 1). At $\Delta B>0$ both third and fifth dispersions are positive definite.

To describe weakly nonlinear waves for the shallow water case with small $|\Delta B|\ll B_{cr}$ we can use the KDV equation with both third and fifth-order dispersions. In dimensionless variables this equation can be written in the form

$$u_t+su_{xxx}+u_{xxxxx}+6uu_x=0, \tag{2.1}$$

where $s=-\mathrm{sign}(\Delta B)$. Recall that the KDV equation is written in the system of reference moving with the "sound" velocity c_s. In this equation both the nonlinear and dispersive terms are assumed to be much smaller in comparison with the rapid propagation at the "sound" velocity (see, e.g. [12–14]). However, these terms may be comparable to each other.

The dispersion relation for linear waves in (2.1),

$$\omega(k)=-sk^3+k^5, \tag{2.2}$$

looks quite similar to the classical KDV dispersion relation close to $k=0$, depending strongly on s. For negative s, their phase velocity

$$c(k)=\frac{\omega(k)}{k}=-sk^2+k^4, \tag{2.3}$$

has the minimum value $c_{\min}=0$ at $k=0$. Therefore the soliton velocity V must be negative in order to exclude the Cherenkov resonance between soliton and linear waves.[1]

For $s<0$ and $V<0$ the function

$$L(k)=c(k)-V$$

is positive and the corresponding linear operator

$$\hat{L}(-i\partial_x)=c(-i\partial_x)-V$$

is positive definite. Not surprisingly, it was shown that classical solitary waves with speed $V<0$ bifurcate from the trivial solution $u=0$ (see for example [48]). At small $V(V\to 0^-)$ the fifth-order dispersion in (2.2) can be neglected. As a result, we arrive at the classical KDV soliton:

$$u=-\frac{\kappa^2}{2\cosh^2(x+4\kappa^2 t)}$$

where $V=-4\kappa^2<0$. This is an asymptotic soliton solution of Eq. (2.1) for $s=-1$. According to [48] these solitary waves are orbitally stable, at least when the speed V is between $-1/4$ and 0. When the speed crosses $-1/4$, the real eigenvalues V of

$$-u_{xx}+u_{xxxx}=Vu$$

become complex. It was shown in [49] that this transition leads to a plethora of multi-modal homoclinic orbits.

For $s=+1(\Delta B<0)$ the dispersion curve (2.2) has a minimum $c_{\min}=-1/4$, which is attained at $k_0=\pm 1/\sqrt{2}$. The corresponding linear operator

$$\hat{L}(-i\partial_x)=\partial_x^2+\partial_x^4-V\equiv(\partial_x^2+k_0^2)^2+c_{\min}-V \tag{2.4}$$

is positive definite if the soliton velocity V is less than the minimal phase velocity $c_{\min}=V_{cr}$ (see, e.g. [50,19,22]). Above this critical value, the horizontal line $V=const$ always intersects the dispersion curve (2.2) and therefore solitons are impossible for $V>V_{cr}$, due to Cherenkov resonance (1.1). Thus, in the present case the touching $V=const$ occurs from below, which corresponds to the bifurcation point for solitons.

[1] It is worth noting that Eq. (1.1) in the present case always has one root $k=0$, but as it will be shown later this root represents the removable singularity for the KDV type equations.

2.1. Behavior of solitons near bifurcation point

Before finding soliton solutions to Eq. (2.1) at $s = +1$, we first recall some general features of the fifth-order KDV equation. This equation, like the classical KDV equation, belongs to the Hamiltonian equations:

$$u_t = J\frac{\delta H}{\delta u}, \quad J = \frac{\partial}{\partial x}, \tag{2.5}$$

where the Hamiltonian H is given by the expression

$$H = \int_{-\infty}^{+\infty} \left[\frac{1}{2}u_{xx}^2 - \frac{1}{2}u_x^2 - u^3\right] dx, \tag{2.6}$$

or in the equivalent form

$$u_t + u_{xxx} + u_{xxxxx} + 6uu_x = 0. \tag{2.7}$$

Besides the Hamiltonian H, this equation has another integral of motion, the momentum

$$P = \frac{1}{2}\int u^2 dx.$$

Consider now solitary wave solutions $u = u_s(x - Vt)$ with condition $u \to 0$ as $|x| \to \infty$, assuming velocities $V < c_{\min}$. After one integration the equation for the soliton shape is written by means of the positive definite operator $\hat{L}$ (2.4):

$$\hat{L}u \equiv -Vu + u_{xx} + u_{xxxx} = -3u^2. \tag{2.8}$$

Sign positiveness of the operator L plays an essential role, not only for the existence of solitary waves but also for their stability [22].

First of all, we demonstrate how the soliton shape can be found near the critical velocity V_{cr} assuming that the difference between the solitary wave velocity and V_{cr} is small enough:

$$\frac{V_{cr} - V}{|V_{cr}|} = \epsilon^2 \ll 1. \tag{2.9}$$

Taking the Fourier transform of Eq. (2.8) yields

$$u_k = \frac{1}{L(k)}(u^2)_k,$$

where

$$L(k) = (k^2 - k_0^2)^2 + (V_{cr} - V) \tag{2.10}$$

is the expression for the operator L in k-space. Hence one can see that when V approaches V_{cr} the Fourier spectrum of u_k is concentrated near $k = k_0$ with the characteristic width $\delta k \sim \epsilon$. On the other hand, the quadratic nonlinearity in (2.7) produces all combined harmonics with $k = \pm nk_0$ where n is an integer. If one assumes now that the amplitude of the main harmonics vanishes (this assumption is later verified), then one should seek for a solitary wave solution of the Eq. (2.7) in the form of the sum of harmonics nk_0:

$$u = u_0 + \sum_{n=1}^{\infty}\left(\psi_n(X)e^{ink_0x} + c.c.\right). \tag{2.11}$$

In this expression we introduced the slow coordinate $X = \epsilon x$ and assumed that $\psi_n \sim \epsilon^n$ and $u_0 \sim \epsilon^2$. Applying the standard procedure of multi-scale expansion (see, for instance, [51,52]), one arrives at the equation

$$|V_{cr}|\epsilon^2\psi_1 - 2\epsilon^2\psi_{1XX} = -6(u_0\psi_1 + \psi_2\psi_1^*). \tag{2.12}$$

The amplitudes of the zeroth and second harmonics are given by

$$u_0 = -24|\psi_1|^2, \qquad \psi_2 = -\frac{4}{3}\psi_1^2. \tag{2.13}$$

Substitution of (2.13) into (2.12) yields for the fundamental harmonic amplitude the stationary nonlinear Schrödinger (SNLS) equation

$$|V_{cr}|\epsilon^2\psi_1 - 2\psi_{1xx} - 152|\psi_1|^2\psi_1 = 0. \tag{2.14}$$

Hence one can see that the nonlinearity as well as the dispersion provide the existence of localized solutions in the form of solitary waves of the focusing NLS equation. After rescaling, this equation is written as

$$-\epsilon^2\psi + \psi_{xx} + 2|\psi|^2\psi = 0. \tag{2.15}$$

Its solution is given by the classical NLS soliton

$$\psi = \frac{\epsilon e^{i\varphi}}{\cosh(\epsilon x)}, \tag{2.16}$$

which depends on one free parameter, i.e. the phase φ.

Thus, as $V \to V_{cr}$ solitons undergo the supercritical bifurcation: the soliton amplitude is proportional to $\varepsilon \sim (V_{cr} - V)^{1/2}$ and its width increases like $\varepsilon^{-1} \sim (V_{cr} - V)^{-1/2}$, and while approaching the critical velocity the soliton solution transforms into the wave train: the train envelope coincides with the NLS soliton. As shown later (Section 4), the behavior of solitons near the supercritical point found here indeed is universal: it happens not only for shallow gravity–capillary solitary waves but for all solitary waves in conservative media.

In order to investigate the stability of the envelope solitary waves described by the SNLS Eq. (2.14), one needs to introduce the time dependence. It is easy to see that the expansion of the dispersion relation in the coordinate system moving with the solitary wave

$$\Omega = k(c(k) - V) \equiv kL(k)$$

near $k = k_0$ has the form

$$\Omega \approx k_0[2(k - k_0)^2 + (V_{cr} - V)]. \tag{2.17}$$

Assuming further that the amplitudes of the harmonics in the expansion (2.11) depend on the slow time $T = \epsilon^2 t$ and taking into account the approximation (2.17) for frequency, one can easily obtain the time-dependent nonlinear Schrödinger equation for the fundamental harmonic

$$\frac{i}{k_0}\Psi_{1T} - |V_{cr}|\Psi_1 + 2\Psi_{1XX} + 152|\Psi_1|^2\Psi_1 = 0. \tag{2.18}$$

After rescaling this equation can be written in the canonical form

$$i\frac{\partial\psi}{\partial t} - \epsilon^2\psi + \psi_{xx} + 2|\psi|^2\psi = 0. \tag{2.19}$$

It should be noted that, unlike the fifth-order KDV Eq. (2.5), the NLS Eq. (2.19) has one additional symmetry, namely, the gradient symmetry: $\psi \to \psi e^{i\phi}$, that appears as a result of the averaging applied over rapid oscillations. Therefore the envelope solitary wave solutions form a broader class than the solitary wave solutions of the Eq. (2.5). As shown in [53] and discussed above, this has nontrivial consequences on the solutions of (2.7) and on their stability. Just as an illustration of the method that will be used in the next subsection, we proceed with looking at the stability of envelope solitary waves described by the SNLS equation.

2.2. Stability of envelope solitons

It is not difficult to see that the solitary wave solution (2.16) represents a stationary point of the Hamiltonian

$$H = \int_{-\infty}^{+\infty} \left(|\psi_x|^2 - |\psi|^4\right) dx \equiv I_1 - I_2, \tag{2.20}$$

for fixed number of particles $N = \int |\psi|^2 dx$. In other words,

$$\delta F_{\rm NLS} = 0, \quad \text{where } F_{\rm NLS} = H + \epsilon^2 N. \tag{2.21}$$

According to the Lyapunov theorem, a stationary point will be stable if it realizes a minimum (or a maximum) of the Hamiltonian.

In order to prove the stability of the soliton (2.16), it is enough to show that this solution realizes a minimum of the Hamiltonian according to the Lyapunov theorem.

Consider first the scaling transformations that preserve the number of particles:

$$\psi(x) \to a^{-1/2}\psi(x/a). \tag{2.22}$$

Under this transform the Hamiltonian takes a dependence on the scaling parameter a,

$$H = \frac{I_1}{a^2} - \frac{I_2}{a}. \tag{2.23}$$

The function $H(a)$ has a minimum at $a = 1$, which corresponds to the soliton solution (2.16):

$$H_s = -\frac{2\epsilon^3}{3} \quad \text{and} \quad 2I_{1s} = I_{2s} = \frac{4\epsilon^3}{3}. \tag{2.24}$$

The soliton also realizes a minimum of H with respect to another simple transformation, i.e., the gauge one,

$$\psi_0(x) \to \psi_0(x)\exp[i\chi(x)], \tag{2.25}$$

which also preserves N,

$$H = H_s + \int (\chi_x)^2 \psi_0^2 dx.$$

Thus, both simple transformations yield a minimum for the Hamiltonian, thus indicating soliton stability.

Now we give an exact proof of this fact, following [38,22]. The crucial point of this proof is based on integral estimates of the Sobolev type. These inequalities arise as sequences of the general embedding theorem first proved by Sobolev.

The Sobolev theorem states that the space L_p can be embedded into the Sobolev space W_2^1 if the dimension of R^D,

$$D < \frac{2}{p}(p+4).$$

This means that between norms

$$\|u\|_p = \left[\int |u|^p d^D x\right]^{1/p}, \quad (p > 0), \qquad \|u\|_{W_2^1} = \left[\int (\mu^2|u|^2 + |\nabla u|^2) d^D x\right]^{1/2}, \quad (\mu^2 > 0),$$

there exists the following inequality (see, e.g., [41]):

$$\|u\|_p \le M\|u\|_{W_2^1} \tag{2.26}$$

where M is some constant > 0. For the particular case $D = 1$ and $p = 4$ the inequality (2.26) can be rewritten in the form

$$\int_{-\infty}^{\infty} |\psi|^4 dx \le M_1 \left[\int_{-\infty}^{\infty} (\mu^2|\psi|^2 + |\psi_x|^2) dx\right]^2. \tag{2.27}$$

Hence one can easily obtain a multiplicative variant of the Sobolev inequality, the so-called Sobolev–Gagliardo–Nirenberg inequality [40] (see also [41,54,38]).

Let us apply in Eq. (2.26) the scaling transform $x \to \alpha x$. Then instead of (2.27) we have

$$\int_{-\infty}^{\infty} |\psi|^4 dx \le M_1 \left[\mu^2 \int_{-\infty}^{\infty} |\psi|^2 dx \cdot \alpha + \int_{-\infty}^{\infty} |\psi_x|^2 dx \cdot \frac{1}{\alpha}\right]^2.$$

This inequality holds for any (positive) α including a minimal value for the r.h.s. Calculating its minimum yields the desired inequality:

$$\int_{-\infty}^{\infty} |\psi|^4 dx \le CN^{3/2} \left[\int_{-\infty}^{\infty} |\psi_x|^2 dx\right]^{1/2}, \tag{2.28}$$

where C is a new positive constant. One can arrive at the same inequality by considering the following set of inequalities [41]:

$$\begin{aligned} \int_{-\infty}^{\infty} |\psi|^4 dx &\le \max_x |\psi|^2 \int_{-\infty}^{\infty} |\psi|^2 dx = \int_{-\infty}^{x_{\max}} \frac{d|\psi|^2}{dx} dx \int_{-\infty}^{\infty} |\psi|^2 dx \\ &\le 2N \int_{-\infty}^{x_{\max}} |\psi||\psi_x| dx \le 2N \int_{-\infty}^{\infty} |\psi||\psi_x| dt \le 2N^{3/2} \left[\int_{-\infty}^{\infty} |\psi_x|^2 dt\right]^{1/2} \end{aligned} \tag{2.29}$$

where $C = 2$. This inequality can be improved by finding the best constant C in (2.28). Evidently, the maximum value of the functional

$$G[\psi] = \frac{I_2}{N^{3/2} I_1^{1/2}}$$

yields the best constant. In order to find the maximum of $G[\psi]$ it is sufficient to consider all stationary points of this functional and among all of them we must choose the one which realizes the needed maximum. It is easy to check that all stationary points of $G[\psi]$ are defined by the equation which coincides with that for the solitary wave solution (2.16):

$$-\lambda^2\psi + \psi_{xx} + 2|\psi|^2\psi = 0.$$

Hence one can see that the maximum of $G[\psi]$ is attained on the real solitary wave solution (2.16) which, moreover, is unique up to a constant phase multiplier:

$$\psi_s = \frac{\lambda}{\cosh(\lambda x)}.$$

Then all integrals contained in $G[\psi]$ are easily calculated:

$$N = 2\lambda, \qquad I_{1s} = \frac{2}{3}\lambda^3, \qquad I_{2s} = \frac{4}{3}\lambda^3,$$

and the inequality (2.28) finally reads:

$$\int_{-\infty}^{\infty} |\psi|^4 dx \leq \frac{1}{\sqrt{3}} N^{3/2} \left[\int_{-\infty}^{\infty} |\psi_x|^2 dx\right]^{1/2}. \tag{2.30}$$

Substituting now this inequality into (2.20) we obtain the following estimate:

$$H \geq H_s + (\sqrt{I_1} - \sqrt{I_{1s}})^2, \tag{2.31}$$

where $H_s = -\frac{2}{3}\varepsilon^3 < 0$ is the value of the Hamiltonian on the solitary wave solution. This estimate becomes precise on the solitary wave solution. That, according to the Lyapunov theorem, proves the solitary wave stability. It is necessary to remind that the 1D NLS equation can be integrated by means of the inverse scattering transform (IST) [5]. For many models integrable by the IST such as the 1D NLS equation, solitons are structurally stable entities which retain their shape after scattering with another solitons or with waves from continuous spectra.

Thus, we have demonstrated that the envelope solitary wave (2.16) is stable with respect to partial-modulation type perturbations described by the NLS equation (2.19). However we have no answer about the stability for the original model (2.5). We repeat here that the purpose of this Section is to illustrate the method based on Lyapunov's theory to obtain some stability results on the solitary wave solutions of (2.1).

2.3. Solitary wave stability

Let us now come back to the stationary KDV equation (2.8) for the solitary wave shape. It is easy to see that this equation is nothing more than the Euler–Lagrange equation for the functional

$$F = H + VP,$$

where H is given by the formula (2.6) and P is the momentum for (2.5). In other words, any solitary wave solution is a stationary point of the Hamiltonian H for fixed momentum P. The soliton velocity V in this case plays the role of Lagrange multiplier. So if we now show that the Hamiltonian (2.6) is bounded from below (its unboundedness from above is obvious) then the stationary point (solitary wave) corresponding to its minimum value will be stable according to the Lyapunov theorem. It is interesting to note that in the NLS limit the functional F reduces to the corresponding functional $F_{\rm NLS}$ (2.21).

First note that the functional F can be written through the mean value of the operator L and the integral of u^3:

$$F = \frac{1}{2}\left\langle \hat{L}(-i\partial_x)\right\rangle - \int_{-\infty}^{+\infty} u^3 dx.$$

In the region $V < V_{cr}$ the mean value of the operator L is always positive. This is a key point to demonstrate that the functional F is bounded from below.

Consider the mean value of the operator $\hat{L}$

$$\left\langle \hat{L}(-i\partial_x)\right\rangle = \int_{-\infty}^{+\infty} u(\partial_x^2 + k_0^2)^2 u dx + (V_{cr} - V)\int_{-\infty}^{+\infty} u^2 dx.$$

Our aim now will be to estimate the integral $\int u^3 dx$ through two other integrals in the functional F, namely through

$$I = \int_{-\infty}^{+\infty} u(\partial_x^2 + k_0^2)^2 u dx \quad \text{and} \quad N = \int_{-\infty}^{+\infty} u^2 dx = 2P.$$

We first estimate the integral I through $I_1 = \int u_x^2 dx$ and N. The needed estimate is given by the Sobolev–Gagliardo–Nirenberg inequality (see, for instance, [38]):

$$\int_{-\infty}^{+\infty} u^3 dx \leq C I_1^{1/4} N^{5/4}, \tag{2.32}$$

where the best constant C can be found similarly to that for (2.30). Simple calculations give

$$C_{\rm best} = \sqrt{6}\cdot 5^{-3/4}.$$

The integral I_1 can be expressed through the integral I if one integrates by parts the integral $\int_{-\infty}^{+\infty} u_x^2 dx$, and uses the Schwartz inequality (see [22]):

$$\int_{-\infty}^{+\infty} u_x^2 dx = -\int_{-\infty}^{+\infty} u(u_{xx} + k_0^2 u)dx + \int_{-\infty}^{+\infty} k_0^2 u^2 dx$$
$$\leq N^{1/2}\left[\int_{-\infty}^{+\infty} u(\partial_x^2 + k_0^2)^2 u dx\right]^{1/2} + k_0^2 N, \tag{2.33}$$

and then substitutes this result into (2.32):

$$\int_{-\infty}^{+\infty} u^3 dx \leq C_{\text{best}}(N^{1/2}I^{1/2} + k_0^2 N)^{1/4} N^{5/4}. \tag{2.34}$$

By means of this inequality the functional F can be estimated as follows:

$$F \geq f(I) = \frac{1}{2}[(V_{cr} - V)N + I] - C_{\text{best}}(N^{1/2}I^{1/2} + k_0^2 N)^{1/4} N^{5/4}. \tag{2.35}$$

As can been seen easily, the function $f(I)$ is bounded from below so that the final answer for F takes the form:

$$F \geq \min f(I).$$

This estimate completes the proof [55].

Thus, we have demonstrated that the Hamiltonian for (2.5) is bounded from below for fixed momentum $P = 1/2N$. If the solitary wave solution yielding the minimum of H is not a separate stationary point, then this minimum can be achieved by means of continuous deformations of some initial distribution with finite norms N and I, as was established in [56] in a small vicinity of solitary wave solutions. As was shown in [50] two branches of symmetric solitary waves with exponentially decaying oscillatory tails bifurcate from infinitesimal periodic waves at the minimum phase speed. At this point, unfortunately, we cannot conclude which one of the two branches of solitary wave solutions found in [50] realizes the minimum of H. Probably, the branch which was found to be linearly stable in [57] realizes the minimum of the Hamiltonian. It remains to be justified numerically. As seen above, in the small amplitude limit, the solitary waves can be viewed as modulated wave packets. Using a two-scale perturbation expansion near the maximum of the phase speed, one sees that in a frame moving with the wave, the envelope is governed to leading order by a steady version of the nonlinear Schrödinger (SNLS) equation [52]. But here arises an apparent contradiction: while in [50] only two branches were found, a one-parameter family of (generally asymmetric) solutions arises from the SNLS equation since the envelope of a solitary wave can be shifted relative to its carrier oscillations by an arbitrary amount. In trying to understand this apparent contradiction, Yang & Akylas [53] discovered that the actual structure of solitary wave solution branches near the maximum phase speed is quite complex. They carried out the two-scale expansion underlying the NLS equation beyond all orders using techniques of exponential asymptotics. Out of the one-parameter family of solitary waves solutions of the SNLS equation, only the symmetric branches arising when the phase-shift parameter is such that the maximum of the envelope coincides with either a crest or a trough of the carrier are true solutions of the fifth-order KDV Eq. (2.1). For all other values of the phase shift, there are in fact growing oscillations of exponentially small amplitude in the tails and the resulting wave is not locally confined. Nevertheless, two or more of these nonlocal wave packets can be pieced together if a certain selection criterion is satisfied [53]. The asymptotic results are in agreement with the numerical results in [58], where it was shown that a plethora of multi-pulse solitary wave solutions, including symmetric as well as asymmetric waves, exist.

At the end of this Section it is necessary to underline once more that the Lyapunov stability of the solitary wave solution reaching the minimal value of H for fixed P implies the stability of this solitary wave not only with respect to small perturbations but also relative to finite ones. This stability criterion can in fact be considered as an energy principle. For continuous media as considered here, the solitary wave state corresponding to the minimum of the Lyapunov functional can be achieved by means of radiation of small amplitude waves which provide the relaxation process to the ground solitary wave. This is quite different from the behavior for Hamiltonian systems with a finite number of degrees of freedom.

3. Optical solitons and their bifurcations

3.1. Introducing remarks

In this section we consider how the mechanism of soliton bifurcations discussed in the previous Section for gravity–capillary waves is modified in nonlinear optics, mainly following the paper by Zakharov and Kuznetsov [22].

Solitons propagating in nonlinear optical media, especially in optical fibers, have been a very popular research topic since the beginning of the seventies when solitons in the KDV equation [4] and in the NLS equation [5] were shown to be structural stable and when, a bit later, Hasegawa and Tappet [6] suggested to use optical solitons as the information bit in fiber communications (for a recent review of this field see, e.g. [9]). For optical fibers these solitons are considered as

1D pulses which can be described by the 1D Maxwell equations. This Section mainly deals with 1D optical solitons. The multi-dimensional solitons and their stability will be considered in the next sections.

As is well known in optics (see, e.g. [42]), the spatial dispersion effects are small in comparison with the temporal dispersion ones (their ratio is a small relativistic factor, $\sim v/c$ where v is the characteristic electron velocity in atoms and c the light velocity, thus, this ratio is of order $\alpha = 1/137$). Therefore the expansion of the electric induction $D(t, r)$ in terms of the electric field $E(t, r)$ represents an infinite set with respect to powers of the electric field, evaluated at the same point as the electric induction. Each term of this set contains only time convolutions. This is why in nonlinear optics the NLS equation, for example, is usually written for the electric field envelope, where the spatial coordinate z plays the role of time in the usual NLS equation and t represents the analog of coordinate (see, e.g., [9]). As shown in this Section, this peculiarity also changes the Hamiltonian formulation of the equations of motion, and introduces some pure optical features for solitons, their bifurcations and stability.

When one talks about optical solitons, the soliton spectrum is assumed to be concentrated inside some transparency window where linear damping is small enough and dispersion effects are prevalent. Typically, the soliton spectrum width $\delta\omega$ is supposed to be small enough compared to the frequency band $\Delta\omega$ of this window: $\delta\omega \ll \Delta\omega$. In real systems, however, the frequency band $\Delta\omega$ is narrow relative to the mean window frequency $\overline{\omega}$, $\Delta\omega \ll \overline{\omega}$. Thus, we have the following hierarchy of the characteristic inverse times:

$$\delta\omega \ll \Delta\omega \ll \overline{\omega}. \tag{3.1}$$

Each of these criteria allows one to consider a slow ($\tau^{-1} \sim \delta\omega$) dynamics of soliton propagation in terms of the amplitude envelope as well as more rapid but still slow pulse dynamics with times $\sim 1/\Delta\omega$. In particular, to derive the NLS equation, a basic model for the description of optical envelope solitons, one has to approximate the wave number by a quadratic polynomial

$$\delta k = \frac{1}{v_{gr}}\delta\omega - \frac{1}{2}\frac{\omega''}{v_{gr}^3}(\delta\omega)^2. \tag{3.2}$$

Here $\delta k = k - k_0$ and $\delta\omega = \omega - \omega_0$, $v_{gr} = \frac{\partial\omega}{\partial k}$ is the group velocity, k_0 and ω_0 are the wave number and the frequency of the soliton carrier wave. On the frequency interval $\Delta\omega$, however, the wave dispersion can differ significantly from the quadratic approximation (3.2) remaining still small in the sense of the criterion (3.1). It should also be noted that the current experimental situation (see, for instance, [59,9]) makes it possible to generate very short optical pulses such that $\frac{\delta\omega}{\omega_0} \ll 1$. On the other hand, the efficiency of optical fibers as a transmission medium for information is inversely proportional to soliton length. Hence, the needs of practice dictate using solitons that are as short as possible. The properties of "short" and "long" solitons of course will be quite different. For "short" solitons the expansion (3.2) is no longer valid and it has to be replaced by a more general formula

$$\delta k - \frac{1}{v_{gr}}\delta\omega = -F(\delta\omega). \tag{3.3}$$

Here the function F must be taken from the microscopic consideration or extracted from experimental data. In spite of the fact that the function F may be far from the parabolic dependence (3.2), averaging over the rapid time $1/\omega_0$ can be performed nevertheless, leading to the description of a slow soliton dynamics by means of the generalized nonlinear Schrödinger (GNLS) equation. This average results in the appearance of a new integral of motion, i.e., the adiabatic invariant, which is related to the pulse energy. Due to this invariant, the GNLS equation admits a soliton solution for the envelope of the electromagnetic field $E(x, t)$ in the form of a propagating pulse with an additional phase multiplier $e^{i\lambda x}$:

$$E(x, t - x/v_{gr}) = e^{i\lambda x}\psi(t - x/v_{gr} + \beta x), \quad v_{gr}^{-1} \gg \beta.$$

As seen in the previous Section, solitons can exist if the function $L(\zeta) = \lambda - \beta\zeta + F(\zeta)$ is positive (or negative) definite for all ζ. This criterion coincides with the familiar one for the gravity–capillary solitons in shallow water: it is the main selection rule for optical solitons also. If these criteria are not satisfied, the soliton radiates its energy due to the Cherenkov effect and ceases to exist after a certain time. This phenomenon takes place, in particular, if $F(\zeta)$ is a third-order polynomial. Even if $L(\zeta) > 0$ and solitons exist, the question about their stability is far from being trivial. In this section we establish that the soliton is stable if $L(\zeta)$ is a positive definite fourth-order polynomial when the main nonlinearity is connected with the Kerr effect. The stability proof is based on the boundedness of the Hamiltonian for the fixed adiabatic invariant. This proof is valid for all physical dimensions including $d = 3$.

One more point that we would like to emphasize is that the objects traditionally called "solitons" in nonlinear optics are not solitons in the rigorous meaning of this word. These are "quasisolitons"—approximate solutions of the Maxwell equations, depending on *four* arbitrary parameters. Real "stationary" solitons which propagate at a constant velocity without change of their shapes are *exact* solutions of the Maxwell equations depending at most on *two* parameters. The latter exist if the dielectric permittivity $\varepsilon(\omega)$ has a *maximum* in a considered frequency range for focusing nonlinearities and a *minimum* if the medium is defocusing. In pure conservative media quasisolitons exist only for a finite amount of time and radiate due to multi-photon processes. In practice, however, this time happens to be much larger than the lifetime due to the linear damping, and a difference between solitons and quasisolitons is not significant.

3.2. Stationary solitons

Now we demonstrate how soliton solutions can be found directly from the Maxwell equations. We consider the simplest model for one-dimensional pulse propagation assuming that the polarization is linear and perpendicular to the propagation axis. In this case the Maxwell equations, which read (c denotes the speed of light)

$$\frac{\partial E}{\partial x} = -\frac{1}{c}\frac{\partial B}{\partial t}, \qquad \frac{\partial B}{\partial x} = -\frac{1}{c}\frac{\partial D}{\partial t} \tag{3.4}$$

can be reduced to the wave equation for the electric field $E(x, t)$:

$$\frac{\partial^2 D}{\partial t^2} - c^2\frac{\partial^2 E}{\partial x^2} = 0 \tag{3.5}$$

where the electric induction D is assumed to be connected with the electric field through the relation

$$D(x, t) = \hat{\varepsilon}(t)E(x, t) + \chi E^3(x, t). \tag{3.6}$$

In this expression $\hat{\varepsilon}$ is the integral operator; the Fourier transform of its kernel, $\varepsilon(\omega)$, is the dielectric permittivity. The second term in (3.6) corresponds to the Kerr effect; χ is the Kerr constant.

The function $\varepsilon(\omega)$ is extended analytically to the upper half-plane of ω (see, for instance, [42]). For real values of ω, $\varepsilon(\omega)$ obeys the Kramers–Kronig relations. In particular, it follows from these relations that on the real axis the imaginary part of ε, ε'', responsible for the dissipation of electromagnetic waves, cannot be equal to zero for all frequencies. Moreover we will suppose that there exists some frequency band, $\Delta\omega$, where the imaginary part of the permittivity is small enough and inside this band ε'' will be neglected.

Consider the propagation of the wave packet with spectra within this transparency window, assuming that the frequency width of the pulse spectrum is small compared with $\Delta\omega$. Only under such conditions can one expect a soliton solution. As mentioned above, two kinds of solitons are possible. The first type of solitons is a stationary soliton which propagates at a constant velocity without change of its shape. Another kind of soliton is the quasi-soliton. Quasisolitons have some internal dynamics and only on average do they propagate with a constant velocity. The classical examples of quasisolitons are breathers in the sine–Gordon equation (for details see, e.g., [12,14,13]).

Stationary solitons are exact solutions of the Eq. (3.5) of the form

$$E = E(x - Vt), \tag{3.7}$$

where V is a constant velocity and E is assumed to vanish at infinity. Substituting (3.7) into Eq. (3.5) allows the equation to be integrated twice:

$$\hat{L}E(x) = \alpha E^3(x), \quad \alpha = \chi\frac{V^2}{c^2} \tag{3.8}$$

where the operator $\hat{L}$ is equal to

$$\hat{L} = 1 - \frac{V^2}{c^2}\hat{\varepsilon}. \tag{3.9}$$

In the Fourier representation the operator $\hat{L}$ can be written in the form:

$$L(\omega) = 1 - \frac{V^2\varepsilon(\omega)}{c^2} \tag{3.10}$$

where the frequency ω and the wave number k are linked through the relation $\omega = kV$. The second term in (3.10) is the square of the ratio between V and the phase velocity for an electromagnetic monochromatic wave of small amplitude,

$$V_{ph} = \frac{c}{\sqrt{\varepsilon(\omega)}}. \tag{3.11}$$

Hence it is easy to see that the operator $\hat{L}$ becomes positive definite if and only if for all ω

$$V_{ph}^2(\omega) > V^2 \tag{3.12}$$

and, respectively, negative definite in the opposite case:

$$V_{ph}^2(\omega) < V^2. \tag{3.13}$$

Only in the case when conditions (3.12) or (3.13) are fulfilled a soliton solution is possible. Suppose that the conditions (3.12) and (3.13) are not satisfied, i.e., the equation

$$\frac{V^2\varepsilon(\omega)}{c^2} = 1 \tag{3.14}$$

has a solution (for simplicity suppose that this solution is unique: $\omega = \omega_0$). Then the Eq. (3.7) can be rewritten as follows

$$E(x - Vt) = E_0(x - Vt) + \hat{L}^{-1}(1 - \hat{P})\alpha E^3(x - Vt). \tag{3.15}$$

Here

$$E_0(x - Vt) = \mathrm{Re}\,(A \exp[-i\omega_0(t - x/V)])$$

is a solution of the homogeneous linear equation

$$\hat{L}E_0 = 0 \tag{3.16}$$

and $\hat{P}$ is the projector to the state $E_0(x - Vt)$ so that $(1 - \hat{P})\chi E^3(x - Vt)$ is orthogonal to E_0 and therefore on this class of functions the operator $\hat{L}$ is invertible. To find an explicit solution of the Eq. (3.15) one can apply, for instance, the iteration scheme by taking E_0 as a zeroth approximation. As a result of such an iteration the solution will be a nonlinear periodic wave with period corresponding to the frequency ω_0. Its Fourier spectrum will be a set of delta functions $\delta(\omega - n\omega_0)$ with integer n. Of importance is that by so doing we will necessarily come to a *nonlocalized* solution which will depend on two parameters, i.e. the imaginary and real parts of the complex amplitude A. Hence one can make the following conclusion: *The stationary* Eq. (3.7) *can have a soliton solution if the operator* $\hat{L}$ *is sign definite*. If Eq. (3.16) has a nontrivial (real) solution, which is equivalent to stating that the phase velocity V_{ph} and the velocity V coincide,

$$V_{ph} = V, \tag{3.17}$$

a stationary soliton solution is absent. Note that this conclusion is based on the fact that the singularity in the right hand side of Eq. (3.15) $(E^3)_\omega/L(\omega)$ cannot be removed. As shown below, such singularities can be removed if the matrix element of the four-wave interaction (in the given case—χ) has some dependence on frequencies.

Eq. (3.17) is nothing more than the condition (1.1) for the Cherenkov radiation for a moving object. And it does not matter what is the nature of this object. It may be a charged particle, a ship, or, for instance, a soliton. In any case this object will lose its energy. In the given case this means that an electromagnetic soliton (or better to say—a pulse) which moves with the velocity V satisfying the condition (3.13) will necessarily radiate waves and, consequently, such a pulse cannot exist as a stationary object. Thus, we arrive at the following criterion for the soliton existence: a soliton solution in some model can exist when the equation

$$\omega(k) = kV \tag{3.18}$$

has no (real) solution. Here $\omega = \omega(k)$ is the dispersion relation. For electromagnetic waves $\omega(k)$ is determined from the equation

$$\omega^2 = \frac{k^2c^2}{\varepsilon(\omega)}. \tag{3.19}$$

The relation (3.18) has a simple interpretation in the ω—k plane (see Fig. 1). The r.h.s. of (3.18) corresponds to the straight line going through the origin and, respectively, the velocity V in this plane is equal to the tangent of the slope angle ϕ,

$$V = \tan\phi.$$

The existence of a solution to Eq. (3.18) means an intersection of the curve $\omega = \omega(k)$ with the straight lines that define the whole cone of angles Ω where the stationary soliton solutions are impossible. Possible soliton solutions correspond to the cone complementary to Ω, $\tilde{\Omega}$. In this case, $\partial\Omega$, straight lines $\omega = kV$ touch the dispersive curve $\omega = \omega(k)$ on the boundaries of the cones. At the touching points k_i, the group and phase velocities coincide:

$$\left.\frac{\omega(k)}{k}\right|_{k_i} = \left.\frac{\partial\omega(k)}{\partial k}\right|_{k_i}. \tag{3.20}$$

For the dispersion law (3.19) this relation yields

$$\left.\frac{d\varepsilon(\omega)}{d\omega}\right|_{\omega_i} = 0. \tag{3.21}$$

At these critical points solitons undergo bifurcation, since outside the cone $\tilde{\Omega}$ stationary soliton solutions are absent. For supercritical bifurcations, it is shown below that the behavior of the optical soliton solutions near these critical points is universal, like for the capillary–gravity solitons in shallow water considered in the previous Section. We demonstrate this fact by using the example of the stationary Eq. (3.7) but it is important to emphasize that the proof is quite general and can be extended to many other models. It is necessary to repeat that this fact was first investigated for capillary–gravity solitons in deep water [60,16,19]. For capillary–gravity waves the dispersion law has a minimum phase velocity in the region connecting the gravity and capillary spectrum ranges (see Fig. 1).

Suppose for simplicity that Eq. (3.21) has only one solution $\omega = \omega_0$ and the cone of angles $\tilde{\Omega}$ lies below the critical velocity,

$$V < V_{cr} = \frac{c}{\sqrt{\varepsilon(\omega_0)}},$$

namely, the function $\varepsilon(\omega)$ has two equal maxima at two symmetric points so that

$$\frac{d^2\varepsilon(\pm\omega_0)}{d\omega^2} < 0.$$

In this case the operator $\hat{L}$ is invertible and Eq. (3.8) can be written as

$$E_\omega = \frac{1}{L(\omega)}\alpha(E^3)_\omega. \tag{3.22}$$

Near the critical velocity, $V_{cr} - V \ll V_{cr}$, $L(\omega)$, which is an even function of ω, is close to zero near two symmetrical points $\omega = \pm\omega_0$. Therefore, according to Eq. (3.22), the distribution of $E(\omega)$ will follow, to a large extent, that for the function $1/L(\omega)$. This means that in the t-representation a solution should be closed to the monochromatic wave. The wave monochromaticity increases as V approaches V_{cr}. Secondly, we will assume that the soliton amplitude vanishes smoothly while approaching the critical velocity V_{cr}. In fact this assumption means that the bifurcation we expect is supercritical. Therefore, $E(t')$ $(t' = t - x/V)$ can be sought as an expansion over the harmonics $n\omega_0$ (compare with (2.11)):

$$E(t) = \sum_{n=0}^{\infty}[E_{2n+1}(\tau)e^{-i(2n+1)\omega_0 t'} + c.c.]. \tag{3.23}$$

Here we formally introduced the small parameter

$$\epsilon = \sqrt{1 - \frac{V}{V_{cr}}} \tag{3.24}$$

and the slow time $\tau = \epsilon t'$ so that the coefficients $E_{2n+1}(\tau)$ are the amplitude envelopes for each harmonics. Such representation means also that the frequency width of each harmonics, $\delta\omega \sim \epsilon$, is small in comparison with the frequency ω_0, i.e., the Fourier spectrum of (3.23) represents a set of narrow peaks. Two main peaks correspond to the first harmonics. Therefore the action of the operator $\hat{L}$ on (3.23) can be also expanded into a series with respect to ϵ. Assuming the amplitude E_{2n+1} to be of order ϵ^{2n+1} and substituting (3.23) into the stationary Eq. (3.8) together with (3.21), leads to the stationary NLS equation at leading order:

$$\epsilon^2 E_1 - S\frac{\partial^2 E_1}{\partial t'^2} - \frac{3}{2}\alpha|E_1|^2 E_1 = 0 \tag{3.25}$$

where

$$S = -\frac{V_{cr}^2}{4c^2}\frac{d^2\varepsilon(\omega_0)}{d\omega^2} > 0. \tag{3.26}$$

Eq. (3.25) has a soliton solution only if $\alpha > 0$:

$$E_1(t') = \frac{2\epsilon}{\sqrt{3\alpha}}\,\mathrm{sech}\left[\frac{\epsilon(t - x/V - t_0)}{\sqrt{S}}\right]. \tag{3.27}$$

This solution is unique up to a constant phase multiplier. It follows the universal asymptotics for stationary solitons: its amplitude vanishes like $\sqrt{V_{cr} - V}$ as V approaches V_{cr} and the soliton duration Δt grows inversely to the factor $\Delta t = \sqrt{S}/\epsilon$. Thus, near the bifurcation point, optical solitons have the same behavior as the capillary–gravity solitons in the shallow water case.

For times larger than Δt it is necessary to take into account the next terms in the expansion, in particular the third-order dispersion and corrections to the cubic nonlinearity. Already the soliton behavior becomes non-universal in this temporal region,

It is important to note that for $\epsilon^2 = 1 - V/V_{cr} < 0$ Eq. (3.25) has no soliton type solutions. When the touching of the dispersive curve occurs from above and the parameter S becomes negative, solitons exist only for defocusing media ($\chi < 0$).

3.3. Quasisolitons and higher-order dispersion

In this subsection which deals with the example of the generalized nonlinear Schrödinger (GNLS) equation we discuss what is the difference between the solitons considered in the above subsection and quasisolitons. The GNLS equation has a wider class of soliton solutions than the original Maxwell equations. Unlike stationary solitons (3.27), these solutions are approximate, and depend on four parameters. However, as shown below, the selection rule for quasi-soliton solutions remains the same as for the stationary solitons considered previously. Let the transparency window $\Delta\omega$ be small compared with the mean value of frequency ω_0 : $\omega_0 \gg \Delta\omega$. In this case one can introduce the envelope $E_1(x, t)$ for the whole region. In order to derive the equation for envelopes the most convenient approach is the one based on the Hamiltonian formalism [33].

Consider Eq. (3.5) which can be conveniently represented as a system of two equations:

$$\frac{\partial \rho}{\partial x} + \frac{\partial^2 \phi}{\partial t^2} = 0, \qquad \frac{\partial \phi}{\partial x} + \frac{1}{c^2}\left(\hat{\varepsilon}\rho + \frac{4\pi\chi}{c^2}\rho^3\right) = 0. \tag{3.28}$$

The potential ϕ and the "density" ρ are related to the electric (E) and magnetic (B) fields as follows:

$$E = \frac{\sqrt{4\pi}}{c}\rho, \qquad B = \sqrt{4\pi}\frac{\partial \phi}{\partial t}. \tag{3.29}$$

The Eq. (3.28) can be written in the Hamiltonian form:

$$\frac{\partial \rho}{\partial x} = \frac{\delta \mathcal{H}}{\delta \phi}; \qquad \frac{\partial \phi}{\partial x} = -\frac{\delta \mathcal{H}}{\delta \rho}, \tag{3.30}$$

where x plays the role of time and the Hamiltonian is the integral over time

$$\mathcal{H} = \int \left[\frac{1}{2}\left(\frac{\partial \phi}{\partial t}\right)^2 + \frac{1}{2c^2}\rho\hat{\varepsilon}\rho + \frac{\pi\chi}{c^4}\rho^4\right] dt \equiv \frac{1}{8\pi}\int \left(B^2 + E\hat{\varepsilon}E + \frac{1}{2}\chi E^4\right) dt. \tag{3.31}$$

The quadratic part of $\mathcal{H}$ yields the linear dispersion law $k = k(\omega)$, which coincides with (3.19). The transition to the normal variables $a_\omega(x)$ is obtained through the change

$$\rho_\omega = \sqrt{\frac{\omega^2}{2k(\omega)}}(a^*_\omega + a_{-\omega}); \qquad \phi_\omega = -i\sqrt{\frac{k(\omega)}{2\omega^2}}(a^*_\omega - a_{-\omega}), \tag{3.32}$$

where ρ_ω and ϕ_ω are the Fourier images of the "density" ρ and the potential ϕ. In these formula $k(\omega)$ is understood as a positive root of the dispersive relation (3.19). Substituting these relations into Eq. (3.30) gives the equations of motion in terms of a_ω:

$$\frac{\partial a_\omega}{\partial x} = i\frac{\delta \mathcal{H}}{\delta a^*_\omega}, \tag{3.33}$$

where the Hamiltonian $\mathcal{H}$ takes the standard form (compare with [33]):

$$\mathcal{H} = \int k(\omega)|a_\omega|^2 d\omega + \frac{1}{2}\int T_{\omega_1\omega_2\omega_3\omega_4} a^*_{\omega_1} a^*_{\omega_2} a_{\omega_3} a_{\omega_4} \delta_{\omega_1+\omega_2-\omega_3-\omega_4} \prod_i d\omega_i. \tag{3.34}$$

The matrix element T is given by the formula

$$T_{\omega_1\omega_2\omega_3\omega_4} = \frac{3\chi}{4c^4}\left[\frac{\omega_1^2\omega_2^2\omega_3^2\omega_4^2}{k(\omega_1)k(\omega_2)k(\omega_3)k(\omega_4)}\right]^{1/2}. \tag{3.35}$$

If the susceptibility χ depends on frequencies then the constant χ in the matrix element (3.35) must be changed into $\chi(\omega_1, \omega_2, \omega_3, \omega_4)$ with the necessary symmetry properties (see, for instance, [42,24]). Consequently the matrix element T has the following symmetry relations:

$$T_{\omega_1\omega_2\omega_3\omega_4} = T_{\omega_2\omega_1\omega_3\omega_4} = T_{\omega_1\omega_2\omega_4\omega_3} = T^*_{\omega_3\omega_4\omega_1\omega_2}. \tag{3.36}$$

In the Hamiltonian (3.34) we kept only the terms responsible for wave scattering and neglected all other processes which contribute at next (sixth) order in the wave amplitude.

The Hamiltonian formulation of the equations of motion (3.33) guarantees "conservation" (independence on x) of the Hamiltonian $\mathcal{H}$ and also of the "momentum"

$$P = \int \omega|a_\omega|^2 d\omega, \tag{3.37}$$

which exactly coincides with the integration over time of the Pointing vector

$$P = \frac{c}{4\pi}\int_{-\infty}^{\infty} EBdt \equiv \int_{-\infty}^{\infty} \rho\frac{\partial \phi}{\partial t}dt.$$

Let us now derive the equations for the envelopes by introducing the envelope amplitude for the wave packet

$$\psi(t,x)=\frac{1}{\sqrt{2\pi}}\int a_\omega e^{-i(\omega-\omega_0)t-ik_0(\omega_0)x}d\omega.$$

Here we assume that the spectrum of a_ω is concentrated in the narrow interval $\delta\omega$ near the frequency ω_0, $\omega_0\gg\delta\omega$. The amplitude $\psi(t,x)$ is a slowly varying function of coordinate and time.

Next one expands $k(\omega)$ and $T_{\omega_1\omega_2\omega_3\omega_4}$ in series with respect to $\Omega=\omega-\omega_0$ near the point ω_0,

$$\kappa(\Omega)=k(\omega)-k(\omega_0)=\frac{1}{V_{gr}}\Omega-k_0S\Omega^2-\gamma\Omega^3+\delta\Omega^4+\cdots, \tag{3.38}$$

$$\begin{aligned}T_{\omega_1\omega_2\omega_3\omega_4} &= T_0+\frac{\partial T}{\partial\omega_1}(\Omega_1+\Omega_2+\Omega_3+\Omega_4)+\frac{1}{2}\frac{\partial^2T}{\partial\omega_1^2}(\Omega_1^2+\Omega_2^2+\Omega_3^2+\Omega_4^2)\\ &\quad+\frac{\partial^2T}{\partial\omega_1\partial\omega_2}(\Omega_1\Omega_2+\Omega_3\Omega_4)+\frac{\partial T}{\partial\omega_1\partial\omega_3}(\Omega_1\Omega_3+\Omega_1\Omega_4+\Omega_2\Omega_3+\Omega_2\Omega_4)+\cdots.\end{aligned} \tag{3.39}$$

We kept all terms up to fourth order in Ω in the expression for $k(\omega)$ and all terms up to second order in Ω in the matrix element T. For the expansion of the matrix element we assumed T to be real for the sake of simplicity and used its symmetry properties (3.36). In this case the coefficients are equal to

$$\begin{aligned}&T_0=T_{\omega_0\omega_0\omega_0\omega_0}=\frac{3\chi}{4c^4}\frac{\omega_1^4}{k^2(\omega_1)};\\ &\frac{\partial T}{\partial\omega_1}=\left.\frac{\partial T_{\omega_1\omega_2\omega_3\omega_4}}{\partial\omega_1}\right|_{\omega_k=\omega_0}=\frac{\partial T_{\omega_1\omega_2\omega_3\omega_4}}{\partial\omega_1}=\frac{3\chi}{4c^4}\frac{\partial}{\partial\omega_1}\left[\frac{\omega_1^2\omega_2^2\omega_3^2\omega_4^2}{k(\omega_1)k(\omega_2)k(\omega_3)k(\omega_4)}\right]^{1/2}\\ &\frac{\partial^2T}{\partial\omega_i}\partial\omega_j=\left.\frac{\partial^2T_{\omega_1\omega_2\omega_3\omega_4}}{\partial\omega_i\partial\omega_j}\right|_{\omega_k=\omega_0}\frac{\partial}{\partial\omega_1}\frac{\omega_1}{k(\omega_1)^{1/2}}=\frac{1}{k(\omega_1)^{1/2}}\left[1-\frac{V_{ph}}{2V_{gr}}\right].\end{aligned}$$

Taking the Fourier transform with respect to Ω yields the generalized NLS equation for ψ:

$$\begin{aligned}&i\left(\frac{\partial\psi}{\partial x}+\frac{1}{V_{gr}}\frac{\partial\psi}{\partial t}\right)+k_0S\psi_{tt}+\beta_1|\psi|^2\psi=-i\gamma\psi_{ttt}-4i\beta_2|\psi|^2\psi_t\\ &\quad-\delta\psi_{tttt}+(\beta_3-\beta_4)\left[(\psi^2\psi_t^*)_t-(\psi_t)^2\psi^*\right]+(\beta_3+\beta_5)\psi^*(\psi^2)_{tt}-\beta_6|\psi|^4\psi.\end{aligned} \tag{3.40}$$

The left hand side of this equation corresponds to the classical NLS equation: the second term in this part describes the propagation of the wave packet as a whole and therefore can be excluded when going to the moving system of coordinates. The next term ($\sim S$) is responsible for the quadratic dispersion. In the case $d\varepsilon(\omega_0)/d\omega_0=0$ the coefficient S coincides with the expression (3.26). The last term in this group defines the nonlinear frequency shift for a monochromatic wave. The first two terms in the right hand side are proportional to $(\delta\omega/\omega_0)^3$. It is important that there are only two such terms and that the coefficient $\beta_2=2\pi\frac{\partial T}{\partial\omega}$ is not equal to zero, even for a constant susceptibility χ. At $\chi=const$ the coefficient β_2 can vanish only if $k\sim\omega^2$. All the other terms are proportional to $(\delta\omega/\omega)^4$. Among them there is the term $\sim|\psi|^4\psi$ which possesses the same order of magnitude.

The coefficients β_i in Eq. (3.40) have a simple form for the matrix element (3.35):

$$\beta_1=\frac{3}{2}\pi k_0^2\chi\left(\frac{V_{ph}}{c}\right)^4;\qquad \beta_2=\frac{\beta_1}{\omega_0}\left(1-\frac{V_{ph}}{2V_{gr}}\right); \tag{3.41}$$

$$\beta_3=\beta_1\frac{k^{1/2}}{\omega_0}\frac{\partial^2}{\partial\omega_0^2}\left(\frac{\omega_0}{k^{1/2}}\right);\qquad \beta_4=\beta_5=\frac{\beta_1}{\omega_0^2}\left(1-\frac{V_{ph}}{2V_{gr}}\right)^2.$$

Eq. (3.40) has an Hamiltonian formulation:

$$i\frac{\partial\psi}{\partial x}=-\frac{\delta\mathcal{H}}{\delta\psi^*}. \tag{3.42}$$

Here the Hamiltonian $\mathcal{H}$ is represented as a sum of Hamiltonians $\mathcal{H}_i$:

$$\mathcal{H}=\mathcal{H}_1+\mathcal{H}_2+\mathcal{H}_3+\mathcal{H}_4+\cdots,$$

where

$$\mathcal{H}_1=\frac{i}{V_{gr}}\int\psi^*\psi_t dt, \tag{3.43}$$

$$\mathcal{H}_2 = -\int \left(k_0 S|\psi_t|^2 - \frac{\beta_1}{2}|\psi|^4 \right) dt, \tag{3.44}$$

$$\mathcal{H}_3 = \int [i\gamma\psi^*\psi_{ttt} + i\beta_2(\psi^*\psi_t - \psi\psi_t^*)|\psi|^2]dt, \tag{3.45}$$

$$\mathcal{H}_4 = \int \left[\delta|\psi_{tt}|^2 - \frac{\beta_3}{2}|\psi|^2(\psi\psi_{tt}^* + c.c.) - \frac{\beta_4}{2}(\psi_t^2\psi^{*2} + c.c.) - \frac{\beta_5}{2}\psi^{*2}\partial_t^2\psi^2 + \frac{\beta_6}{3}|\psi|^6 \right] dt. \tag{3.46}$$

$\mathcal{H}_2$ corresponds to the classical NLS equation, and the next ones to the complex MKDV equation. It is important that each Hamiltonian is small compared to the previous one. However, such a situation can change if, for some reason, some coefficients have an additional smallness. As seen from (3.27), the soliton width decreases with the coefficient of quadratic dispersion S. So, at small S (such situation occurs near the so-called zero dispersion point) it is necessary to take into account only the cubic dispersion ($\sim\gamma$) and to neglect all higher-order terms, and also the term proportional to β_2. On the opposite, if the coefficient β_1 is small, one needs to take into account the nonlinear dispersion $\sim\beta_2$ and to neglect the cubic linear dispersion. As shown in the next sections, one also needs to keep the last term in (3.46) in the latter case.

Let us now analyze the solutions for the generalized NLS equation. We start from the NLS equation with quadratic dispersion (corresponding to the Hamiltonian (3.44))

$$i\frac{\partial\psi}{\partial x} + \psi_{tt} + 2|\psi|^2\psi = 0 \tag{3.47}$$

to illustrate how the mechanism (3.27) presented above works. Here the dimensionless variables are used and the nonlinearity is assumed to be focusing: $S\chi > 0$.

It should be noted that, unlike the wave Eq. (3.5), the general NLSE (3.40) and, in particular, the NLSE with quadratic dispersion has one additional symmetry, namely, the gauge symmetry: $\psi \to \psi e^{i\phi}$, that appears as a result of the averaging over rapid oscillations. Therefore the envelope soliton solutions form a broader class than solutions to the wave equation (3.5) and, according to our definition, they must be related to quasisolitons. To find the corresponding solution one should put $\psi(x,t) = e^{i\lambda x}\psi_s(t+\beta x)$ where ψ_s obeys the equation:

$$L(i\partial_t)\psi \equiv -i\beta\psi_t + \lambda\psi - \psi_{tt} = 2|\psi|^2\psi, \tag{3.48}$$

where the subscript s is omitted. Here the parameter β, which corresponds to an inverse velocity for the Eq. (3.40) in dimensional variables, is equal to the difference between the soliton and the group velocities divided by v_{gr}^2.

According to (3.18), the condition for Cherenkov radiation will be written as follows:

$$\beta\Omega = k(\Omega) \quad \text{or} \quad L(\Omega) = 0, \tag{3.49}$$

where the dispersion law for the Eq. (3.48) takes the form:

$$k(\Omega) = \lambda + \Omega^2. \tag{3.50}$$

Hence one can see that for $\lambda < 0$ the resonance condition (3.49) is satisfied for all values of the "velocity" β ! (see Fig. 2) and consequently solitons do not exist in this case. This can be checked directly by solving the Eq. (3.48): for $\lambda < 0$ all solutions are periodic or quasi-periodic. Soliton solutions are possible for $\lambda > 0$. Their velocities lie in the range $-2\sqrt{\lambda} \geq \beta \geq 2\sqrt{\lambda}$ (see Fig. 3). At the points $\Omega = \pm\sqrt{\lambda}$ the dispersive curve $k = k(\Omega)$ touches the straight line $k = \beta_{cr}\Omega$. At these points the solution must vanish in agreement with the general theory. It directly follows from the exact solution of (3.48):

$$\psi = e^{i\lambda x}\frac{e^{i\beta t'}\Delta\Omega}{\cosh(\Delta\Omega t')}, \quad \Delta\Omega = \sqrt{\lambda - \beta^2/4}. \tag{3.51}$$

Solitons exist only for $\lambda > \beta^2/4$. The upper boundary in this inequality defines the critical velocity:

$$\beta_{cr} = \pm 2\sqrt{\lambda}.$$

Note also that at $\lambda > \beta^2/4$ the operator L in Eq. (3.48) is positive definite.

Now let us consider the case of third-order dispersion. Suppose as before that the desired solution depends on x exponentially:

$$\psi(x,t) = e^{i\lambda x}\psi_s(t'), \quad t' = t + \beta x. \tag{3.52}$$

The corresponding operator $L(i\partial_t)$ has the form

$$L(\Omega) = -\beta\Omega + \lambda + S\Omega^2 + \gamma\Omega^3. \tag{3.53}$$

This operator for any values λ, β, S and $\gamma \neq 0$ is not sign definite. This means that the equation $L(\Omega) = 0$ or its equivalent form

$$\beta\Omega = \lambda + S\Omega^2 + \gamma\Omega^3$$

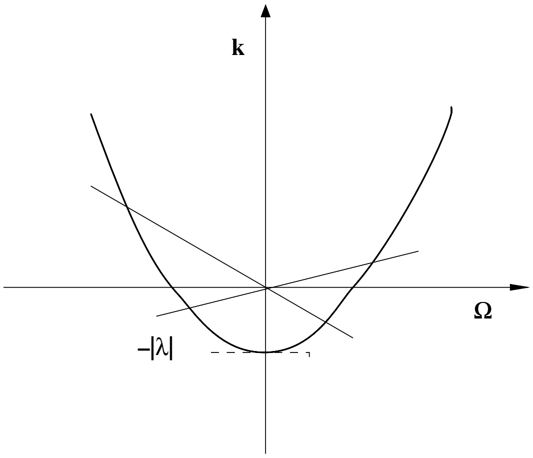

Fig. 2. The dispersive curve (3.50) for negative λ. Any straight lines leaving the origin always intersect the curve.

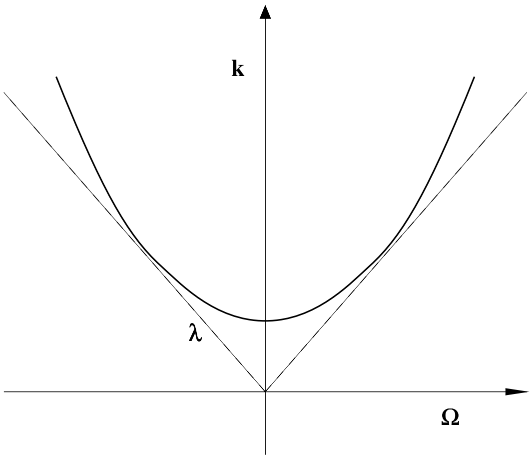

Fig. 3. The dispersive curve (3.50) for positive λ. Dashed lines tangent to the curve correspond to the critical velocities $\beta = \pm 2\sqrt{\lambda}$. The soliton velocity cone is limited by these two dashed lines.

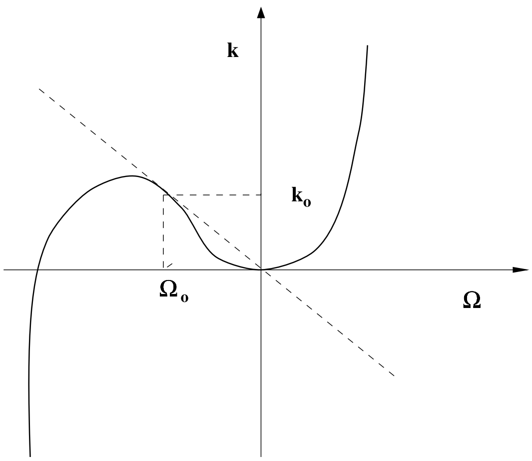

Fig. 4. The third-order dispersion $k = D\Omega^2 + \gamma\Omega^3$. The straight line dispersive curve (3.50) for positive λ. The straight line (dashed) touches the dispersive curve at the point $\Omega = \Omega_0$ but intersects the curve at $\Omega = 0$.

has at least one real solution: any straight line going through the origin always intersects the dispersive curve $k(\Omega) = \lambda + S\Omega^2 + \gamma\Omega^3$. For example, at $\lambda = 0$ and $\beta \geq \beta_0 = -S^2/(4\gamma)$ all straight lines $k = \beta\Omega$ intersect twice the curve $k = k(\Omega)$. At $\beta < \beta_0$ straight lines have one intersection, and touching takes place at $\beta = \beta_0$ (see Fig. 4). However, one intersection is enough to exclude solitons. From a different point of view, the example of the KDV equation, which simultaneously has cubic linear dispersion and solitons, would seem to be in contradiction with what we just said. But there are no contradictions. The reason is the dependence of the matrix element on wave vectors that provides the cancelation of singularities in the equation of type (3.22).

Let us show on the example of the KDV equation

$$U_t + U_{xxx} + 6UU_x = 0, \tag{3.54}$$

how such a cancelation of singularities occurs. For solitons moving with velocity v

$$L(k) = ik(v + k^2).$$

If $v > 0$ the equation $L(k) = 0$ has one real root $k = 0$. In this case the analog of (3.22) reads

$$U_k = \frac{3ik(U^2)_k}{L(k)},$$

which, evidently, does not contain a singularity at $k = 0$. A similar situation occurs for other equations of the KDV type (see, for instance, [61]). It is necessary to mention the papers [62] where the soliton type solutions were found for the GNLSE with third-order dispersion and the corresponding nonlinearity ((3.44), (3.45)). Being formally correct these solutions have a spectrum concentrated in the frequency range $\Omega \sim 1/\gamma,\ 1/\beta_2$, comparable with the frequency ω_0. In other words they have parameters which are not compatible with perturbation theory. The existence of such solitary waves indicates that the cancelation of singularities also takes place. The soliton spectrum is shifted by a small value when there is the following relation between the coefficients S, γ, β_1 and β_2:

$$\frac{k_0 S}{\beta_1} = \frac{3\gamma}{4\beta_2}.$$

This situation is quite special: Eq. (3.40) (written in dimensionless variables)

$$i\psi_x + \psi_{tt} + 2|\psi|^2\psi = i\epsilon(\psi_{ttt} + 6|\psi|^2\psi_t) \tag{3.55}$$

allows the application of the inverse scattering transform (see, for instance, [12]). In this case the Hamiltonians (3.44) and (3.45) are conserved separately. Both Hamiltonians are generated by the same auxiliary operator, namely the Zakharov–Shabat operator [5]. The parameter ϵ in this equation is of order $\delta\omega/\omega$ and ψ takes values of order one. The soliton solution for this equation was first presented in the papers [63]. The simplest solution among them is

$$\psi = e^{i\mu^2 x}\frac{\mu}{\cosh[\mu(t - \epsilon\mu^2 x)]},$$

which transforms at $\epsilon = 0$ into the stationary NLS soliton (3.51).

From the above results one can make the following conclusion: the existence of soliton solutions for the operators L of third order is linked to presence of the derivative in the nonlinear term or, in another words, to the matrix elements dependence on frequencies. If such dependence is absent or not essential (as it is, for instance, near a zero dispersion point) then there are no reasons for the cancelation of singularities in equations of type (3.22).

In the following we restrict ourselves to the case where the nonlinearity dispersion is absent or not essential. In such a situation the third-order dispersion cannot provide soliton existence—one needs to take into account the next order terms.

For the fourth-order dispersion the corresponding operator L reads

$$L(\Omega) = -\beta\Omega + \lambda + S\Omega^2 + \gamma\Omega^3 + \delta\Omega^4. \tag{3.56}$$

The sign definiteness of L will now be defined by the sign of the parameter δ: at $\delta > 0$ the function L will be positive for large Ω and, respectively, negative in the opposite case.

By an appropriate frequency shift, $\Omega \to \Omega + \nu$, one can always exclude the cubic term from the operator L. Secondly, with the help of simple rescaling and sign change, $L(\Omega)$ can be transformed to the two following canonical forms:

$$L(\Omega) = -\beta\Omega + k(\Omega) = -\beta\Omega + \lambda + (\Omega^2 - \nu_0^2)^2, \tag{3.57}$$

$$L(\Omega) = -\beta\Omega + k(\Omega) = -\beta\Omega + \lambda + (\Omega^2 + \nu_0^2)^2. \tag{3.58}$$

Applying the criterion (3.49) to the dispersion law (3.57) for $\lambda < 0$ yields satisfaction of the resonance condition (3.49) for all values of β and, thus, in this parameter region solitons are impossible. For positive $\lambda = \mu^4 > 0$ solitons are possible at the region $-\beta_{cr} \le \beta \le \beta_{cr}$ where

$$\beta_{cr} = 4\Omega_0(\Omega_0^2 - \nu_0^2) \quad \text{or} \quad \Omega_0^2 = \frac{1}{6}\left(2\nu_0^2 + \sqrt{16\nu_0^4 + 12\mu^4}\right). \tag{3.59}$$

Near the critical velocity (3.59) the dispersion is positive and therefore localized solitons can only exist for the focusing ($\delta\chi > 0$) nonlinearity, which in contrast, would be defocusing in the case of quadratic dispersion.

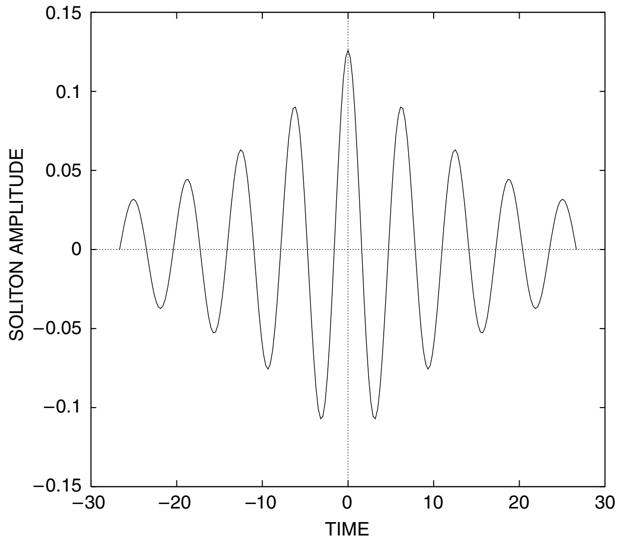

Fig. 5. The soliton shape at $\mu/\nu_0 = 1/3$. The soliton amplitude and time are measured in units ν_0^2 and ν_0^{-1}, respectively. The soliton envelope has a form which is close to the sech form.

The soliton shape is determined from the equation:

$$L(i\partial_t)\psi = 2\sigma|\psi|^2\psi, \tag{3.60}$$

where $L(i\partial_t)$ is given by formulas (3.57) and (3.58); $\sigma = \mathrm{sign}(\delta\chi)$ defines the character of the nonlinear interaction: focusing for $\sigma = 1$ and, respectively, defocusing for negative σ. Soliton solutions are possible only for the focusing case. The simplest solutions of (3.60) are standing solitons (they have zero speed). Their form for L (3.58) is found by integrating the equation

$$\mu^4\psi + (\partial_t^2 + \nu_0^2)^2\psi - 2|\psi|^2\psi = 0. \tag{3.61}$$

It is important that a moving soliton with fourth-order dispersion has a shape which is different from that of the soliton for the NLSE with quadratic dispersion and cannot be distorted into a standing soliton through phase transform and simple rescaling.

In order to find solutions, Eq. (3.61) should be supplemented by the boundary conditions:

$$\psi, \psi_t \to 0 \quad \text{as}\ t \to \pm\infty.$$

The symmetry of Eq. (3.61) with respect to t allows one to look for real symmetric solutions: $\psi(t) = \psi(-t) = \psi^*(t)$. At infinity ($t \to \pm\infty$) these solutions must vanish exponentially, $\psi \sim e^{\nu t} \to 0$, where the exponents ν are defined from the equation

$$\mu^4 + (\nu^2 - \nu_0^2)^2 = 0.$$

The roots of this equation are given by the expressions

$$\nu = \pm\left[\frac{1}{2}\left(\sqrt{\mu^4 + \nu_0^4} + \nu_0^2\right)\right]^{1/2} \pm i\left[\frac{1}{2}\left(\sqrt{\mu^4 + \nu_0^4} - \nu_0^2\right)\right]^{1/2}. \tag{3.62}$$

Namely, all the roots are complex. In particular, this means that standing soliton solutions must have an oscillating structure. If $\mu \sim \nu_0$, the ratio between the imaginary and real parts of the exponents is of the same order. The critical touching takes place at $\mu = 0$. Near this point the real part ν' vanishes, but the imaginary part remains:

$$\nu \approx \pm\mu^2/\nu_0 \pm i\nu_0. \tag{3.63}$$

Only in this limit one can obtain the envelope soliton with the sech form (3.27).

For large $\mu (\mu \gg \nu_0)$ the roots have the asymptotic behavior

$$\nu = \mu\frac{\pm 1 \pm i}{\sqrt{2}}.$$

Figs. 5–7 show solitons for different values of μ and ν_0. In the limit $\mu \to 0$ (Fig. 5), the soliton shape has the form of the envelope soliton (3.27).

As μ increases, the number of oscillations decreases on the soliton size (Fig. 6) and at large $\mu (\mu \gg \nu_0)$ the soliton has only one oscillation on this scale (Fig. 7). At a large distance all solitons possess exponentially decreasing oscillating tails.

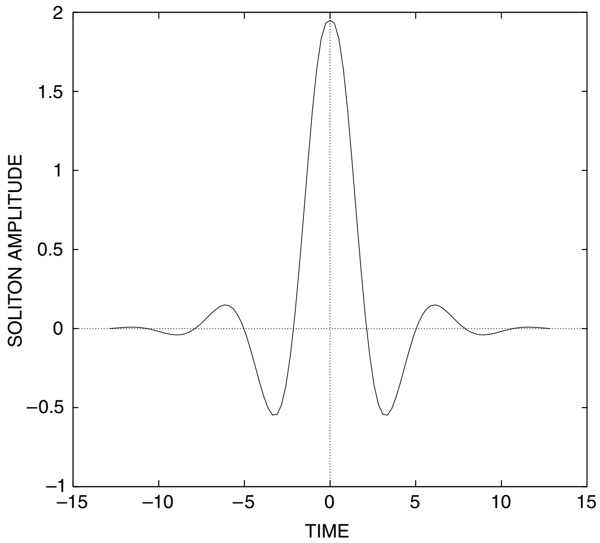

Fig. 6. The soliton shape at $\mu/\nu_0 = 1$. The soliton amplitude (in units ν_0^2) grows and its duration (in units ν_0^{-1}) decreases. On the soliton size there are still some oscillations.

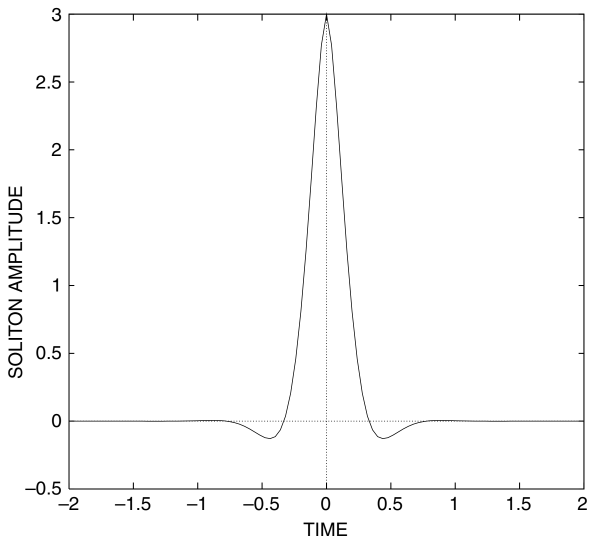

Fig. 7. The soliton shape at $\mu/\nu_0 = 10$. The oscillating tail is no longer visible.

When the ratio μ/ν_0 increases, the soliton amplitude increases and its duration decreases. The solitons obtained here as real solutions of Eq. (3.61) are simultaneously solutions in the form of stationary solitons for the wave Eq. (3.5) with dielectric permittivity:

$$\varepsilon(\omega) = \varepsilon_0 - a(\omega^2 - \omega_0^2)^2 \quad \text{and} \quad \chi a > 0.$$

As for the case of second dispersion (3.58), the situation is familiar to the one for solitons with quadratic dispersion (3.50). Solitons here are possible only for $\lambda > -\gamma_0^4$, the only difference with the quadratic dispersion solitons is connected with a change in the critical velocity value. Near these points the structure of solitons has the universal form (3.27).

3.4. Stability of solitons

Consider the stability of solitons found in the previous subsection. As already demonstrated in Section 2 the standing soliton (2.16) for the NLSE with quadratic dispersion is stable. We first show that the propagating solitons are also stable. The Hamiltonian for this equation is of the form

$$H = \int (|\psi_t|^2 - |\psi|^4) dt \equiv I_1 - I_2, \tag{3.64}$$

and soliton solutions (3.52) represent stationary points of the Hamiltonian H for fixed momentum $P = -i\int \psi\psi_t^* dt$ and number of particles (power) $N = \int |\psi|^2 dt$:

$$\delta(H + \beta P + \lambda N) = 0. \tag{3.65}$$

To prove the stability of the soliton let us demonstrate, following [38], that the soliton realizes the minimum of the Hamiltonian for fixed P and N. In order to do that, it is convenient to represent the parameter λ as the sum of $\beta^2/4$ and the positive quantity μ^2 and then consider the functional $F = H + \beta P + (\beta^2/4)N$ which has the meaning of a Hamiltonian in the moving system of reference. Through the change of the wave function $\psi \to \psi e^{it\beta/2}$, F is transformed into H (3.64) and respectively the variational problem (3.65) is written as

$$\delta[H + (\lambda - \beta^2/4)N] = 0;$$

it corresponds to the moving soliton solution (3.51). Thus, we reduce the stability of the moving NLS solitons to the stability of the standing one; that proves their stability.

Let us now consider the fourth-order dispersion. The corresponding functional $F = H + \beta P + \lambda N$ can be represented as the sum of the mean value of the positive definite operator $L(i\partial_t)$ (3.56) and the nonlinear term:

$$F = \int \psi^* L(i\partial_t)\psi\, dt - \int |\psi|^4 dt. \tag{3.66}$$

In order to obtain the stability proof for solitons one needs to find the analog of the estimate (2.30) for the mean value of the operator $L(i\partial_t)$.

Let $L(\Omega)$ be a positive definite polynomial of $\Omega \in (-\infty, \infty)$ of even degree $N = 2l$:

$$L_{2l}(\Omega) = C_{2l}\Omega^{2l} + C_{2l-1}\Omega^{2l-1} + \cdots + C_0.$$

Then $L(\Omega)$ can be expanded as follows

$$L_{2l}(\Omega) = \sum_{p=0}^{l} L_{2l-2p}(\Omega_p) \prod_{i=1}^{p-1} (\Omega - \Omega_i)^2 \tag{3.67}$$

where Ω_i and polynomials $L_{2l-2p}(\Omega)$ are constructed from $L_{2l}(\Omega)$ by the following way.

Let $\Omega = \Omega_0$ be the minimal point of $L_{2l}(\Omega) : \min L_{2l}(\Omega) = L_{2l}(\Omega_0)$. The latter means that $L_{2l}(\Omega)$ can be written as

$$L_{2l}(\Omega) = L_{2l}(\Omega_0) + (\Omega - \Omega_0)^2 L_{2l-2}(\Omega)$$

where $L_{2l-2}(\Omega)$ is the nonnegative polynomial of degree $2l-2$. Expanding the polynomial $L_{2l-2}(\Omega)$ yields a new nonnegative polynomial with degree $2l-4$. Further recursion leads us to the formula (3.67). What is important is that all coefficients in this expansion are nonnegative: $L_{2l-2p}(\Omega_p) \geq 0$. Note that $L_0(\Omega_l) = C_{2l}$.

The expansion (3.67) generates the following expansion for the mean value of the operator $L_{2l}(i\partial_t)$:

$$\begin{aligned}\langle L_{2l}(i\partial_t)\rangle &\equiv \int \psi^* L_{2l}(i\partial_t)\psi\, dt \\ &= L_{2l}(\Omega_0)N_0 + L_{2l-2}(\Omega_1)N_1 + \cdots + L_0(\Omega_l)N_l \end{aligned} \tag{3.68}$$

where

$$N_p = \int |\psi_p|^2 dt; \qquad \psi_p = \prod_{q=0}^{p-1} (i\partial_t + \Omega_q)\psi,\ p \geq 1;\ \psi_0 \equiv \psi.$$

This representation exhibits how the square norm of the positive definite polynomial operator can be expanded through the norms N_p with nonnegative coefficients $L_{2l-2p}(\Omega_p)$.

The expansion (3.67) to fourth-order positive definite dispersion (3.56)

$$L(\Omega) = \lambda - \beta\Omega + D\Omega^2 + \gamma\Omega^3 + \Omega^4$$

reads

$$L(\Omega) = \mu^4 + \eta^2(\Omega - \Omega_0)^2 + (\Omega - \Omega_0)^2(\Omega - \Omega_1)^2, \tag{3.69}$$

where μ^4 replaces $L_4(\Omega_0)$ and $L_2(\Omega_1)$ is replaced by η^2. Without loss of generality, in Eq. (3.69) one can put $\Omega_0 = -\Omega_1 = \nu_0$ (it corresponds to the change $\psi \to \psi \exp\{-i\frac{1}{2}(\Omega_0 + \Omega_1)t\}$) so that the formula (3.69) takes the form

$$L(\Omega) = \mu^4 + \eta^2(\Omega - \nu_0)^2 + (\Omega^2 - \nu_0^2)^2. \tag{3.70}$$

The difference between dispersions (3.57) and (3.58) is in the sign of the quantity $2\nu_0^2 - \eta^2$ positive or negative ($2\nu_0^2 > \eta^2$ for (3.57) and $2\nu_0^2 < \eta^2$ for (3.58). The integral expansion for the norm of the operator L corresponding to (3.70) is written as follows

$$\langle L(i\partial_t)\rangle = \mu^4 N + \eta^2 J_1 + J_2, \tag{3.71}$$

where

$$J_1 = \int |(i\partial_t + \nu_0)\psi|^2 dt, \qquad J_2 = \int |(\partial_t^2 + \nu_0^2)\psi|^2 dt.$$

This representation means that moving solitons can be considered as stationary points of a new Hamiltonian

$$H' = \eta^2 J_1 + J_2 - \int |\psi|^4 dt \tag{3.72}$$

for fixed number of particles N:

$$\delta(H' + \mu^4 N) = 0. \tag{3.73}$$

If the Hamiltonian H' is bounded from below for a fixed N and its lower boundary corresponds to the soliton, then one has soliton stability. In terms of the new Hamiltonian the soliton solutions obey the equation

$$\mu^4\psi_s + \eta^2(i\partial_t + \nu_0)^2\psi_s + (\partial_t^2 + \nu_0^2)^2\psi_s - 2|\psi_s|^2\psi_s = 0. \tag{3.74}$$

Multiplying this equation by ψ_s^* and integrating over t leads to the following relation between integrals contained in H':

$$\mu^4 N_s + \eta^2 J_{1s} + J_{2s} - 2\int |\psi_s|^4 dt \equiv H_s' + \mu^4 N_s - \int |\psi_s|^4 dt = 0.$$

Another relation follows after multiplying Eq. (3.74) by $t\partial_t\psi_s^*$ and integrating:

$$(\mu^4 + \eta^2\nu_0^2 + \nu_0^4)N_s + (2\nu_0^2 - \eta^2)\int |\partial_t\psi_s|^2 dt - 3\int |\partial_t^2\psi_s|^2 dt - \int |\psi_s|^4 dt = 0.$$

Combining both relations yields

$$H_s' = (\eta^2\nu_0^2 + \nu_0^4)N_s + (2\nu_0^2 - \eta^2)\int |\partial_t\psi_s|^2 dt - 3\int |\partial_t^2\psi_s|^2 dt.$$

For both dispersions the Hamiltonian H_s' on the soliton solution is bounded from above by the number of particles N_s multiplied by some positive coefficient: for (3.57) we have

$$H_s' \le \left[\frac{1}{12}(2\nu_0^2 - \eta^2)^2 + \eta^2\nu_0^2 + \nu_0^4\right] N_s$$

and for (3.58)

$$H_s' \le (\eta^2\nu_0^2 + \nu_0^4)N_s.$$

We now prove the boundedness from below of the Hamiltonian H' for fixed N. In order to do that we first estimate two integrals J_1 and J_2 through two other integrals N and $I_2 = \int |\psi|^4 dt$. It is easy to see that, for the first integral J_1, the following estimate (2.30) holds:

$$\int_{-\infty}^{\infty} |\psi|^4 dt \le \frac{1}{\sqrt{3}} N^{3/2} \left[\int_{-\infty}^{\infty} |(i\partial_t + \nu_0)\psi|^2 dt\right]^{1/2}. \tag{3.75}$$

Using again the inequality (2.30) leads to the desirable estimate for J_2: one first integrates by parts the integral $\int |\psi_t|^2 dt$, then uses the Schwartz inequality

$$\begin{aligned}\int |\psi_t|^2 dt &= -\int \psi^*(\psi_{tt} + \nu_0^2\psi)dt + \int \nu_0^2|\psi|^2 dt \\ &\le N^{1/2}\left[\int |(\partial_t^2 + \nu_0^2)\psi|^2 dt\right]^{1/2} + \nu_0^2 N\end{aligned}$$

and finally substitutes this result into (2.30):

$$J_2 \ge \frac{1}{N}\left(\frac{3I_2^2}{N^3} - \nu_0^2 N\right)^2. \tag{3.76}$$

Through the inequalities (3.75) and (3.76), the Hamiltonian H' can be estimated as follows:

$$H' \ge f(I_2) = \frac{3I_2^2}{N^3} + \frac{1}{N}\left(\frac{3I_2^2}{N^3} - \nu_0^2 N\right)^2 - I_2. \tag{3.77}$$

Continuing this inequality, one can write

$$f(I_2) \geq 2\frac{\sqrt{3}I_2}{N^2}\left(\frac{3I_2^2}{N^3} - \nu_0^2 N\right) - I_2.$$

Hence we finally arrive at the desired inequality, i.e., at the boundedness of the Hamiltonian:

$$H' \geq -\frac{4\sqrt{3}N}{9}\left[1 + \frac{\sqrt{3}N}{6\nu_0^2}\right]^{3/2}. \tag{3.78}$$

According to the Lyapunov theorem this proves the stability of the stationary point of the Hamiltonian corresponding to its minimum. This minimum point is some soliton solution of Eq. (3.74) which, in principle, is unique. Also important is that according to the estimate (3.78) the Hamiltonian can take negative values. If initially the Hamiltonian $H' < 0$, one can use the mean value theorem to obtain an estimate from below for the maximum value of $|\psi|^2$ (compare with [38])

$$\max_t |E|^2 \geq \frac{|H'|}{N}.$$

Thus the maximum of intensity which existed initially cannot disappear as distance grows. The radiation of small amplitude waves must provide, due to the boundedness of the Hamiltonian from below, relaxation of such initial conditions to the soliton state.

We conclude this section with a few words about the stability of stationary solitons (3.27). Near the critical velocity this question can be considered in the framework of the parabolic NLS Eq. (3.50) for which we already know the answer. In order to investigate the stability of solitons far from the critical velocity, the next order terms in the dispersion must be included. As seen above, the fourth-order terms which provided the positiveness of the corresponding operator L also provide stability for solitons. We guess that the positive definite polynomial operators of even order must provide the soliton stability. Probably solitons will be unstable only for operators growing proportionally to $\sqrt{|\Omega|}$ and slowly at infinity, $|\Omega| \to \infty$.

Note that the analysis for solitons based on the criteria (3.12) and (3.13) is valid for any dimension. It is essential that the requirement for soliton existence remains correct: the corresponding operator L must be sign definite. Moreover, fourth-order dispersion in all physical dimensions d provides the existence of stable solitons of the GLNS equation with cubic nonlinearity. This follows by estimating the fourth-order dispersive term in the Hamiltonian through the integrals I_2 and N. In this case the inequality (2.30) reads

$$\int |\psi|^4 d^D x \leq C\left[\int |\Delta\psi|^2 d^D x\right]^{D/4}\left[\int |\psi|^2 d^D x\right]^{2-D/4}. \tag{3.79}$$

Substituting this estimate into the Hamiltonian

$$H = \int |\Delta\psi|^2 d^D x - \int |\psi|^4 d^D x$$

provides its boundedness from below:

$$H \geq \int |\Delta\psi|^2 d^D x - C\left[\int |\Delta\psi|^2 d^D x\right]^{D/4}\left[\int |\psi|^2 d^D x\right]^{2-D/4}$$
$$\geq -(4/D - 1)\left(\frac{4}{CD}\right)^{4/(D-4)} N^{(8-D)/(4-D)}.$$

Besides soliton stability, this proves also that for media with Kerr nonlinearity wave collapse is stopped by fourth-order dispersion in all physical dimensions.

4. Supercritical bifurcations: general consideration

In the two previous sections we have examined two types of solitons undergoing supercritical bifurcations and demonstrated that solitons behave similarly near the bifurcation point. Now we will show that such behavior of solitons near their supercritical bifurcation point is intrinsic for any type of solitons.

Let us consider a purely conservative nonlinear wave medium which can be described by the Hamiltonian

$$H = \int \omega_k |a_k|^2 d\mathbf{k} + H_{int}, \tag{4.1}$$

where ω_k is the dispersion law of small amplitude waves, a_k are the normal amplitudes of the waves and the Hamiltonian H_{int} describes the nonlinear interaction of the waves. The expansion of the interaction Hamiltonian H_{int} in powers of a_k and a_k^*,

$$H_{int} = H_1 + \cdots,$$

starts from the cubic term

$$H_1 = \int (V_{kk_1k_2} a_k^* a_{k_1} a_{k_2} + c.c.)\delta_{k-k_1-k_2} d\mathbf{k} d\mathbf{k}_1 d\mathbf{k}_2 + \frac{1}{3}\int (U_{kk_1k_2} a_k^* a_{k_2}^* a_{k_2}^* + c.c.)\delta_{k+k_1+k_2} d\mathbf{k} d\mathbf{k}_1 d\mathbf{k}_2 \tag{4.2}$$

which describes the three-wave interaction. Here the matrix elements $V_{kk_1k_2}$ and $U_{kk_1k_2}$ have symmetry properties in accordance with the definition of H_1:

$$V_{kk_1k_2} = V_{kk_2k_1}, \qquad U_{kk_1k_2} = U_{kk_2k_1} = U_{k_1kk_2}.$$

Since the variables a_k and their complex conjugates a_k^* are classical analogs of the quantum annihilation and creation operators, one can classify all nonlinear processes corresponding to each term in the interaction Hamiltonian. For instance, the first term proportional to $V_{kk_1k_2}$ is responsible for the process of decay of one wave into two waves and the second term in (4.2) is responsible for the process $0 \to 3$ (creation of three waves from a "vacuum"). Among the fourth-order terms the most important one is the Hamiltonian

$$H_2 = \frac{1}{2}\int T_{k_1k_2k_3k_4} a_{k_1}^* a_{k_2}^* a_{k_3} a_{k_4} \delta_{k_1+k_2-k_3-k_4} \Pi d\mathbf{k}_i \tag{4.3}$$

responsible for the processes of scattering of waves, $2 \to 2$.

The equations of motion of the medium can be written in terms of the amplitudes a_k in the standard Hamiltonian form [33]

$$\frac{\partial a_k}{\partial t} + i\omega_k a_k = -i\frac{\delta H_{int}}{\delta a_k^*}, \tag{4.4}$$

so that in the absence of interactions the system consists of a collection of noninteracting oscillators (waves):

$$a_k(t) = a_k(0)e^{-i\omega_k t}.$$

Eq. (4.4) describes the dynamics in wave number space. To go back to the physical space one needs to perform the inverse Fourier transform

$$\psi(\mathbf{x}, t) = \frac{1}{(2\pi)^{d/2}}\int a_k(t)e^{i\mathbf{k}\cdot\mathbf{r}} d\mathbf{k}. \tag{4.5}$$

Originally, the function $\psi(\mathbf{x}, t)$ is related to the characteristics of the medium (fluctuations of the density and velocity of the medium, electric and magnetic fields, and so on) by a linear transformation (see, for example, Ref. [33]). It is important that if $\psi(\mathbf{x}, t)$ is a periodic function of the coordinates, then its spectrum $a_k(t)$ consists of a sum of δ-functions. For localized distributions $\psi(\mathbf{x}, t) \to 0$ as $|\mathbf{x}| \to \infty$. The Fourier amplitude $a_k(t)$ being a localized function of $\mathbf{k}$, it does not contain δ-function singularities.

Let us now consider the solution of Eq. (4.4) in the form of a soliton propagating with constant velocity $\mathbf{V}$:

$$\psi(\mathbf{x}, t) = \psi(\mathbf{x} - \mathbf{V}t).$$

In this case the whole dependence of a_k on time t is contained in the oscillating exponent:

$$a_k(t) = c_k e^{-i\mathbf{k}\cdot\mathbf{V}t},$$

where by virtue of Eq. (4.4) the amplitude c_k satisfies the equation

$$(\omega_k - \mathbf{k}\cdot\mathbf{V})c_k = -\frac{\delta H}{\delta c_k^*} \equiv f_k. \tag{4.6}$$

The difference $\omega_k - \mathbf{k}\cdot\mathbf{V}$ appearing in this equation is positive for all $\mathbf{k}$ if the soliton velocity is less than the minimum phase velocity

$$|\mathbf{V}| < \min(\omega_k/k). \tag{4.7}$$

Conversely, the difference is negative for all $\mathbf{k}$ if the soliton velocity is greater than the maximum phase velocity:

$$|\mathbf{V}| > \max(\omega_k/k). \tag{4.8}$$

We show that a soliton solution is possible if the condition (4.7) (or (4.8)) is satisfied. Let us assume the opposite to be true—let both conditions (4.7) and (4.8) be violated, i.e. equation (1.1) has a solution. For simplicity, we assume that it is unique: $\mathbf{k} = \mathbf{k}_0$. Then, since $x\delta(x) = 0$, the homogeneous linear equation

$$(\omega_k - \mathbf{k}\cdot\mathbf{V})C_k = 0$$

possesses a nontrivial solution in the form of a monochromatic wave

$$C_k = A\delta(\mathbf{k} - \mathbf{k}_0).$$

In this case, the Fredholm alternative allows us to write Eq. (4.6) as

$$c_k = A\delta(\mathbf{k}-\mathbf{k}_0) + \frac{f_k}{\omega_k - \mathbf{k}\cdot\mathbf{V}} \quad \text{with}\, f_{k_0} = 0 \tag{4.9}$$

(compare with (3.15)). This equation, in contrast to Eq. (4.6), contains a free parameter — the complex amplitude A. It can be solved iteratively, for example, taking $A\delta(\mathbf{k}-\mathbf{k}_0)$ as the zeroth term. It is important that because of the nonlinearity one obtains multiple harmonics with $\mathbf{k} = n\mathbf{k}_0$ where n is integer as a result of iterations. The solution consists of a sum of δ-functions. In physical space the solution is a periodic function of the coordinates, i.e., it is nonlocalized. Hence the first selection rule for solitons follows: *the difference $\omega_k - k\cdot V$ must be sign definite,*which is equivalent to the (4.7) or (4.8). In other words it means that Cherenkov radiation is absent.

In this whole scheme, there is however an important exception. Having represented Eq. (4.6) in the form (4.9), we have in fact assumed that the singularity in the expression

$$\frac{f_k}{\omega_k - \mathbf{k}\cdot\mathbf{V}} \tag{4.10}$$

is nonremovable. This may not be the case—the singularity in the denominator of Eq. (4.10) could be canceled with the numerator, i.e., it could be removable [22]. As seen in the previous section, this happens for the classical soliton of the KDV equation, for equations which are generalizations of the KDV equation [61], for the combination of the 1D NLS and MKDV equations integrated by the same Zakharov–Shabat operator [5] and so on. In all of these cases cancelation occurs as a result of the k dependence of the matrix elements. However, even in these cases, the selection rule for solitons remains the same: after the resonance (1.1) is removed—the part remaining in the denominator must be sign definite.

In what follows the singularities in Eq. (4.10) are assumed to be nonremovable in the forbidden region, and we study the behavior of the soliton solution as the soliton velocity approaches the critical value. For definiteness, it is assumed that the plane $\omega = \mathbf{k}\cdot\mathbf{V}$ is tangent to the dispersion surface $\omega = \omega_k$ from below, i.e., the criterion (4.7) holds. Let touching occur at the point $\mathbf{k} = \mathbf{k}_0$. Then, instead of Eq. (4.9), in the allowed region

$$c_k = \frac{f_k}{\omega_k - \mathbf{k}\cdot\mathbf{V}}.$$

As the velocity $\mathbf{V}$ approaches the critical value $\mathbf{V}_{cr}$, the denominator in this expression becomes small near the touching point, so that c_k exhibits a sharp peak at this point

$$c_k = \left[\frac{1}{2}\omega_{\alpha\beta}\kappa_\alpha\kappa_\beta + k_0(\mathbf{V}_{cr} - \mathbf{V})\right]^{-1} f_k. \tag{4.11}$$

Here $\omega_{\alpha\beta} = \partial^2\omega/\partial k_\alpha \partial k_\beta$ is a symmetric, positive definite tensor of the second derivatives, evaluated at $\mathbf{k} = \mathbf{k}_0$, and $\boldsymbol{\kappa} = \boldsymbol{k} - \boldsymbol{k}_0$.

It is evident from Eq. (4.11) that as V approaches the critical velocity the width of the peak narrows as $\sqrt{V_{cr} - V}$, and the distribution corresponding to the main peak $\mathbf{k} = \mathbf{k}_0$ approaches a monochromatic wave. Accounting for nonlinearity, the spectrum contains harmonics which are multiples of $\mathbf{k} = \mathbf{k}_0$. If it is assumed that the amplitude of the soliton vanishes gradually as $\mathbf{V} \to \mathbf{V}_{cr}$ (which would correspond to a second-order phase transition), then the solution $\psi(\mathbf{x})$ (or, equivalently, c_k) can be sought as an expansion in terms of harmonics:

$$\psi(\mathbf{x}') = \sum_{n=-\infty}^{\infty} \psi_n(\mathbf{X}) e^{in\mathbf{k}_0\cdot\mathbf{x}'}, \quad \mathbf{x}' = \boldsymbol{x} - \boldsymbol{V}t. \tag{4.12}$$

Here the small parameter

$$\lambda = \sqrt{1 - V/V_{cr}} \tag{4.13}$$

and the slow coordinate $\mathbf{X} = \lambda\mathbf{x}'$ are formally introduced, so that $\psi_n(\mathbf{X})$ is the amplitude of the envelope of nth harmonic. The assumption that the soliton amplitude vanishes continuously at $V = V_{cr}$ means that the leading term of the series in Eq. (4.12) corresponds to the first harmonic, and all other harmonics are small in the parameter λ. This is the condition under which the NLS equation is derived (see, for example, Refs. [22,64,13]). In the present case, we arrive at the stationary NLS

$$-k_0 V_{cr}\lambda^2\psi_1 + \frac{1}{2}\omega_{\alpha\beta}\frac{\partial^2\psi_1}{\partial X_\alpha \partial X_\beta} + B|\psi_1|^2\psi_1 = 0 \tag{4.14}$$

at leading order in λ, where B is related to the matrix $\tilde{T}_{k_1k_2k_3k_4}$ of four-wave interactions as

$$B = -(2\pi)^d \tilde{T}_{k_0k_0k_0k_0}. \tag{4.15}$$

In this approximation the leading term in the interaction Hamiltonian has the form

$$H_{int} = \frac{\tilde{T}_{k_0k_0k_0k_0}}{2}\int c^*_{k_1}c^*_{k_2}c_{k_3}c_{k_4}\delta_{\mathbf{k}_1+k_2-k_3-k_4}d\mathbf{k}_1d\mathbf{k}_2d\mathbf{k}_3d\mathbf{k}_4 = -\frac{B}{2}\int|\psi_1|^4 d\mathbf{x}, \tag{4.16}$$

and the tilde means renormalization of the four-wave matrix elements due to the three-wave interaction—in the present case the interaction with the zeroth and second harmonics. According to [22] $\tilde{T}_{k_0k_0k_0k_0}$ is given by the following expression:

$$\tilde{T}_{k_0k_0k_0k_0} = T_{k_0k_0k_0k_0} - 2\frac{\left|U_{-2k_0k_0k_0}\right|^2}{2\omega_{k_0}+\omega_{2k_0}} - 2\frac{\left|V_{2k_0k_0k_0}\right|^2}{\omega_{2k_0}-2\omega_{k_0}} - 4\lim_{\kappa\to 0}\frac{\left|V_{k_0k_0\kappa}\right|^2}{\omega_\kappa - \kappa v_{gr}}. \tag{4.17}$$

Here the second and third terms represent the contribution from the interaction with the second harmonic. The last term, in the form of a limit as $\kappa \to 0$, accounts for the interaction with the zeroth harmonic where $v_{gr} = \partial\omega/\partial\mathbf{k}$ is the group velocity taken at $\mathbf{k} = \mathbf{k}_0$. According to Goldstone's theorem (cf. [65]) the frequency ω_κ can vanish or tend to a constant value as $\kappa \to 0$. The same statement is valid for the matrix element $V_{kk_1k_2}$ if one of the wave vectors $\mathbf{k}$, $\mathbf{k}_1$ or $\mathbf{k}_2$ tends to zero. Thus, in the expression (4.17) for the matrix element of $\tilde{T}$, there are indeterminacies when $\mathbf{k}_i = \mathbf{k}_0$. As a result we arrive at the limit

$$\lim_{\kappa\to 0}\frac{|V_{k_0k_0\kappa}|^2}{\omega(\kappa) - \boldsymbol{\kappa}\boldsymbol{v}_{gr}}.$$

For example, for surface waves on the deep water

$$\omega(\kappa) = (g\kappa)^{1/2}, \qquad V_{k_0k_0\kappa} \sim \kappa^{3/4}$$

and all the indeterminacies vanish. For finite depth one has $V_{k_0k_0\kappa} \sim \kappa^{1/2}$, $\omega(\kappa) \sim \kappa$, so that this limit is finite in each direction, however, the quantity $\tilde{T}_{k_0k_0k_0k_0}$ remains undetermined due to the angular dependence between the vectors $\mathbf{k}_0$ and $\boldsymbol{\kappa}$. Indeterminacies of this type are related to the excitation of induced motion of the medium as a whole. Such a situation occurs for all waves whose dispersion laws ω_k become linear as $k \to 0$. In addition to the surface waves considered above, such waves include ion-acoustic waves in plasma, sound waves in a solid, etc. The additions to the four-wave matrix element $\tilde{T}$ will be finite far from the resonances

$$2\omega_{k_0} - \omega_{2k_0} = 0 \tag{4.18}$$

and

$$\omega(\kappa) - \boldsymbol{\kappa}\boldsymbol{v}_{gr} = 0. \tag{4.19}$$

The first resonance corresponds to the excitation of the second harmonics, while the second one relates to the resonant interaction of wave packets with low-frequency waves. In nonlinear optics the latter corresponds to the Mandelstamm–Brillouin scattering.

As already noted, $\omega_{\alpha\beta}$ in Eq. (4.14) is a symmetric positive definite tensor. For this reason, performing a rotation to its principal axes and carrying out the corresponding extensions along each axis, Eq. (4.14) can be transformed into the standard form

$$-\lambda^2\psi + \Delta\psi - 2\mu|\psi|^2\psi = 0, \tag{4.20}$$

where $\mu = \mathrm{sign}(\tilde{T}\omega_{\alpha\alpha})$. Hence it follows, in the first place, that solitons are possible only if μ is negative (focusing nonlinearity when the product $\tilde{T}\omega_{\alpha\alpha}$ is negative) and, in the second place, that the amplitude of the solitons is proportional to

$$\lambda = \sqrt{1 - V/V_{cr}}, \tag{4.21}$$

i.e., the amplitude vanishes according to a square-root law, the size of the soliton increases as $1/\lambda$ as the velocity approaches the critical value. This is in complete correspondence with both examples considered in the previous sections.

4.1. Stability of solitons in multi-dimensions

It is obvious that for soliton stability the most dangerous disturbances will be those having wave numbers close to $\mathbf{k} = \mathbf{k}_0$ moving together with the soliton, i.e., modulation type disturbances. In order to include these perturbations one needs to introduce the time dependence in the averaged equations. In this case the amplitudes ψ_n in the expansion (4.12) must be assumed to depend not only on the "slow" coordinate X but also on the slow time $T = \lambda^2 t$. Then using multi-scale expansions, we obtain the nonstationary NLS equation instead of the stationary NLS equation (4.20):

$$i\psi_t - \lambda^2\psi + \Delta\psi - 2\mu|\psi|^2\psi = 0 \tag{4.22}$$

(compare with Eq. (2.19)).

The problem of the stability of solitons solution to this equation has been well studied (see, for example, Refs. [22,38]). We recall the basic points in the investigation of stability. Eq. (4.22) as an equation for envelopes inherits the canonical Hamiltonian form (4.4)

$$i\frac{\partial\psi}{\partial t}=\frac{\delta\tilde{H}}{\delta\psi^*}, \tag{4.23}$$

where the Hamiltonian

$$\tilde{H}=\lambda^2N+\int(|\nabla\psi|^2-|\psi|^4)d\mathbf{r},\qquad(\mu=-1) \tag{4.24}$$

arises as a result of averaging the initial Hamiltonian. Eq. (4.22) preserves, besides $\tilde{H}$, the total number N of particles (adiabatic invariant), so that solitons are stationary points of the functional $H=\tilde{H}-\lambda^2N$ (which we also call Hamiltonian) with a fixed number of particles:

$$\delta(H+\lambda^2N)=0. \tag{4.25}$$

The number of particles (or intensity) on the soliton solution as a function of λ has the form

$$N_s=\int|\psi_s|^2d\mathbf{x}=\lambda^{2-d}\int|g(\xi)|^2d\boldsymbol{\xi}, \tag{4.26}$$

where d is the dimension of the space, $\psi_s=\lambda g(\lambda x)$ and $g(\xi)$ satisfies the equation

$$-g+\Delta g+2|g|^2g=0.$$

In the 1D case, $g=\sqrt{2}\,\mathrm{sech}\,\xi$ and, correspondingly, $N_s=4\lambda$. In the 2D case N_s is independent of λ for the entire family of solitons, while in the 3D case N_s decreases as λ increases. The dependence of N_s on λ^2 is crucial from the point of view of soliton stability. The (linear) instability of 3D solitons follows from the so-called Vakhitov–Kolokolov (VK) criterion [39].

4.1.1. Vakhitov–Kolokolov criterion

Now we will give the derivation of this criterion, following the review [38] for the NLSE solitons. The crucial point in its derivation is based on the oscillation theorem for the stationary Schrödinger operator. This theorem establishes the one-to-one correspondence between a level number and a number of nodes of the eigenfunction. As well known, this theorem is valid only for scalar (one-component) Schrödinger operators and cannot be extended, for example, to the analogous matrix operators. This means that the Vakhitov–Kolokolov type of criteria, as a rule, define only sufficient conditions for soliton instability and cannot necessarily determine the stability of solitons.

As known, the NLSE is invariant with respect to a Galilean transformation. Therefore it is enough to consider only the rest soliton solution $\psi=\psi_s(\mathbf{r})e^{i\lambda^2t}$. where the function $\psi_s(r)$ is assumed to be radial symmetric without no nodes (ground soliton solution). Letting,

$$\psi(\mathbf{r},t)=(\psi_s(r)+u+iv)e^{i\lambda^2t},\quad \psi_s\gg u,v$$

in the Eq. (4.22) and linearizing on the background of the soliton solution leads to coupled linear equations for real-valued functions u and v. These equations are Hamiltonian:

$$u_t=\frac{1}{2}\frac{\delta H^{(2)}}{\delta v},\qquad v_t=-\frac{1}{2}\frac{\delta H^{(2)}}{\delta u}. \tag{4.27}$$

Here $H^{(2)}$ represents the second variation of $H+\lambda^2N$:

$$\tilde{H}=\langle v|L_0|v\rangle+\langle u|L_1|u\rangle \tag{4.28}$$

with

$$L_0=\lambda^2-\Delta-2\psi_0^2,\qquad L_1=\lambda^2-\Delta-6\psi_0^2,$$

where $\langle v|L_0|v\rangle=\int v(\mathbf{r})L_0v(\mathbf{r})d\mathbf{r}$ and $\langle u|L_1|u\rangle=\int u(\mathbf{r})L_0u(\mathbf{r})d\mathbf{r}$, respectively.

The first term in (4.28), the mean value of the operator L_0, can be interpreted as a kinetic energy and the second one, $\langle u|L_1|u\rangle$, as a potential energy. Thus, soliton stability or instability are determined by the properties of the operators L_0 and L_1.

The first property of L_0 follows directly from the soliton equation (4.20):

$$L_0\psi_s=\lambda^2\psi_s-\Delta\psi_s-2\psi_s^3\equiv 0, \tag{4.29}$$

which shows ψ_s is an eigenfunction of the operator L_0. Moreover, ψ_s is the ground state, i.e., it has no nodes. Thus, due to the oscillation theorem, L_0 as a Schrödinger operator has no energy levels with negative values and so the mean value

$\langle v|L_0|v\rangle \geq 0$. Hence the question about soliton stability or instability will define by a sign of the potential energy $\langle u|L_1|u\rangle$. If we will find such perturbations u for which this mean value will be negative then we shall have instability and vice versa.

The (Schrödinger) operator L_1 has also the eigenfunction with zero eigenvalue corresponding to the s-state:

$$L_1\nabla\psi_s = 0$$

which can be checked by differentiation of Eq. (4.20) with respect to $\mathbf{r}$. The function $\phi_1 = \nabla\psi_s$ represents a neutral mode corresponding to the soliton shift as a whole. Unlike ψ_s for L_0, this eigenfunction has one node (at $r = 0$) and therefore according to the oscillation theorem below the level $E = 0$ we have one (ground) state ϕ_0. Hence, we could make a conclusion about instability because for $u \sim \phi_0$ the mean value of the L_1 operator is negative. However, it is a fictitious instability because on u there exists the constraint,

$$\langle u|\psi_s\rangle \equiv \int u\psi_s d\mathbf{r} = 0 \tag{4.30}$$

and therefore the minimal mean value of the operator L_1 must be sought in the class of functions orthogonal to ψ_s. The condition is a direct consequence of the conservation law for particle number $N = \int |\psi|^2 d\mathbf{r} : \delta N = 2\langle u|\psi_s\rangle = 0$. For the linearized system (4.27) this restriction serves as a solvability condition. Thus, the stability problem reduces to solution of the following eigenvalue problem:

$$L_1|\phi\rangle = E|\phi\rangle + C|\psi_0\rangle \tag{4.31}$$

where C is the Lagrange multiplier which should be found from the solvability condition (4.30) $\langle\phi|\psi_0\rangle = 0$. If we shall show now that this eigenvalue problem contains one negative eigenvalue $E < 0$ then this means a soliton instability. If in the spectrum of (4.31) there are no states with negative energies then the soliton solution is stable.

Expanding $|\phi\rangle$ over the complete set of eigenfunctions $\{\phi_n\}$ of the operator $L_1 (L_1\phi_n = E_n\phi_n)$,

$$\phi = \sum_n C_n\phi_n,$$

from (4.31) for the coefficients C_n we get

$$C_n = C\frac{\langle\phi_n|\psi_s\rangle}{E_n - E}, \qquad C_1 \equiv 0.$$

The solvability condition (4.30) gives the dispersion relation

$$f(E) \equiv {\sum_n}' \frac{\langle\psi_o|\phi_n\rangle\,\langle\phi_n|\psi_0\rangle}{E_n - E} = 0. \tag{4.32}$$

Prime here means that in the sum the state with $n = 1(\phi_1 = \nabla\psi_s)$ is absent due to the orthogonality condition $\langle\phi_1|\psi_s\rangle = 0$.

Consider now the energy interval between the ground-state energy $E_0 < 0$ and the first positive level E_2. In this interval the function $f(E)$ monotonically $(\partial f/\partial E > 0)$ increases from minus infinity at $E = E_0$ up to plus infinity at $E = E_2$. If the function $f(E)$ at $E = 0$ takes negative values then the dispersion equation has no negative eigenvalues and, thus we have a stable situation. If $f(0) > 0$ then the eigenvalue problem (4.31) contains negative E and the soliton undergoes instability.

In order to find $f(0)$ first note that

$$f(0) = {\sum_n}' \frac{\langle\psi_o|\phi_n\rangle\,\langle\phi_n|\psi_0\rangle}{E_n} \equiv \langle\psi_0|L_1|\psi_0\rangle\,.$$

Next, by differentiating the soliton Eq. (4.20) with respect to λ^2 one can get

$$L_1(\partial\psi_0/\partial\lambda^2) = -\psi_0$$

or

$$\langle\psi_0|L_1|\psi_0\rangle = -\left\langle\psi_0|\partial\psi/\partial\lambda^2\right\rangle = -\frac{1}{2}\frac{\partial N_s}{\partial\lambda^2}.$$

Hence we have the following (Vakhitov–Kolokolov) linear stability criterion for solitons [39]:

if

$$\frac{\partial N_s}{\partial\lambda^2} > 0 \tag{4.33}$$

then the soliton is stable and respectively unstable if this derivative is negative.

This criterion has a simple physical meaning. The value $-\lambda^2$ for stationary nonlinear Schrödinger Eq. (4.20) can be interpreted as the energy of the bound state soliton. If we add one "particle" to the system and the energy of this bound state will decrease then we will have a stable situation. If by adding one "particle" the level $-\lambda^2$ will be pushed toward the continuous spectrum, then such a soliton will be unstable.

At $d = 3$ the derivative $\partial N_s/\partial\lambda^2$ is negative and therefore 3D solitons are unstable (the modulational instability). For the 2D case the Vakhitov–Kolokolov criterion (4.33) provides an absence of linear exponential instability. A more detailed analysis in this case yields the power type instability (for details see the survey [38] and Ref. [66]). At $d = 1$ the derivative $\partial N_s/\partial\lambda^2 > 0$ and 1D solitons occur linearly stable in the full agreement with the Lyapunov stability considered above.

It is necessary to mention that the VK criterion (4.33) is valid for a more general NLS equation than (4.22), for instance, for the case when one changes $|\psi|^2$ by arbitrary function $f(|\psi|^2)$ (for more details see [38]).

4.1.2. Lyapunov stability

The VK stability criterion for the NLS equation has another interpretation. According to (4.25) the envelope solitons are stationary points of the energy E for a fixed number of waves N. Therefore such solutions will be stable in the Lyapunov sense if they realize a minimum (or a maximum) of the energy for fixed N.

Under the scaling transformations leaving N unchanged,

$$\psi(\mathbf{x}) = \frac{1}{a^{d/2}}\psi_s\left(\frac{\mathbf{x}}{a}\right), \tag{4.34}$$

where ψ_s is the solitonic solution, the Hamiltonian H becomes a function of the scaling parameter a:

$$H(a) = \frac{I_{1s}}{a^2} - \frac{I_{2s}}{a^d}, \tag{4.35}$$

where $I_{1s} = \int |\nabla\psi_s|^2 d\mathbf{x}$, $I_{2s} = \int |\psi_s|^4 d\mathbf{x}$ and $\mu = -1$. Hence, as already demonstrated for solitons in the 1D case, the energy (4.35) is bounded from below and has a global minimum at $a = 1$ corresponding to the soliton solution with

$$H_s = -\frac{2\lambda^3}{3} \quad \text{and} \quad 2I_{1s} = I_{2s} = \frac{4\lambda^3}{3}.$$

In the 3D case, the opposite occurs: the function $H(a)$ in (4.35) has a maximum, corresponding to the soliton solution, and is unbounded from below as $a \to 0$. The gauge transformation (2.25) gives a minimum of E and therefore all soliton solutions at $d = 3$ represent saddle points of the energy. This indicates a possible instability of solitons in this case.

4.2. About collapses

The nonlinear stage of this instability for $d \geq 2$ leads to wave collapse, i.e., the formation of singularity in a finite time. One of the main criteria for wave collapse is connected with the unboundedness of H (or E) which takes place as $a \to 0$ [67] (see also [68]). In such a case wave collapse can be understood as the process of some particle falling in the unbounded self-consistent potential.

To clarify the latter we apply the variational approach and take a trial function for the NLSE (4.22) in the form

$$\psi(\mathbf{r}, t) = a^{-d/2}\psi_s\left(\frac{\mathbf{r}}{a}\right)\exp(i\lambda^2 t + i\mu r^2),$$

where $a = a(t)$ and $\mu = \mu(t)$ are assumed to be unknown functions of time. After substitution of this ansatz into the action

$$S = \frac{i}{2}\int(\psi_t\psi^* - c.c.)dtd\mathbf{r} - \int Hdt$$

and integration over spatial variables we arrive at Newton's equation for a:

$$C\ddot{a} = -\frac{\partial H}{\partial a}, \tag{4.36}$$

where $C = \int \xi^2|\psi_0(\xi)|^2 d\xi$ plays the role of a particle mass and the function (4.35) has a meaning of potential energy. The behavior of $a(t)$ depends on the total energy,

$$E = C\frac{\dot{a}^2}{2} + H(a)$$

and the dimension d. For $d = 1$ the soliton realizes the minimum value of the potential energy $H(a)$ and it is one of the reasons why 1D solitons are stable. For $d = 3$ if a "particle" stands at the maximum point of $H(a)$ initially, then depending on the its direction of motion (toward or away from the center $a = 0$) the system will collapse ($\psi \to \infty$) or expand ($\psi \to 0$). For the collapsing regime (falling at the center) $a(t)$ behaves like

$$a(t) \sim (t_0 - t)^{2/5} \tag{4.37}$$

near the singularity, where t_0 is the collapse time. As shown in [67], this asymptotic behavior for $a(t)$ near the singular time coincides with that following from the exact semi-classical collapsing solution which asymptotically (as $t \to t_0$) tends to the compact distribution:

$$|\psi| \to \lambda\sqrt{1-\xi^2} \quad \text{for } \xi = r/a(t) \leq 1$$

with $\lambda \sim (t_0 - t)^{-3/5}$.

Hence we can make a few conclusions. First, the influence of nonlinearity grows with increasing spatial dimension d. As a consequence, stable solitons are intrinsic for low-dimensional systems while for higher dimensions instead of solitons we expect blow-up events. Secondly, one of the main criteria of collapse is the unboundedness of the Hamiltonian. In this case, the collapse can be interpreted as the fall of a particle to an attracting center in a self-consistent potential [67].

Thus, the Hamiltonian unboundedness can be considered as one of the main criteria of the existence of wave collapse, and the collapse in such systems can be represented as a process of falling down of some "particle" in a self-consistent unbounded potential. However the global picture is more complicated. From the very beginning we have a spatially distributed system with an infinite number of degrees of freedom and therefore, rigorously speaking, it is hardly feasible to describe such a system by its reduction to a system of ODEs like (4.36). The NLS equation is a wave system and we deal primarily with waves. Waves may propagate, radiate and so on. To illustrate the importance of this point, let us try to understand the influence of wave radiation on wave collapse.

Let Ω be an arbitrary region with a negative Hamiltonian $H_\Omega < 0$. Then using the mean value theorem for the integral I_2,

$$\int_\Omega |\psi|^4 d\mathbf{r} \le \max_{x\in\Omega} |\psi|^2 \int_\Omega |\psi|^2 d\mathbf{r},$$

we have

$$\max_{x\in\Omega} |\psi|^2 \ge \frac{|H_\Omega|}{N_\Omega}. \tag{4.38}$$

This estimate shows that wave radiation promotes collapse: far from the region Ω radiative waves can be considered almost linear, nonlinear effects are small for them. These waves carry out the positive portion of Hamiltonian making H_Ω more negative with simultaneous vanishing of the number of waves N_Ω that results in growth of the r.h.s. of (4.38) [69,64,70]. This is why we can say that wave radiation promotes collapse, which plays the role of friction in nonlinear wave dynamics. Simultaneously radiation turns out to accelerate the compression of the collapsing area with the self-similar behavior

$$r \sim (t_0 - t)^{1/2}, \tag{4.39}$$

different from that given by the semi-classical answer (4.37).

4.3. Virial theorem

The exact criterion for singularity formation within the NLS equation can be obtained from the virial theorem. In classical mechanics the virial theorem requires first the calculation of the second time derivative of the moment of inertia and then its averaging. It gives the relation between mean kinetic and potential energies of particles if the interaction between particles is of power type.

In 1971 Vlasov, Petrishchev and Talanov [71] found that this theorem can be applied also to the 2D NLS equation. The resulting relation is written for the mean square size $\langle r^2\rangle = N^{-1}\int r^2|\psi|^2 d\mathbf{r}$ of the distribution as follows:

$$\frac{d^2}{dt^2}\int r^2|\psi|^2 d\mathbf{r} = 8H. \tag{4.40}$$

This equality can be verified by direct calculation. In this relation $N\langle r^2\rangle$ has the meaning of the inertia moment.

Since H is a conserved quantity, Eq. (4.40) can be integrated twice to yield

$$\int r^2|\psi|^2 d\mathbf{r} = 4Ht^2 + C_1 t + C_2, \tag{4.41}$$

where $C_{1,2}$ are the additional integrals of motion. The existence of these integrals is explained by two Noether symmetries: the lens transform (this fact was established by V.I. Talanov [72]) and the scaling transformation [73,74].

Hence one can easily see that the mean square size $\langle r^2\rangle$ of any field distribution with negative Hamiltonian

$$H < 0, \tag{4.42}$$

vanishes in a finite time independently on $C_{1,2}$, which, with the conservation of N, means the formation of a singularity of the field ψ [71]. This is the famous Vlasov–Petrishchev–Talanov (VPT) criterion, which is nowadays a cornerstone in the theory of wave collapse. This was the first rigorous result for nonlinear wave systems with dispersion, which showed the possibility of the formation of a wave field singularity in finite time, despite the presence of the linear dispersion of waves, an effect impeding the formation of point singularities (focii) in linear optics.

One can make the following two concluding statements:

(i) Solitons near supercritical bifurcation are stable only in the 1D case, while in the 2D (critical) and 3D cases solitons are unstable and can be considered as separatrix solutions separating collapsing solutions from the dispersive ones [75].

(ii) This is probably the simplest method for explaining the well-known empirical fact that solitons, as a rule, exist only in 1D systems. For multi-dimensional systems stable solitons are rare and can only appear as a result of topological constraints or of a mechanism that removes Cherenkov singularities (which is discussed in the previous Section). The latter, as can be easily understood, is due to the existence of a certain symmetry class.

5. From supercritical to subcritical bifurcations

For subcritical bifurcation at the critical velocity the soliton undergoes a jump in its amplitude. In this case the corresponding theory can be developed near the transition point between subcritical and supercritical bifurcations (in analogy with the tricritical point for phase transitions) when one can use the finite set of terms in the Hamiltonian expansion. This happens when the coefficient μ in Eq. (4.22) changes its sign taking positive values. In this case Eq. (4.22) no longer possesses stationary localized (vanishing at infinity) solutions. In order for them to exist it is necessary to take into account the next higher-order terms in the expansion of the Hamiltonian in terms of the parameter $\Delta k/k_0$, where Δk is the width of the main peak. If the jump in the soliton amplitude at $V = V_{cr}$ is large (of the order of 1), then the entire series must used and it is no longer possible to count on a systematic theory based on an expansion of the Hamiltonian. Only if the matrix element $\tilde{T}_{k_0k_0k_0k_0} = \tilde{T}_0$ is small, i.e., there is an additional smallness compared with the supercriticality (and as a result the jump is also small), it is sufficient in this case to retain several of the next terms in the expansion. We shall study only one-dimensional solitons, since, as we have seen above, multi-dimensional solitons are unstable for a soft transition (supercritical bifurcation). This same tendency also remains for subcritical bifurcations (hard regimes). It is easy to see that in this situation two additional terms make the main contribution to the interaction Hamiltonian. The first term is a correction to the local four-wave Hamiltonian, $-B/2\int|\psi|^4\,dx$. It arises because a term linear in $\kappa_i = k_i - k_0$ is retained in the expansion of the matrix element $\tilde{T}_{k_1k_2k_3k_4}$

$$\tilde{T}_{k_1k_2k_3k_4} = \tilde{T}_0 + \mathrm{Re}\left(\frac{\partial\tilde{T}}{\partial k_1}\right)(\kappa_1+\kappa_2+\kappa_3+\kappa_4), \tag{5.1}$$

where

$$\frac{\partial\tilde{T}}{\partial k_1} \equiv \left.\frac{\partial\tilde{T}_{k_1k_2k_3k_4}}{\partial k_1}\right|_{k_i=k_0}.$$

Note that this expansion coincides, up to notations, with Eq. (3.39) of Section 3. It is necessary to underline that Eq. (5.1) assumes analyticity of $\tilde{T}_{k_1k_2k_3k_4}$ relative to κ_i $(i = 1, 2, 3, 4)$. As we will see later for the interfacial waves (propagating along the interface between two ideal fluids in the presence of gravity and capillarity) in the expansion (5.1) in this expansion there exists the nonlocal (non-analytical) contribution proportional to $|\kappa_i|$.

As a result of (5.1), the Hamiltonian of the four-wave interaction in the envelope approximation can be written in the x representation as

$$H^{(4)} = \pi\int\left\{\tilde{T}_0|\psi|^4 + 2i\mathrm{Re}\left(\frac{\partial\tilde{T}}{\partial k_1}\right)(\psi_x^*\psi - \psi_x\psi^*)|\psi|^2\right\}dx. \tag{5.2}$$

The expression $i(\psi_x^*\psi - \psi_x\psi^*)$ in this integral is well known in the theory of phase transitions (see Ref. [18]) — it is the so-called Lifshitz invariant.

The second term – local in ψ – is a six-wave interaction

$$H^{(6)} = -C\int|\psi|^6 dx. \tag{5.3}$$

As will be evident from what follows, the interaction constant C can be both negative and positive — the combination of both contributions (5.2) and (5.3) will be important.

The total Hamiltonian in dimensionless variables will depend on three constants μ, β, and C

$$H = \lambda^2 N + \int\left[|\psi_x|^2 + \frac{\mu}{2}|\psi|^4 + i\beta(\psi_x^*\psi - \psi_x\psi^*)|\psi|^2 - C|\psi|^6\right]dx. \tag{5.4}$$

The constant μ is assumed to be small, and the constants β and C do not contain any additional smallness.

The equations of motion for ψ that correspond to this Hamiltonian can be written according to Eq. (4.23) as

$$i\psi_t - \lambda^2\psi + \psi_{xx} - \mu|\psi|^2\psi + 3C|\psi|^4\psi + 4i\beta|\psi|^2\psi_x = 0. \tag{5.5}$$

This equation can be used for description of the envelope solitary water waves (WW) in finite depth h when the coefficient μ changes its sign at $\theta_{cr} = k_0 h \approx 1.363$ [76] while ω_0'' is always negative. Thus the nonlinearity belongs to the focusing type for $\theta(= kh) > \theta_{cr}$ and respectively becomes defocusing in the region $\theta < \theta_{cr}$ [76,77]. (Here k is the carrying wave number.) According to [77], for dimensionless variables (when $\mu = \mathrm{sign}(\theta_{cr} - \theta)$) the coefficients β and C for the WW case equal

$$\beta \approx -0.397, \qquad C \approx 0.176. \tag{5.6}$$

In nonlinear optics, as shown in [23], a decrease of μ (Kerr constant) can be provided by the interaction of light pulses with acoustic waves (Mandelstamm–Brillouin scattering).

5.1. Soliton solutions for local nonlinearity

The stationary (independent of t) soliton solutions of Eq. (5.5) will be determined from the following ordinary differential equation:

$$-\lambda^2\psi + \psi_{xx} - \mu|\psi|^2\psi + 3C|\psi|^4\psi + 4i\beta|\psi|^2\psi_x = 0. \tag{5.7}$$

We recall that by construction these solitons move with a constant velocity. The Eq. (5.5) itself, however, contains a larger class of localized solutions. However, these solutions are all nonstationary—their phase and group velocities are different.

Eq. (5.7) can be integrated easily, if the amplitude $r = |\psi|$ and phase $\varphi = \arg\psi$ are introduced instead of ψ: $\psi = re^{i\varphi}$. Next, substituting ψ in Eq. (5.7) and separating real and imaginary parts we obtain an equation for the imaginary part

$$\varphi_x = -\beta r^2. \tag{5.8}$$

After eliminating the phase, the equation for r reduces to Newton's equation

$$2r_{xx} = -\partial U/\partial r \tag{5.9}$$

with the potential

$$U = -\lambda^2 r^2 - \frac{\mu}{2}r^4 + C_1 r^6,$$

where the interaction constant C is renormalized as $C_1 = C + \beta^2$. Then Eqs. (5.8) and (5.9) can be integrated using the energy integral:

$$r^2 = \frac{4\lambda^2}{\sqrt{16\lambda^2 C_1 + \mu^2}\cosh(2\lambda x) - \mu}, \tag{5.10}$$

$$\varphi = -\frac{\beta^2}{\sqrt{C_1}}\tan^{-1}\left[\frac{\sqrt{16\lambda^2 C_1 + \mu^2}e^{2\lambda x} - \mu}{4\lambda\sqrt{C_1}}\right]. \tag{5.11}$$

This soliton type solution exists only if $C_1 > 0$. It is interesting to note that the renormalization of the interaction constant C is due to the β term in the Hamiltonian. This can be seen directly from Eq. (5.4), rewriting H in terms of the amplitude and phase as

$$H = \lambda^2 N + \int\left[r_x^2 + r^2(\dot{\varphi}_x + \beta r^2)^2 + \frac{\mu}{2}r^4 - (C+\beta^2)r^6\right]dx. \tag{5.12}$$

It is also easy to see that the soliton solution (5.10) is a stationary point of H. Indeed, the variation of H with respect to φ leads to Eq. (5.8), and Newton's equation (3.3) arises as a result of varying H with respect to r.

The solutions (5.10) and (5.11) with $\lambda = 0$ and $\mu > 0$ degenerate into a soliton that decays in a power-law fashion [45]

$$r_{\lim}^2 = \frac{2\mu}{\mu^2 x^2 + 4C_1}, \qquad \varphi_{\lim} = -\frac{\beta}{\sqrt{C_1}}\tan^{-1}\frac{\mu x}{2\sqrt{\beta^2 + C}}. \tag{5.13}$$

Thus, as the velocity passes through V_{cr}, the soliton undergoes a jump. The amplitude of the soliton has its maximum value at the jump

$$(\Delta A)^2 = \frac{\mu}{2C_1}.$$

For negative μ, as λ increases, the amplitude of the soliton increases according to a square-root law, and the soliton size decreases as λ^{-1}, in accordance with the general behavior (4.21) for supercritical bifurcations.

An important feature of the solution (5.10) is the existence of a nonlinear coordinate dependence (called a chirp in optics) of the phase φ. The maximum change in phase (from $-\infty$ to $+\infty$ in x)

$$\Delta\varphi = -\frac{\beta\pi}{\sqrt{\beta^2 + C}}$$

is reached at the jump for $V = V_{cr}$. It can be both greater and less than π, depending on the sign of the constant C.

The solution (5.10) can also be used for negative but small values of μ. In this case, as should be, the soliton solution softly splits off zero at the point $V = V_{cr}$. Its amplitude then grows for large λ exactly in the same manner as for $\mu > 0$.

The integral characteristics of both solutions (with $\mu > 0$ and $\mu < 0$) are different. Thus, the total number of particles on the soliton solution for $\mu > 0$,

$$N = \frac{2}{\sqrt{C_1}}\left[\frac{\pi}{2} - \tan^{-1}\left\{\frac{\sqrt{16\lambda^2 C_1 + \mu^2} - \mu}{4\lambda\sqrt{C_1}}\right\}\right], \tag{5.14}$$

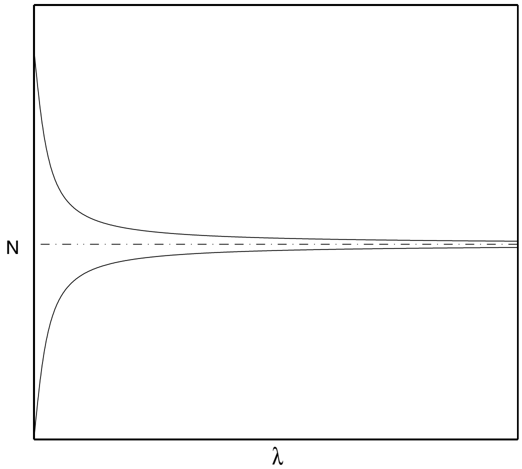

Fig. 8. The dependence (5.14) of the number of particles N versus λ for two soliton families: $\mu < 0$ (lower curve) and $\mu > 0$ (upper curve).

reaches its maximum value

$$2N_{cr} = \frac{\pi}{\sqrt{C_1}} \tag{5.15}$$

at $\lambda = 0$ and decreases smoothly to N_{cr} as $\lambda \to \infty$ (see Fig. 8). For negative μ the number of particles N for small λ increases as λ and then asymptotically approaches $N = N_{cr}$ from below. It is important that the derivatives $\partial N/\partial\lambda$ has different signs: for solitons with a jump this derivative is negative, while for solitons with $\mu < 0$ it is positive. For both branches at large λ the number of waves $N = \int |\psi|^2 dx$ approaches from below and above the same value N_{cr} which coincides with the number of waves N on the solitons with $\mu = 0$. This property for solitons in fibers, with large enough Mandelstamm–Brillouin scattering, means that the energy of optical pulse saturates, tending to the constant value with a decrease of the pulse duration.

5.2. Lyapunov stability of solitons with local nonlinearity

As noted above, both types of solitons (with $\mu > 0$ and $\mu < 0$) are stationary points of energy E with a fixed number of particles,

$$\delta(E + \lambda^2 N) = \delta H = 0, \tag{5.16}$$

where the energy in accordance with Eq. (5.12) is given by the expression

$$E = \int \left[r_x^2 + \frac{\mu}{2} r^4 - (C + \beta^2) r^6 + r^2 (\varphi_x + \beta r^2)^2 \right] dx. \tag{5.17}$$

As we see in the previous sections (see also Ref. [38]), in the case of Hamiltonian systems a stationary point will be Lyapunov-stable if it provides a minimum or maximum of the energy.

In the case at hand, if one can find soliton solutions for which the energy will be bounded from below for a fixed number of particles (E is obviously the unbounded from above functional), then the stationary point corresponding to the minimum of E will be stable. Since the solution of the variational problem (5.16) is unique (up to a constant phase factor) for fixed λ^2, which is equivalent to fixing N, the soliton solution (5.10) will be Lyapunov-stable in this case.

Under the scaling transformations of the soliton solution that preserve the number of particles the energy E as a function of the scaling parameter a takes the form:

$$E = (I_1 - C_1 I_2)\frac{1}{a^2} + \frac{\mu I_3}{2a},$$

where

$$I_1 = \int r_x^2 dx, \qquad I_2 = \int r^6 dx, \qquad I_3 = \int r^4 dx.$$

(We note that the last integral in the expression for the energy (5.17) is identically zero in the soliton solution.) By virtue of Eq. (5.16)

$$\left.\frac{\partial E}{\partial a}\right|_{a=1} = 0 \quad \text{or} \quad I_1 - C_1 I_2 = -\frac{\mu}{4} I_3.$$

Hence it follows, in the first place, that the soliton energy likewise depends strongly on the constant μ:

$$E_s = \frac{\mu}{4} I_{3s} \quad (I_{3s} > 0).$$

This quantity is positive for solitons with weak repulsion ($\mu > 0$) and negative for solitons with weak attraction ($\mu < 0$). In the second place, for $\mu > 0$ the energy E as a function of the scaling parameter a,

$$E = -\frac{\mu}{4} I_{3s} \left(\frac{1}{a^2} - \frac{2}{a} \right), \tag{5.18}$$

is unbounded from below as $a \to 0$, but for weak attraction ($\mu < 0$) it possesses a minimum corresponding to the soliton solution.

We shall now show that the energy E for $\mu < 0$ has a lower bound for all possible deformations that leave N unchanged.

Let us consider the integral $\int r^6 dx = I_2$. This integral can be estimated using the Sobolev–Gagliardo–Nirenberg inequality in terms of the integral $I_1 = \int r_x^2 dx$ and the number of the particles N:

$$\int r^6 dx \le MN^2 \int r_x^2 dx. \tag{5.19}$$

This inequality can be obtained from the Sobolev embedding theorem (2.26) by the same scheme as the inequality (2.28) was derived. The inequality (5.19) can be also improved by seeking the smallest value of the constant M. To find this best constant one needs to consider the minimum value of the functional

$$M[\psi] = \frac{I_2}{I_1 N^2}.$$

Namely, among all stationary points of the functional $M[\psi]$ one needs to choose the point with minimal value of $M[\psi]$. It is easy to see that this variational problem, $\delta M = 0$, is equivalent to finding the soliton solutions for the ψ^6 model with real ψ:

$$-\psi + \psi_{xx} + 3\psi^5 = 0.$$

This equation has a unique solution $\psi = 1/\sqrt{\cosh 2x}$, whence the best constant is simply found as

$$M_{best} = (2/\pi)^2.$$

As the result, the inequality (5.19) reads

$$\int r^6 dx \le \left(\frac{N}{N_1} \right)^2 \int r_x^2 dx, \tag{5.20}$$

where $N_1 = \pi/2$.

Next, substituting this inequality into Eq. (5.17) we obtain for the energy E the estimate

$$E \ge \left[1 - C_1 \left(\frac{N}{N_1} \right)^2 \right] \int r_x^2 dx + \int r^2 (\varphi_x + \beta r^2)^2 dx + \frac{\mu}{2} \int r^4 dx. \tag{5.21}$$

Hence for $\mu > 0$ follows that the energy E is bounded from below by zero if the coefficient in front of the integral $I_1 = \int r_x^2 dx$ is positive, i.e. when

$$1 - C_1 \left(\frac{N}{N_1} \right)^2 \ge 0.$$

This defines an upper bound on the number of particles

$$N \le \frac{\pi}{2\sqrt{C + \beta^2}} = N_{cr}. \tag{5.22}$$

We recall that N_{cr} is the lower limit for the family of solitons (5.10) with $\mu > 0$. Therefore for such solitons it is impossible to draw any conclusion about their stability. However, for soliton solutions with $\mu < 0$ the inequality (5.22) holds and, as will be seen from the estimates made below, it is possible to prove stability of such solutions.

Thus, let $\mu < 0$ in Eq. (5.21). According to (2.30), we have

$$\int r^4 dx \le \frac{1}{\sqrt{3}} \left(\int r_x^2 dx \right)^{1/2} N^{3/2}.$$

Next, substituting this estimate into Eq. (5.21) one can obtain

$$E \geq \left[1 - C_1\left(\frac{N}{N_1}\right)^2\right] I_1 - \frac{|\mu|}{2\sqrt{3}} N^{3/2} I_1^{1/2} + \int r^2(\varphi_x + \beta r^2)^2 dx$$
$$\geq -\frac{|\mu|^2 N^3}{8\sqrt{3}}\left[1 - C_1\left(\frac{N}{N_1}\right)^2\right].$$

The latter inequality holds only if the criterion (5.22) is satisfied. This means that the energy E has a lower (negative) boundary if

$$N < N_{cr},$$

which is compatible with the entire region of existence of solitons with $\mu < 0$. It should be noted that for $\mu = 0$ the NSE (5.4) is, as is said, a critical equation of the NLS type. For this nonlinearity ($\sim|\psi|^6$ in H) collapse becomes possible if $\beta = 0$ and the energy E is negative, as it follows from the virial theorem (4.40).

If $N < N_{cr}$, dispersion completely spreads out the solution. However, a small negative correction to the Hamiltonian significantly changes the situation. A relatively weak four-wave interaction against the background of strong attraction ($\sim|\psi|^6$), leading to collapse (see, for example, Ref. [74]), is responsible for the existence of stable bound stationary states—solitons. Weak localization appears [78].

5.3. Linear stability criterion

The preceding analysis has answered the question about stability only for solitons with weak attraction. From this answer it is impossible to draw any conclusion about the stability of solitons with weak repulsion ($\mu > 0$). In this subsection we shall consider this question, investigating the linear stability problem.

We shall seek for the solution of Eq. (5.4) in the form

$$\psi = (r + a)e^{i(\phi+\alpha)} \approx (r + a + ir\alpha)e^{i\phi}, \tag{5.23}$$

where r and ϕ are the soliton solution (5.10) and (5.11), and a and α are small deviations of the amplitude and phase of the soliton, respectively.

Linearizing Eq. (4.5) it is easily found that the dynamics of the perturbations a and α is determined by the Hamiltonian equations

$$2r\frac{\partial a}{\partial t} = \frac{\delta \widetilde{H}}{\delta \alpha}, \qquad 2r\frac{\partial \alpha}{\partial t} = -\frac{\delta \widetilde{H}}{\delta a}. \tag{5.24}$$

Here $\widetilde{H} = \delta^2 H$ is the second variation of the Hamiltonian (5.10)

$$\widetilde{H} = \langle a|L|a\rangle + \int r^2(\alpha_x + 2\beta r a)^2 dx, \tag{5.25}$$

where the (Schrödinger) operator L is given by the expression

$$L = -\frac{\partial^2}{\partial x^2} + \lambda^2 + 2\mu r^2 - 15C_1 r^4 \tag{5.26}$$

and $\langle a|L|a\rangle$ is the mean value of the operator L for the given state $|a\rangle$,

$$\langle a|L|a\rangle = \int a(x)La(x)dx.$$

If the quadratic form $\delta^2 H$ is sign definite, then the soliton solution will be stable. We note that the second term in (5.25) is positive. Then the positiveness of the entire quadratic form $\widetilde{H}$ is determined by the average value of the operator L, $\langle a|L|a\rangle$. In this expression the average is taken not with respect to arbitrary states $|a\rangle$ but only with respect to those states that are orthogonal to $|r\rangle$:

$$\langle r|a\rangle = 0. \tag{5.27}$$

This orthogonality condition is a consequence of the conservation of the number of particles N and is one of the solvability conditions for the linear system (5.24). In this case, finding the stability criterion for solitons (5.4) is identical to the derivation of the VK criterion (4.33) (see also [39,38]) for the NLSE. Positiveness of

$$\partial N/\partial \lambda^2 > 0 \tag{5.28}$$

guarantees stability of solitons. This situation occurs for solitons with weak attraction, and as a result they are stable. This conclusion is in complete agreement with the results of the preceding subsection.

For solitons with weak repulsion ($\mu > 0$) the criterion (5.28) gives a sign indefiniteness for the quadratic form $\tilde{H}$. This is a necessary condition for instability. This criterion becomes necessary and sufficient only in the case $\beta = 0$, where the average value of L in Eq. (5.25) can be interpreted as a potential energy, and the integral $\int r^2\alpha_x^2 dx$ can be interpreted as the kinetic energy (compare with the derivation of the VK criterion (4.33) in the previous section).

Certain arguments can be given in support of the fact that a soliton with weak repulsion is nonetheless unstable for $\beta \neq 0$ also. The average value of L can be made negative by taking for a the eigenfunction ξ with $E < 0$. For a given value of ξ it is always possible to find a phase α such that the integral

$$\int r^2(\alpha_x + 2\beta ra)^2 dx$$

vanishes. Thus the Hamiltonian $\tilde{H}$ can be made negative, which can be regarded physically as necessary for instability. However, strictly speaking, this still requires a definite proof. An example that refutes the above argument is well known. The Hamiltonian $H = -p^2/2 - q^2/2$ gives the equation of motion for an ordinary stable oscillator even though H is negative. However, it can be asserted absolutely that instability will remain for small values of β. Whether or not a threshold with respect to β exists is still unknown, but it is likely.

But despite the uncertainty with the linear stability, it should be noted that a soliton with weak repulsion is always unstable with respect to finite disturbances. This follows, specifically, from the fact that under scaling transformations leaving the number of particles N unchanged the energy E (5.18) as a function of the scaling parameter a for solitons with $N > N_{cr}$ is unbounded as $a \to 0$. The latter, as we have mentioned already (see, Section 4 and the reviews [79,80] as well), is a criterion for wave collapse.

5.4. Interfacial waves

As an example of bifurcations for solitons first we consider the interfacial deep water waves (IW) which propagate along the interface ($z = \eta(x, t)$) between two ideal incompressible fluids with respective densities ρ_1 and ρ_2, in the presence of gravity (with the acceleration g acting down the vertical z-axis) and capillarity with interfacial tension σ. We shall assume that the lighter fluid with density ρ_2 occupies the region $\infty > z > \eta(x, t)$, and respectively the heavier fluid occupies the region $-\infty < z < \eta(x, t)$. Flows of both fluids are considered to be potential and two-dimensional. The fluid velocities are given by

$$\mathbf{v}_{1,2} = \nabla\phi_{1,2},$$

where the velocity potentials ϕ_1 and ϕ_2 satisfy Laplace's equation

$$\Delta\phi_{1,2} = 0. \tag{5.29}$$

These equations are subject to the following boundary conditions. Far from the interface as $z \to \pm\infty$

$$\phi_{1,2} \to 0.$$

On the interface $z = \eta(x, t)$ the kinematic conditions hold:

$$\frac{\partial\eta}{\partial t} = (-v_x\eta_x + v_z)_{1,2}. \tag{5.30}$$

The dynamic condition reduces to the discontinuity of pressures across the interface due to capillarity:

$$p_1 - p_2 = -\sigma\frac{\partial}{\partial x}\left(\frac{\eta_x}{\sqrt{\eta_x^2 + 1}}\right).$$

The use of Bernoulli equations in each fluid allows to rewrite the latter equation in terms of potentials and their derivatives:

$$\rho_1\left(\frac{\partial\phi_2}{\partial t} + \frac{1}{2}(\nabla\phi_2)^2 + g\eta\right) - \rho_2\left(\frac{\partial\phi_1}{\partial t} + \frac{1}{2}(\nabla\phi_1)^2 + g\eta\right) = \sigma\frac{\partial}{\partial x}\left(\frac{\eta_x}{\sqrt{\eta_x^2 + 1}}\right). \tag{5.31}$$

The Eqs. (5.29)–(5.31) conserve the total energy:

$$H = K + U, \tag{5.32}$$

where the kinetic energy is equal to

$$K = \int_{z>\eta}\frac{\rho_2(\nabla\phi_2)^2}{2}d\mathbf{r} + \int_{z<\eta}\frac{\rho_1(\nabla\phi_1)^2}{2}d\mathbf{r}$$

and the potential energy is given by the expression

$$U = \int(\rho_1 - \rho_2)\frac{g\eta^2}{2}dx + \int\sigma\left(\sqrt{\eta_x^2 + 1} - 1\right)dx.$$

As shown first in [81] (see also [82,83,33,84]), the equations of motion (5.30) and (5.31) together with the Laplace equations (5.29) represent a Hamiltonian system. The Hamiltonian coincides with the energy (5.32). The new variables $\Psi = (\rho_1\psi_1 - \rho_2\psi_2)$ and the interface shape η are canonical conjugate variables:

$$\frac{\partial \eta}{\partial t} = \frac{\delta H}{\delta \Psi}, \qquad \frac{\partial \Psi}{\partial t} = -\frac{\delta H}{\delta \eta}, \tag{5.33}$$

where $\psi_{1,2} = \phi_{1,2}|_{z=\eta}$. The given Hamiltonian form generalizes Zakharov's canonical form for free surface hydrodynamics [85]. A Hamiltonian formulation of the problem of a free interface between two ideal fluids, under rigid lid boundary conditions for the upper fluid, was also given by Benjamin and Bridges [86]. Craig and Groves [87] give a similar expression, by using the Dirichlet–Neumann operators for both the upper and lower fluid domains (see also [88]).

The Hamiltonian can be expanded in series with respect to powers of the canonical variables. In this case the steepness of the interface plays the role of a small parameter of expansion. The normal variables a_k for interfacial waves are given by the following formulas:

$$\Psi(k) = i\sqrt{\frac{(1+\rho)\omega_k}{2|k|}}(a_k - a^*_{-k}), \tag{5.34}$$

$$\eta(k) = \sqrt{\frac{|k|}{2(1+\rho)\omega_k}}(a_k + a^*_{-k}),$$

where $\rho = \rho_2/\rho_1$. In these formulas

$$\omega_k = \left(\frac{|k|}{1+\rho}[g(1-\rho) + \sigma k^2]\right)^{1/2} \tag{5.35}$$

is the dispersion relation for linear internal waves and k is the wave vector directed along the x-axis (1D case). At $\rho = 0$ the interfacial waves transform into the gravity–capillary waves for the deep water case with

$$\omega_k = \left(gk + \sigma k^3\right)^{1/2}.$$

For the internal wave dispersion (5.35) the maximum solitary wave velocity V coincides with the minimum phase velocity of linear waves:

$$V_{cr} = \min \frac{\omega_k}{k}.$$

It occurs when

$$k = k_0 = \left[\frac{g(1-\rho)}{\sigma}\right]^{1/2}. \tag{5.36}$$

At this point the values of the linear frequency and critical velocity are

$$\omega_0 \equiv \omega(k_0) = \sqrt{2Agk_0} \quad \text{and} \quad V_{cr} \equiv \frac{\omega_0}{k_0} = \sqrt{\frac{2Ag}{k_0}}, \tag{5.37}$$

where

$$A = \frac{1-\rho}{1+\rho}$$

is the Atwood number.

The transformation (5.34) diagonalizes the quadratic part of the Hamiltonian,

$$H_0 = \int \omega_k |a_k|^2 dk,$$

and the equations of motion in the new variables a_k take the standard form (4.4):

$$\frac{\partial a_k}{\partial t} = -i\frac{\delta H}{\delta a_k^*}. \tag{5.38}$$

In the case of internal waves the expansion of H_{int} in the wave amplitude starts with the cubic terms. According to [32], the three-wave matrix elements are given by the expressions

$$U_{123} = \frac{A}{4\sqrt{2\pi}(1+\rho)^{1/2}} \left\{ \left(\frac{k_3\omega_1\omega_2}{2k_1k_2\omega_3}\right)^{1/2} [k_1k_2 + |k_1||k_2|] + \left(\frac{k_1\omega_2\omega_3}{2k_2k_3\omega_1}\right)^{1/2} [|k_2||k_3| + k_2k_3] + \left(\frac{k_2\omega_3\omega_1}{2k_3k_1\omega_2}\right)^{1/2} [|k_3||k_1| + k_3k_1] \right\}, \tag{5.39}$$

$$V_{123} = \frac{A}{4\sqrt{2\pi}(1+\rho)^{1/2}} \left\{ \left(\frac{k_3\omega_1\omega_2}{2k_1k_2\omega_3}\right)^{1/2} [-|k_1||k_2| + k_1k_2] + \left(\frac{k_1\omega_2\omega_3}{2k_2k_3\omega_1}\right)^{1/2} [|k_2||k_3| + k_2k_3] + \left(\frac{k_2\omega_3\omega_1}{2k_3k_1\omega_2}\right)^{1/2} [-|k_3||k_1| + k_3k_1] \right\} \tag{5.40}$$

where $k_1 \equiv 1$, $k_2 \equiv 2$, and $k_3 \equiv 3$. Hence the values of $U_{k_0k_0-2k_0}$ and $V_{2k_0|k_0k_0}$ are equal

$$U_{k_0k_0-2k_0} = \frac{Ak_0\omega_{k_0}}{2\sqrt{2\pi}(1+\rho)^{1/2}} \left(\frac{k_0}{\omega_{2k_0}}\right)^{1/2}, \qquad V_{2k_0k_0k_0} = \frac{Ak_0\omega_{k_0}}{2\sqrt{2\pi}(1+\rho)^{1/2}} \left(\frac{k_0}{\omega_{2k_0}}\right)^{1/2}. \tag{5.41}$$

Respectively the "bare" four-wave constant $T_{k_0k_0|k_0k_0}$ is

$$T_{k_0k_0|k_0k_0} = \frac{5}{32\pi} \frac{k_0^3}{(\rho+1)}. \tag{5.42}$$

Substituting expressions (5.41) and (5.42) into (4.17) gives

$$\widetilde{T}_0 = \frac{k_0^3}{2\pi(1+\rho)} \left(A_{cr}^2 - A^2\right) \equiv \frac{\mu}{2\pi},$$

where the square of the critical Atwood number A_{cr}^2 is equal to $5/16$. This gives for the critical value of ρ

$$\rho_{cr} = \frac{4-\sqrt{5}}{4+\sqrt{5}},$$

in agreement with the paper [29]. For $\rho < \rho_{cr}$, the four-wave coupling coefficient is negative, and the corresponding nonlinearity is of the focusing type. In this case, solitary waves near the critical velocity V_{cr} are described by the stationary NLSE (4.20) and undergo a supercritical bifurcation at $V = V_{cr}$ [29]. For $\rho > \rho_{cr}$ the coupling coefficient changes sign and the bifurcation becomes subcritical. In the particular case of the deep water waves (when $\rho = 0$) the solitons undergo only a supercritical bifurcation.

Now we will give the general structure of the Hamiltonian expansion corresponding to interfacial waves near the critical density ratio assuming the following two dimensionless parameters are small:

$$\lambda = \sqrt{1 - V/V_{cr}} \quad \text{and} \quad \theta = 1 - \rho/\rho_{cr}.$$

In this case the expansion of the four-wave matrix element $\widetilde{T}_{k_1k_2k_3k_4}$ linear in $\kappa_i = k_i - k_0$ contains, besides the local terms proportional to β, the nonlocal ones:

$$\widetilde{T}_{k_1k_2k_3k_4} = \frac{\mu}{2\pi} + \frac{\beta}{2\pi}(\kappa_1 + \kappa_2 + \kappa_3 + \kappa_4) - \frac{\gamma}{8\pi}(|\kappa_1 - \kappa_3| + |\kappa_2 - \kappa_3| + |\kappa_2 - \kappa_4| + |\kappa_1 - \kappa_4|). \tag{5.43}$$

The constants β and γ have different parity relative to reflection $k_0 \to -k_0$. The coefficient β changes its sign, but the coefficient γ retains its sign under this transform. The difference in parities between β and γ gives different contributions to the averaged four-wave Hamiltonian:

$$\overline{H}^{(4)} = \frac{1}{2} \int \left[\mu|\psi|^4 + 2i\beta(\psi_x^*\psi - \psi_x\psi^*)|\psi|^2 - \gamma|\psi|^2\widehat{k}|\psi|^2\right] dx. \tag{5.44}$$

Here $\widehat{k}$ is the positive definite integral operator

$$\widehat{k} = -\partial_x \widehat{H},$$

and $\widehat{H}$ is the Hilbert transform:

$$\widehat{H}f(x) = \frac{1}{\pi}\left(P.V. \int_{-\infty}^{\infty} \frac{f(x')dx'}{x'-x}\right).$$

The Fourier transform of the kernel of the operator $\widehat{k}$ is equal to $|k|$.

Due to this nonlocal interaction the equation of motion (5.5) for ψ takes the form

$$i\frac{\partial\psi}{\partial t}-\lambda^2\omega_0\psi+\frac{\omega_0''}{2}\psi_{xx}-\mu|\psi|^2\psi+4i\beta|\psi|^2\psi_x+\gamma\psi\widehat{k}|\psi|^2+3C|\psi|^4\psi=0. \tag{5.45}$$

The nonlocal term in this equation is analogous to that found first time by Dysthe for gravity waves [46]. The existence of the nonlocal contribution in the expansion (5.43) is connected with non-analytical dependence of the matrix element T in its arguments. For interfacial deep water waves (IW) this non-analyticity originates from the solution of Laplace equation for the hydrodynamic potential and its reduction to the moving interface. For instance, for water waves (WW) with a finite depth the nonlocal term is absent [77] as well as for electromagnetic waves in nonlinear dielectrics [23] because of the analyticity of matrix elements with respect to frequencies, which is a consequence of causality (see, for example, Refs. [42,23]). In the latter case the spatial dispersion effects are relativistically small and can be neglected.

As before, the solitary wave shape in this case will be defined from the solution of the variational problem (5.16),

$$\delta(\overline{E}+\omega_0\lambda^2N)=0, \tag{5.46}$$

where the energy E (5.17) gets the nonlocal addition

$$-\frac{\gamma}{2}\int r^2\widehat{k}r^2dx. \tag{5.47}$$

Thus, in order to solve the variational problem (5.46), we need to know three coefficients: β, γ and C. One can easily see that the contributions from terms proportional to $\beta,\ \gamma$ in $\overline{H}^{(4)}$ and the six-wave Hamiltonian can be determined independently, which makes calculations more simple. According to [32] for IW

$$\beta=\frac{3k_0^2}{16(1+\rho)},\qquad \gamma=\frac{16}{3}\beta,\qquad C=\frac{Mk_0^6}{3\omega_0} \tag{5.48}$$

where $M=\frac{289(21+8\sqrt{5})}{16384}\approx 0.685961$. Thus, the nonlocal ($\sim\gamma$) and six-wave ($\sim C$) interactions correspond to an attraction between waves (focusing nonlinearity).

For $\rho<\rho_{cr}$, the four-wave coupling coefficient μ is negative, and the corresponding nonlinearity is of the focusing type. In this case, solitary waves near the critical velocity V_{cr} are described by the stationary ($\partial/\partial t=0$) NLSE and undergo a supercritical bifurcation at $V=V_{cr}$ [29]. For $\rho>\rho_{cr}$ the coupling coefficient changes sign and, as a result, the bifurcation becomes subcritical.

5.5. Soliton families for IW and their stability

For IW near the transition point, $|\theta|\ll 1$, the product $\omega_0''C$ is positive; moreover γ is also positive for IW, and therefore the corresponding nonlinearities are focusing, thus providing the existence of localized solutions. As for the local case, depending on the sign of μ there exist two branches of solitons. For IW they were found numerically [31,32] using the Petviashvili scheme [89]. Explicit solutions for both kinds of IW solitons can be obtained in the limiting case only when $V\to V_{cr}$. For negative μ these are the classical NLS solitons with a sech shape. For the subcritical bifurcation at $V=V_{cr}$ the soliton amplitude remains finite with algebraic decay ($\sim 1/|x|$) at infinity [31,32] (compare with (5.13)). Like for the local nonlinearities both soliton families for positive and negative values of μ asymptotically tend to the same state at $\lambda\to\infty$ which can be related to solitons for the critical NLS type equation. As the result both dependences of N at large λ converge to the critical value N_{cr}, for the family with $\mu<0$ from below and from above for the family with $\mu>0$ (but here N_{cr} differs from the value (5.15) because of nonlocalities). Thus, these dependences are familiar to those depicted in Fig. 9. From this point of view influence of the nonlocal nonlinearity, being of focusing, is not of principle if it concerns solitonic behavior. More or less the same conclusion can be made for stability of solitons. As shown in [23,31,32,77], for $N<N_{cr}$ solitons corresponding to the supercritical branch realize the minimum values of the energy and therefore they are stable in the Lyapunov sense, i.e. stable with respect to not only small perturbations but also against finite ones. In particular, the boundedness of E from below can be viewed if one considers the scaling transformation $\psi=(1/a)^{1/2}\psi_s(x/a)$ retaining the number of waves N where $\psi=\psi_s(x)$ is the soliton solution. Under this transform $E=E(a)$ has the same form as Eq. (5.18):

$$E\,(a)=\left(\frac{1}{a}-\frac{1}{2a^2}\right)\frac{\mu}{2}\int|\psi_s|^4dx. \tag{5.49}$$

It is worth noting that the dispersion term and all nonlinear terms in E, except $\int\frac{\mu}{2}|\psi|^4dx$, have the same scaling dependence $\propto a^{-2}$. The latter means that at $\mu=0$ Eq. (5.45) can be related to the critical NLS equation like the two-dimensional cubic NLS equation. From Eq. (5.49) it is also seen that for $\mu<0$ $E(a)$ has a minimum corresponding to the soliton. Unlike the supercritical case, the scaling transformation for the other soliton branch with $\mu>0$ gives a maximum of $E(a)$ on solitons and unboundedness of E as $a\to 0$. Under the gauge transformation $\psi=\psi_s e^{i\chi}$, on the contrary, the energy reaches a minimum on soliton solutions and consequently the solitons with $\mu>0$ represent saddle points. This indicates a possible instability of solitons for the whole subcritical branch, at least with respect to finite perturbations.

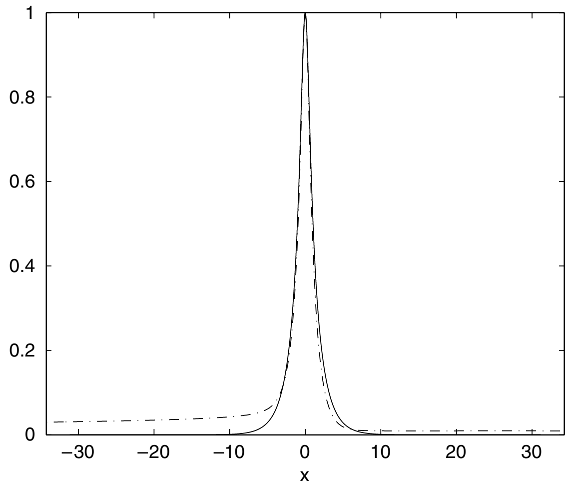

Fig. 9. Initial (solid line) and final (dashed line) at $t = 1.18$ distributions for $|\overline{\psi}|$, interfacial waves, self-similar variables. The soliton amplitude was increased by 1%, $\mu = 1$, $\lambda = 1$. The ratio between final and initial soliton amplitudes in the physical variables is about 11.

We consider this question in more details with the main emphasis to the nonlinear stage of the instability following to [90]. This problem, indeed, is not trivial in spite of a closed similarity with the critical NLSE. It is worth noting that Eq. (5.45) at $\mu = \gamma = C = 0$ represents an integrable model (the so-called derivative NLSE) [91] and exponentially decaying solitons in this model are stable. It is more or less evident also that small coefficients γ, C cannot break the stability of solitons. This means that in the space of parameters we may expect the existence of a threshold. Above this threshold solitons must be unstable and the development of this instability may lead to collapse, i.e. the formation of a singularity in finite time.

Consider the energy (5.17) with account of the nonlocal addition (5.47) written in terms of amplitude r and phase $\varphi(\psi = re^{i\varphi})$:

$$E = \int \left[r_x^2 + \frac{\mu}{2} r^4 - \frac{\gamma}{2} r^2 \widehat{k} r^2 - \frac{1}{3} r^6 + r^2 \left(\varphi_x + \beta r^2 \right)^2 \right] dx, \tag{5.50}$$

where by an appropriate choice of the new dimensionless variables the renormalized constant $\tilde{C} = C + \beta^2$ can be taken equal to 1/3. Hence one can see that the energy takes its minimum value when the last term in (5.50) vanishes, i.e. when

$$\varphi_x + \beta r^2 = 0. \tag{5.51}$$

Integrating this equation gives an x-dependence for the phase, called chirp in nonlinear optics. It is interesting to note that the remaining part of the energy does not contain the phase at all.

First investigate the local model when $\gamma = 0$. Let the energy be negative in some region $\Omega : E_\Omega < 0$. Then, following Refs. [64,80], one can establish that due to radiation of small amplitude waves $E_\Omega < 0$ can only decrease, becoming more and more negative, but the maximum value of $|\psi|$, according to the mean value theorem, can only increase:

$$\max_{x \in \Omega} |\psi|^4 \geq \frac{3|E_\Omega|}{N_\Omega}. \tag{5.52}$$

This process is possible only for energies which are unbounded from below. In accordance with (5.49) such a situation is realized when $\mu > 0$. In this case the radiation leads to the appearance of infinitely large amplitudes r. However, it is impossible to conclude that the singularity formation develops in finite time.

For $\gamma > 0$ the estimations on the maximum value of $|\psi|$ are not as transparent as they are for the local case. Instead of (5.52), it is possible to obtain a similar estimate,

$$\max_x |\psi|^4 \geq \frac{3|E|}{N}.$$

However, it is expressed through the total energy E and the total number of waves N. Besides, two inequalities must be satisfied: $E < 0$ and $N < \frac{2N_2}{\gamma}$. For interfacial waves, $N_2 \approx 1.39035 > N_{cr} \approx 1.3521$. Thus, the maximum amplitude in this case is bounded from below by a conservative quantity and this maximum can never disappear during the nonlinear evolution.

Now we consider the situation where the self-steepening process can be neglected ($\beta = 0$). In this case Eq. (5.45) becomes

$$i\psi_t + \psi_{xx} - \lambda^2 \psi - \mu |\psi|^2 \psi + \gamma \psi \widehat{k} |\psi|^2 + 3C |\psi|^4 \psi = 0.$$

It is possible to obtain a criterion of collapse using the virial equation, compare with (4.41) (for details, see also [71,92,64]). This equation is written for the positive definite quantity

$$R = \int x^2|\psi|^2 dx,$$

which, up to the multiplier N, coincides with the mean square size of the distribution. The second derivative of R with respect to time is defined by the virial equation

$$R_{tt} = 8\left(E - \frac{\mu}{4}\int|\psi|^4 dx\right). \tag{5.53}$$

Hence, for $\mu > 0$ one can easily obtain the following inequality:

$$R_{tt} < 8E,$$

which yields, after double integration, $R < 4Et^2 + \alpha_1 t + \alpha_2$. Here $\alpha_{1,2}$ are constants which are obtained from the initial conditions. Hence, it follows that for the states with negative energy, $E < 0$, there always exists such moment of time t_0 when the positive definite quantity R vanishes. At this moment of time the amplitude becomes infinite. Therefore the condition $E < 0$ represents a sufficient criterion of collapse (compare with [71,92]). However, it is necessary to add that this criterion can be improved by the same way as it was done in Refs. [93,75] for the three-dimensional cubic NLS equation. From Eq. (5.53) one can see that for the stationary case (on the soliton solution) $E_s = \frac{\mu}{4}\int|\psi_s|^4 dx$, in agreement with Eq. (5.49). As we demonstrated before for $\mu > 0$ the soliton realizes a saddle point of E for fixed N. It follows from (5.49) that for small a the energy becomes unbounded from below, but for $a > 1$ it decreases (this corresponds to spreading). Therefore in order to achieve a blow-up regime the system should pass through the energetic barrier equal to E_s. Thus, for this case the criterion $E < 0$ must be changed into the sharper criterion: $E < E_s$. This criterion can be obtained rigorously using step by step the scheme presented in [93,75] and therefore we skip its derivation.

5.6. Numerical results of the soliton collapse

In order to verify all the theoretical arguments about the formation of collapse presented above a numerical integration of the NLSE (5.45) for $\mu > 0$ was performed by using the standard fourth-order Runge–Kutta scheme. The initial conditions were chosen in the form of solitons but with larger amplitudes than for the stationary solitons. The increase in initial amplitude was varied in the interval from 0.1% up to 10%. The initial phase was given by means of Eq. (5.51). In all runs with these initial conditions we observed a high increase of the soliton amplitude up to a factor 14 with a shrinking of its width. In the peak region pulses for both IW and WW cases behaved similarly. Near the maximum the pulse peak was almost symmetric: anisotropy was not visible. The difference was observed in the asymptotic regions far from the pulse core where the pulses had different asymmetries for IW and WW because of the opposite sign for β (see Eqs. (5.48), (5.6)). For the given values of β we did not observe the simultaneous formation of two types of singularities with blowing-up amplitudes and sharp gradients as it was demonstrated in the recent numerical experiments for the three-dimensional collapse of short optical pulses due to self-focusing and self-steepening in the framework of the generalized NLS equation [94] and equations of the Kadomtsev–Petviashvili type [95].

In our numerical computations we found that the amplitude and its spatial collapsing distribution develop in a self-similar manner. Near the collapse point in the equation (with $\mu > 0$) one can neglect the term proportional to μ. In this asymptotic regime Eq. (5.45) admits self-similar solutions,

$$r(x,t) = (t_0 - t)^{-1/4} f\left(\frac{x}{(t_0 - t)^{1/2}}\right), \tag{5.54}$$

where t_0 is the collapse time.

To verify that we approached the asymptotic behavior given by Eq. (5.54), we normalized at each moment of time the ψ−function by the maximum (in x) of its modulus $\max|\psi| \equiv M$ and introduced new self-similar variables,

$$\psi(x,t) = M\overline{\psi}(\xi,\tau), \quad \xi = M^2(x - x_{\max}), \ \tau = \ln M. \tag{5.55}$$

Here $x_{\max}$ is the point corresponding to the maximum of $|\psi|$. In comparison with those given by Eq. (5.54), such new variables are more convenient because they do not require the determination of the collapsing time t_0.

Figs. 9 and 10 show typical dependences of $|\overline{\psi}|$ as a function of the self-similar variable ξ at $t = 0$ (solid line) and at the final time (dashed line) for both the IW and WW cases. In both figures one can see a fairly good coincidence between the initial soliton distribution and the final one at the central (collapsing) part of the pulse and asymmetry of the pulse at its tails due to self-steepening. The latter demonstrates that collapse has a self-similar behavior. The form of the central part of the pulse approaches the soliton shape because asymptotically the NLS model (5.45) tends to the critical NLS system. It is necessary to mention that this has been well known for the classical two-dimensional NLS equation since the paper by Fraiman [96].

Fig. 11 shows how $1/\max|\psi|^4$ depends on time. This dependence is almost linear in the correspondence with the self-similar law (5.54). If the initial amplitudes were less than the stationary soliton values, then the distribution would spread in time dispersively, in full correspondence with qualitative arguments based on the scaling transformations (5.49).

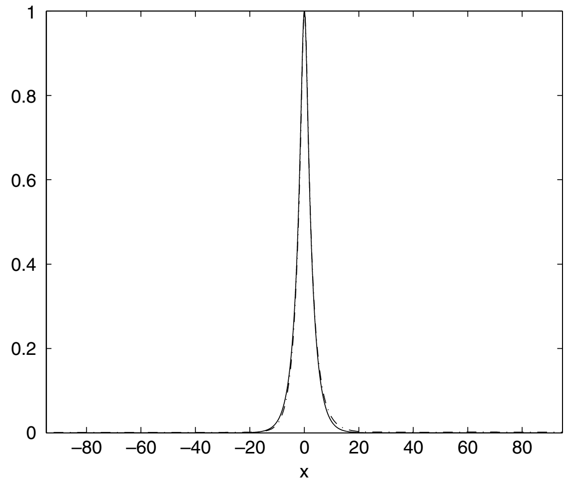

Fig. 10. Initial (solid line) and final (dashed line) at $t = 2.7192$ distributions for $|\overline{\psi}|$, WW solitons, self-similar variables. The soliton amplitude was increased by 1%, $\mu = 1$, $\lambda = 1$. The ratio between final and initial soliton amplitudes in the physical variables is about 11.

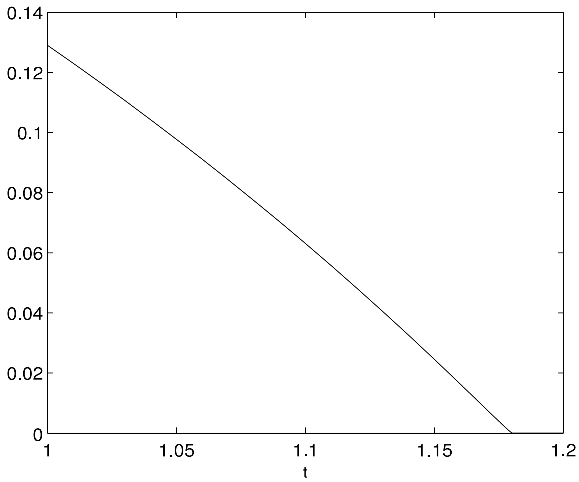

Fig. 11. Dependence of $1/\max|\psi|^4$ on time. Interfacial waves.

6. Bifurcations for the interacting fundamental and second harmonics

In Section 4 we saw that the renormalized four-wave interaction constant $\widetilde{T_0}$ (4.17) becomes infinitely large while approaching the resonance (4.18)

$$2\omega(k_0) = \omega(2k_0). \tag{6.1}$$

In this case the perturbation theory for (4.17) breaks down and we should consider this resonance separately by introducing two envelopes ψ_1 for the fundamental frequency (FF) and ψ_2 for its second harmonic (SH). These envelopes satisfy two equations coupled due to quadratic nonlinearity ($\sim V_{2k_0k_0k_0}$). As well known, such system represents a partial case of the so-called three-wave system [24] when carrier frequencies of three wave packets satisfy the triad resonant condition. It is known also that such wave packets can form bound states – solitons – due to their mutual nonlinear interaction (see e.g. [25]). The three-wave system describes spatial solitons as well as spatial–temporal solitons in $\chi^{(2)}$ media [25,34]. This system couples amplitudes of three quasi-monochromatic waves due to quadratic nonlinearity (see also [97] for the study of the interactions between the fundamental wave and its second harmonic in the presence of symmetries).

When the difference in group velocities between these first and second harmonics is small enough (which is typical for nonlinear optics) it is necessary to take into account wave dispersion. In this case this model can be considered as a vector nonlinear Schrödinger system but with quadratic nonlinearity. The balance between nonlinear interaction and dispersion results in the existence of solitons. If the difference in group velocities is large enough, then each wave packet propagates away with its group velocity, and the system cannot form a bound state between the first and second harmonic. Moreover, in this case this system is close to being integrable. For zero dispersion it can be integrated by means of the inverse scattering transform [98,12].

As it was shown in Section 4, when the soliton velocity V is close to the minimum phase velocity of linear waves, $V_{cr} = \min(\omega(k)/k)$, the Fourier spectrum of the stationary pulse represents a set of peaks. Their positions in the frequency domain correspond to the frequency related to the minimal phase velocity (fundamental harmonic) and to its multiple harmonics. The width of these peaks vanishes when the critical velocity V_{cr} is approached. This allows us to introduce an envelope for each peak, and apply a standard multi-scale expansion. In the case when the carrier frequency, corresponding to the critical velocity does not satisfy the resonance condition (6.1), then near the supercritical bifurcation point the envelope of the fundamental harmonic obeys the stationary (focusing) nonlinear Schrödinger equation (2.14). If the carrier frequency for the fundamental harmonic is close to the resonance condition (6.1) then the corresponding equations for a steady pulse transform into the steady (time-independent) equations for the interacting fundamental and second harmonics. The stability of these soliton solutions with respect to modulation perturbations can be described in the framework of the unsteady system for fundamental and second harmonics.

With account of dispersion (plus diffraction in the multi-dimensional case) the system of interacting fundamental and second harmonics contains three free parameters: the phase mismatch Ω, which characterizes how far the carrier frequencies of the first and second harmonics are from resonance, and two dispersion coefficients ω_1'' and ω_2'' where 1, 2 stand for FF and SH respectively, and prime means derivative with respect to $k_{1,2}$ where $k_2 = 2k_1$. In the multi-dimensional case, instead of dispersion coefficients two dispersion tensors arise. All other parameters can be excluded by simple rescaling. The behavior of solitons depends significantly on these three parameters. In the simplest variant soliton solutions are determined by one internal parameter λ^2, which can be considered as a "chemical potential". Until now, mainly such solutions have been treated. We would like to pay attention to the fact that the system of interacting fundamental and second harmonics (as well as the three-wave system) is not Galilean invariant in the general situation. In particular, this means that the system must have a more broad soliton family than could be considered before. The velocity of the soliton, together with the "chemical potential" are two inner independent parameters of the soliton family. In this Section, following [27], we analyze this two-parameter family both analytically and numerically. These two parameters are not arbitrary indeed: there are some restrictions imposed on them which follow from the conditions for soliton existence. This yields a two-dimensional region in the parameter space. Passing across the boundary of this region, a soliton undergoes bifurcations. In this section we show that for this system two types of bifurcations are possible. The first is a supercritical bifurcation, when the first harmonic amplitude ψ_1 for the soliton solution vanishes smoothly while the second harmonic amplitude depends on ψ_1 quadratically. Near such a bifurcation a soliton solution transforms into the soliton of the nonlinear Schrödinger equation (NLS) that is embedded in the general scheme considered in Section 4. As we showed, such solutions are stable for the one-dimensional case.

Another possibility is a subcritical bifurcation which takes place when the characteristic size for the second harmonic becomes infinite as the soliton parameters approach the boundary. In this case near the boundary the amplitude of the second harmonic remains finite, but the amplitude ψ_1 vanishes. Correspondingly, close to the boundary the Manley–Rowe integral in this case becomes infinite. The derivative of this integral relative to the parameter λ^2 becomes negative, so that, in accordance with the Vakhitov–Kolokolov (VK) type of criteria [39,47], there is soliton instability. In this case, however, this criterion is only a *sufficient* criterion for instability : it cannot be used to establish stability. The original VK stability criterion for the NLS equation [39] is simultaneously necessary and sufficient. The difference in the stability criteria between the NLS and the FF–SH system (being a two-component NLS system), is connected with the vector character of the latter (for details see the next Section and also [47]). Just for this reason, it is impossible to generalize the criterion (4.33) to the FF–SH system completely.

6.1. Basic equations

The equations of motion describing the interaction of the first (fundamental) and second harmonics can be written as follows:

$$i\frac{\partial\psi_1}{\partial t} + \frac{1}{2}\omega_1''\psi_{1xx} = -2\psi_2\psi_1^*, \tag{6.2}$$

$$i\frac{\partial\psi_2}{\partial t} - \Omega\psi_2 + \frac{1}{2}\omega_2''\psi_{2xx} = -\psi_1^2 \tag{6.3}$$

where the parameter Ω in (6.2) characterizes the phase mismatch and the matrix element $V_{2k_0k_0k_0}$ is chosen to be equal -1. Here, for simplicity, we consider only the one-dimensional case. The corresponding generalization to the multi-dimensional case is straightforward. In particular, in the multi-dimensional case one needs to change the 1D operators $\omega_l''\partial_x^2$ $(l = 1, 2)$ in (6.2) and (6.3) to

$$\frac{\partial^2\omega_l(k_l)}{\partial k_{li}\partial k_{lj}}\frac{\partial^2}{\partial x_i\partial xj}.$$

The system under consideration (6.2) and (6.3) is Hamiltonian:

$$i\frac{\partial\psi_l}{\partial t} = \frac{\delta H}{\delta\psi_l^*} \tag{6.4}$$

with the Hamiltonian

$$H = \int \Omega |\psi_2|^2 dx + \sum_l \int \frac{1}{2}\omega_l'' |\psi_{lx}|^2 dx - \int (\psi_1^{*2}\psi_2 + c.c.)dx. \tag{6.5}$$

Besides the Hamiltonian, this system conserves also the Manley–Rowe integral

$$N = \int (|\psi_1|^2 + 2|\psi_2|^2)dx \tag{6.6}$$

which is a consequence of gauge invariance of the system. This integral has also be regarded as an adiabatic invariant which appears as a result of average of the original system over fast oscillations corresponding to the carrier frequencies of two resonant wave packets. From the definition (6.6), the Manley–Rowe integral is a positive quantity.

Another integral of motion for Eqs. (6.2) and (6.3) is the momentum,

$$P = -\frac{i}{2}\sum_l \int (\psi_l^* \psi_{lx} - \psi_{lx}^* \psi_l)dx,$$

which is a consequence of the invariance of the system to spatial translations. The latter, however, does not guarantee that the equations of motion will be Galilean invariant. To check this, let us perform two transformations. The first is passing to the coordinate system moving with velocity V,

$$x' = x - Vt, \quad t' = t, \tag{6.7}$$

and another is a simple gauge transform:

$$\psi_1 \to \psi_1 e^{-i\omega t + ikx}, \qquad \psi_2 \to \psi_2 e^{-i2\omega t + i2kx}. \tag{6.8}$$

Then we require that the obtained equations have a similar form to the original system. Simple calculations show that this demand can be satisfied if and only if

$$2p_1 = p_2 \tag{6.9}$$

where $p_l = 1/\omega_l''$. Under this condition only, we have Galilean invariance. In all other cases the system (6.2) and (6.3) is *not Galilean invariant*. The same situation occurs for the three-wave system which describes the interaction of three resonant wave packets (compare with [47]).

The absence of Galilean invariance in the general case for the system (6.2) and (6.3) means that movable soliton solutions cannot be transformed by means of the transformations (6.7) and (6.8) to the rest soliton. In other words, the soliton velocity V itself is a new independent parameter, which together with the energy of solitons (as bound states) $\varepsilon = -\lambda^2$ defines a two-parameter soliton family. This family is given as follows:

$$\psi_1(x,t) = \psi_{1s}(x - Vt)e^{i\lambda^2 t}, \qquad \psi_2(x,t) = \psi_{2s}(x - Vt)e^{2i\lambda^2 t},$$

where the amplitudes ψ_{1s} and ψ_{2s} satisfy the equations:

$$-\lambda^2\psi_1 - iV\partial_x\psi_1 + \frac{1}{2}\omega_1''\partial_x^2\psi_1 = -2\psi_2\psi_1^*, \tag{6.10}$$

$$-2\lambda^2\psi_2 - iV\partial_x\psi_2 - \Omega\psi_2 + \frac{1}{2}\omega_2''\partial_x^2\psi_2 = -\psi_1^2. \tag{6.11}$$

Here and below the index s for ψ_{1s} and ψ_{2s} is temporarily omitted.

It is easy to verify that a solution of the stationary equations is a stationary point of the Hamiltonian H for fixed N and momentum P: Eqs. (6.10) and (6.11) follow from the variational problem

$$\delta(H + \lambda^2 N - VP) = 0. \tag{6.12}$$

In order to have localized solution of the stationary system it is necessary to require that two operators

$$A_1 = -\lambda^2 - iV\partial_x + \frac{1}{2}\omega_1''\partial_x^2,$$

$$A_2 = -2\lambda^2 - iV\partial_x - \Omega + \frac{1}{2}\omega_2''\partial_x^2$$

must be sign (negative or positive) definite to provide exponentially decreasing behavior at infinity. Physically, this requirement means absence of Cherenkov radiation by solitons. Indeed, the condition on the signature of the operators is more restrictive: the operators A_1 and A_2 must be simultaneously negative (or positive) definite. This follows from the two integral relations which can be obtained from the variational principle (6.12) and directly from the stationary equations

(6.10) and (6.11) after multiplying the first equation by ψ_1^*, the second one by $2\psi_2^*$, with summation of the obtained results, followed by their integration over x (for details, see [47]).

In the case when the operators are negative definite, the following conditions must be fulfilled:

$$\omega_{1,2}'' > 0, \tag{6.13}$$

$$-\lambda^2 + \frac{1}{2}\frac{V^2}{\omega_1''} < 0, \tag{6.14}$$

$$2\lambda^2 - \Omega + \frac{1}{2}\frac{V^2}{\omega_2''} < 0. \tag{6.15}$$

These requirements guarantee absence of Cherenkov radiation by stationary propagating solitons (for details, see papers [22,23]).

The conditions (6.13)–(6.15) define the region of parameters where soliton solutions are possible. To find this region it is convenient to introduce instead of λ^2 a new quantity $\Lambda^2 = \lambda^2 - p_1V^2/2$ which in accordance with (6.14) has to be a positive quantity. As a result, the last inequality (6.15) reads as

$$2\Lambda^2 + (2p_1 - p_2)\frac{V^2}{2} + \Omega > 0. \tag{6.16}$$

Depending on the signs of $\kappa = 2p_1 - p_2$ and Ω we have four possibilities:

(1) $\kappa > 0, \Omega > 0$. In this case the inequality (6.16) is satisfied automatically. The allowed region for the soliton parameters is the quarter-plane $\Lambda_1^2 > 0, V^2 \geq 0$.
(2) $\kappa > 0, \Omega < 0$. The allowed region for the soliton parameters is the quarter-plane $\Lambda^2 > 0$ and $V^2 \geq 0$ except the triangular region near origin bounded by the straight line $2\Lambda^2 + \kappa V^2/2 = |\Omega|$.
(3) $\kappa < 0, \Omega > 0$. The allowed region for the parameters is the quarter-plane $\Lambda^2 > 0$ and $V^2 \geq 0$ except the region below the straight line $2\Lambda^2 = |\kappa|V^2/2 - \Omega$.
(4) $\kappa < 0, \Omega < 0$. The allowed region is the quarter-plane $\Lambda^2 > 0$ and $v^2 \geq 0$ except the region below the straight line $2\Lambda^2 = |\kappa|V^2/2 + |\Omega|$. In this case Λ^2 cannot reach zero value.

It is interesting to note that $\kappa = 0$ recovers Galilean invariance of the Eqs. (6.2), (6.3). In this case both criteria (6.13) and (6.16) when expressed through Λ^2 do not contain the velocity:

$$\Lambda^2 > 0, \quad 2\Lambda^2 + \Omega > 0.$$

6.2. Solitons

Next we shall analyze the soliton solutions which are defined from the system (6.10), (6.11). By the transformation,

$$\psi_l \to \psi_1 e^{ip_1Vx}, \qquad \psi_2 \to \psi_2 e^{2ip_1Vx},$$

and rescaling the amplitudes ψ_l, this system can be rewritten as follows:

$$-\Lambda_1^2\psi_1 + \partial_x^2\psi_1 = -2\psi_2\psi_1^*, \tag{6.17}$$

$$-\Lambda_2^2\psi_2 + (\partial_x - ik)^2\psi_2 = -\psi_1^2 \tag{6.18}$$

where $k = \kappa V$ and

$$\Lambda_1^2 = 2p_1\Lambda^2, \qquad \Lambda_2^2 = 2p_2(2\Lambda^2 + \kappa V^2/2 + \Omega).$$

The ordinary differential equations (6.17) and (6.18) have two integrals. The first is

$$\psi_1^*\psi_{1x} - \psi_{1x}^*\psi_1 + 2(\psi_2^*\psi_{2x} - \psi_{2x}^*\psi_2) - 4ik|\psi_2|^2 = M, \tag{6.19}$$

which has a meaning of "angular momentum" for the system (6.17), (6.18). Another integral, "energy", is:

$$|\psi_{1x}|^2 + |\psi_{2x}|^2 - \Lambda_1^2|\psi_1|^2 - (\Lambda_2^2 + k^2)|\psi_2|^2 + \psi_2^*\psi_1^2 + \psi_2\psi_1^{*2} = E. \tag{6.20}$$

For a soliton solution ($\psi_{1,2} \to 0$ as $|x| \to \infty$) both integrals are equal to zero. Here we have two cases $k = 0$ and $k \neq 0$. In the first case a soliton solution can be taken purely real and so the first integral becomes equal to zero identically. Thus, in this case only the "energy" integral survives, which can be used to reduce the order of the system of ordinary differential equations. In the second case the solution remains complex.

It is worth noting that the conditions (6.13)–(6.15) for localized solutions of the system (6.17), (6.18) correspond simply to the positivity of Λ_1^2 and Λ_2^2. Thus, the boundary of the soliton parameter region is given by the two equations $\Lambda_1^2 = 0$ (when $\Lambda_2^2 > 0$) and $\Lambda_2^2 = 0$ (when $\Lambda_1^2 > 0$). Accordingly, we have two variants of soliton degeneracy.

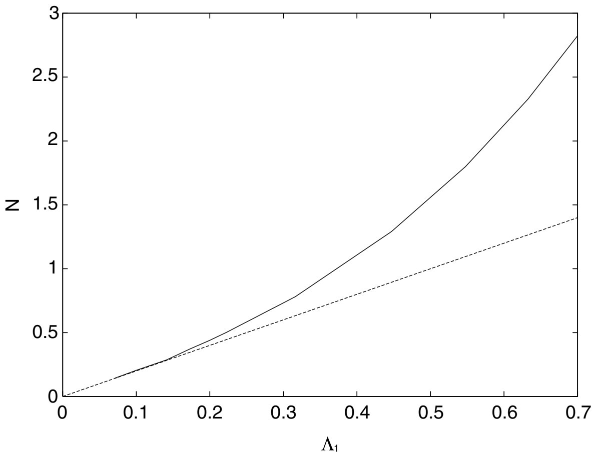

Fig. 12. The dependence of N on Λ_1 for the case $k = 0$ and $\Lambda_2^2 = 1$ (near the supercritical bifurcation boundary). The straight (dashed) line corresponds to the analytical result (6.23).

6.3. Supercritical bifurcations

Consider first how the soliton family looks like near the parameter boundary $\Lambda_1^2 = 0$ (when $\Lambda_2^2 > 0$). First, note that in Eq. (6.17) ψ_1 near this boundary changes on the scale $l \sim 1/\Lambda_1$. In this limit the wave function ψ_2 has the same characteristic scale, i.e, $l \sim 1/\Lambda_1$. Therefore in Eq. (6.18) we can neglect the derivatives and, as a result, obtain a local relation between ψ_1 and ψ_2:

$$(\Lambda_2^2 + k^2)\psi_2 = \psi_1^2.$$

Substitution of this expression into (6.17) leads to the stationary nonlinear Schrödinger equation (NLS):

$$-\Lambda_1^2\psi_1 + \partial_x^2\psi_1 + \frac{2}{\Lambda_2^2 + k^2}|\psi_1|^2\psi_1 = 0. \tag{6.21}$$

Its solution is the NLS soliton:

$$\psi_1 = \frac{\Lambda_1}{\sqrt{\Lambda_2^2 + k^2}}\,\mathrm{sech}(\Lambda_1 x). \tag{6.22}$$

In this case the second harmonic amplitude is given by the expression

$$\psi_2 = \frac{\Lambda_1^2}{(\Lambda_2^2 + k^2)^2}\,\mathrm{sech}^2(\Lambda_1 x).$$

In this asymptotic regime, the main contribution in the Manley–Rowe integral is given by the first harmonic (6.22):

$$N \approx \frac{2\Lambda_1}{\Lambda_2^2 + k^2}. \tag{6.23}$$

In Fig. 12 we show the dependence of the Manley–Rowe integral N on Λ_1 for the case $k = 0$ and $\Lambda_2^2 = 1$, obtained by numerical integration of the system (6.17), (6.18).

For small $\Lambda_1 N$ changes linearly in accordance with the analytical dependence (6.23). For larger Λ_1 we have a positive deviation from this linear dependence. In all our numerical work we also checked that for $\Lambda_1^2 < 0$, as well as for $\Lambda_1^2 < 0$, soliton solutions are absent, which is in complete agreement with the definition of the soliton region given by (6.13)–(6.15).

Thus, while approaching the boundary $\Lambda_1^2 = 0$ the first harmonic amplitude undergoes a supercritical bifurcation: $\max|\psi_1| \sim \Lambda_1$ and ψ_2 vanishes like Λ_1^2. For $\Lambda_1^2 < 0$ Eq. (6.21) has no localized solution at all. Note that this case is completely embedded in the general situation for this type of bifurcations of solitons (cf. [22,23]).

To conclude this subsection, we would like to point out that the reduction of the FF–SH system to the NLS was first obtained in the paper [25] at the case when the phase mismatch parameter is large enough. Later it was discussed in many other papers (see, for instance, the paper [99] and the review [100]).

6.4. Subcritical bifurcations

Consider now how solitons behave near the other boundary $\Lambda_2^2 = 0$ (when Λ_1^2 and k are not equal to zero). In the special case $k = 0$ the system (6.17), (6.18) takes the form:

$$-\Lambda_1^2\psi_1 + \partial_x^2\psi_1 = -2\psi_2\psi_1, \tag{6.24}$$

$$-\Lambda_2^2\psi_2 + \partial_x^2\psi_2 = -\psi_1^2. \tag{6.25}$$

Here without lose of generality we put $\psi_1 = \psi_1^*$ so that Eq. (6.24) transforms into the stationary Schrödinger equation for ψ_1 and $U(x) = 2\psi_2(x)$ there serves as a potential. The latter quantity is found from the second Eq. (6.25) by means of a Green's function:

$$\psi_2 = \frac{1}{2\Lambda_2}\int_{-\infty}^{\infty} e^{-\Lambda_2|x-x'|}\psi_1^2(x')dx'. \tag{6.26}$$

Thus one can see that $\psi_2(x)$ decreases exponentially for large x and the small parameter Λ_2 defines the largest scale $L = \Lambda_2^{-1}$ in this problem:

$$\psi_2 \approx \frac{e^{-\Lambda_2|x|}}{2\Lambda_2}\int_{-\infty}^{\infty}\psi_1^2(x')dx' \quad \text{for } |x| \sim L. \tag{6.27}$$

On the other hand, in the stationary Schrödinger equation (6.24) the value Λ_1^2 can be considered as the energy of the bound state and will yield the smallest scale $l = \Lambda_1^{-1} \ll L$ in this problem. Further, as we show below, the characteristic scale a of $\psi_1(x)$ lies between l and L:

$$L \gg a \gg l. \tag{6.28}$$

This allows one to neglect the second derivative term in (6.24) and so estimate the maximum value of ψ_2 (attained at $x = 0$):

$$\max(\psi_2) \approx \Lambda_1^2/2. \tag{6.29}$$

Comparing (6.27) and (6.29) we arrive at the following estimate for the integral

$$\int_{-\infty}^{\infty}\psi_1^2 dx \approx \Lambda_1^2\Lambda_2,$$

Thus, this integral vanishes as $\Lambda_2 \to 0$. At the same time the integral of ψ_2^2 becomes infinitely large: the maximum value remains constant and the characteristic scale becomes infinite as Λ_2 tends to zero. By this argument, the corresponding Manley–Rowe integral will diverge as

$$N \approx 2\int \psi_2^2 dx \approx \frac{\Lambda_1^4}{2\Lambda_2}. \tag{6.30}$$

The main contribution to this integral is given by the second harmonic, and correspondingly the contribution from the first harmonic is small.

Let us next consider the behavior of $\psi_1(x)$. To estimate its amplitude and to find its characteristic size we shall assume that

$$\psi_1(x) = A\phi_1(\xi), \qquad \text{where } \xi = \alpha x \quad \text{and} \quad \alpha = \frac{1}{a}. \tag{6.31}$$

Substituting (6.31) into the integral (6.26) and taking into account the relations between the scales (6.28) we find that

$$\psi_2 = \frac{A^2C_0}{2\alpha\Lambda_2}exp(-\Lambda_2|x|) + \frac{A^2}{\alpha^2}\phi_2(\xi), \quad \text{where } C_0 = \int_{-\infty}^{\infty}\phi_1^2(\xi)d\xi \quad \text{and} \quad \phi_{2\xi\xi} \approx -\phi_1^2.$$

For large $|x|(\sim L)$, this expression has the same asymptotic behavior as (6.27). For small $|x| \sim a$ the first term can be expanded so that the potential $(U = 2\psi_2)$ can be represented as follows

$$U(x) = U_0 + \frac{A^2}{\alpha^2}V(\xi),$$

where

$$U_0 = \frac{A^2C_0}{\alpha\Lambda_2}, \quad \text{and} \quad V(\xi) = \left[-|\xi| + 2\phi_2(\xi)\right].$$

Here U_0 gives the constant background and the potential $V(\xi)(\ll U_0)$ provides the bound state for ψ_1. In this case U_0 is approximately equal to Λ_1^2 which coincides with (6.29) for the maximum value of ψ_2. The small difference between Λ_1^2 and $2\max(\psi_2)$ is just α^2, so that:

$$\Lambda_1^2 = \frac{A^2C_0}{\alpha\Lambda_2} + \alpha^2, \quad \text{and} \quad A^2 = \alpha^4.$$

As a result, for these scales Eq. (6.24) takes the form:

$$-\phi_1 + \partial_\xi^2\phi_1 + V(\xi)\phi_1 = 0.$$

We now see that the scaling of the small parameters yields the relations:

$$\frac{A^2 C_0}{\alpha} \approx \Lambda_1^2\Lambda_2$$

from which it follows that:

$$C_0\alpha^3 \approx \Lambda_1^2\Lambda_2, \tag{6.32}$$

and the amplitude of ψ_1 is of the order:

$$A \approx \frac{\Lambda_1^{4/3}\Lambda_2^{2/3}}{C_0^{2/3}}.$$

Thus, we have shown that for the case $k = 0$ the amplitude of the second harmonic (SH) remains constant at the bifurcation point, but its size becomes infinitely large. In contrast to SH, the FF amplitude vanishes as $\Lambda_2^{2/3}$, but its size grows as $\Lambda_2^{-1/3}$. This means that we have in this case a subcritical bifurcation.

In spite of this behavior for ψ_1, the main contribution to the Manley–Rowe integral comes from the SH, since the input from the FF is small. This statement holds also at $k \neq 0$. To establish this fact it is not necessary to know the solution for ψ_1, it is enough to estimate contributions from the FF and SH in the Manley–Rowe integral.

At $k \neq 0$ it is possible to arrive almost at the same conclusion as for $k = 0$.

Assuming that the amplitude ψ_1 has a characteristic scale larger than Λ_1^{-1} and k^{-1}, one can get an estimate for ψ_2 at $x = 0$ (analogous to (6.29)):

$$\max|\psi_2| \simeq (\Lambda_1^2 + k^2/4)/2, \tag{6.33}$$

so that it remains finite at $\Lambda_2 = 0$. As in the case $k = 0$, the characteristic size of $|\psi_2|$ grows like Λ_2^{-1}. From this point of view, the type of bifurcation remains the same as it was at $k = 0$: this is the subcritical bifurcation. As far as ψ_1 is concerned, its contribution to the Manley–Rowe integral becomes infinitely small with respect to that from SH:

$$\int |\psi_1| dx \sim \Lambda_2.$$

This follows from comparison of Eq. (6.27) at $x = 0$ and Eq. (6.33). The latter indicates that the FF amplitude must vanish at $\Lambda_2 = 0$.

Thus, for both cases ($k = 0$ and $k \neq 0$) the Manley–Rowe integral diverges like $1/\Lambda_2$ for small Λ_2:

$$N \approx 2\int |\psi_2|^2 dx \sim 1/\Lambda_2.$$

Thus, the function $N(\Lambda_2)$ has a negative derivative near $\Lambda_2 = 0$, which, due to the VK type criterion (see the next section and [47]), corresponds to instability of solitons. At $k = 0$ it is possible to prove. For $k \neq 0$ it is an open problem, but it looks very reasonable to expect instability also in this case because the instability holds for small k.

In Fig. 13 we show the dependence of N on Λ_2 for the case $\Lambda_1^2 = \Lambda_2^2 + 1$ and $k = 0$, obtained from our numerical solutions. It was found that such behavior of N with respect to Λ_2 is typical for all cases including $k = 0$: for small values of $\Lambda_2 N$ diverges, for intermediate values it has a minimum at $\Lambda_2 = \Lambda_{2min}$ and grows for larger Λ_2. When $N \to \infty$ as $\Lambda_2 \to 0$, the dependence $N(\Lambda_2)$ can be approximated by the function

$$f(\Lambda_2) = \frac{a}{\Lambda_2} + b \tag{6.34}$$

where for $\Lambda_1^2 = 1 + \Lambda_2^2$ and $k = 0$, $a = 0.537,\ b = 3.33$. So we can see a nice correspondence with theoretical formula (6.30) where $a = 1/2$.

In Fig. 14 we show the dependence of $N(\Lambda_2)$ for this case, where the dashed line corresponds to the approximation (6.34). The same dependence $N(\Lambda_2)$ is displayed on Fig. 15for $k = 1$ with the same value of $\Lambda_1 = 1$ (in this case $a = 0.763$ and $b = 4.299$).

As $\Lambda_2 \to 0$ our numerics demonstrate that the maximum amplitude of the second harmonic tends to a constant (Fig. 16) but its width grows (Fig. 17) which causes the divergence of N for small Λ_2. Simultaneously the first harmonic amplitude vanishes and its width increases slightly as Λ_2 approaches zero (see also Fig. 18). We note that such a tendency was also observed numerically in [101] for the FF–SH solitons in one particular case, but the authors did not give any explanation of this fact. As a result, the contribution of the first harmonic to N becomes small compared with that from the second harmonic. Thus, the function $N(\Lambda_2)$ for the soliton solution has a negative derivative in the band $0 < \Lambda_2 < \Lambda_{2min}$ where, according to the VK type criterion [47], the considered soliton solutions should be expected to be unstable against small perturbations. It

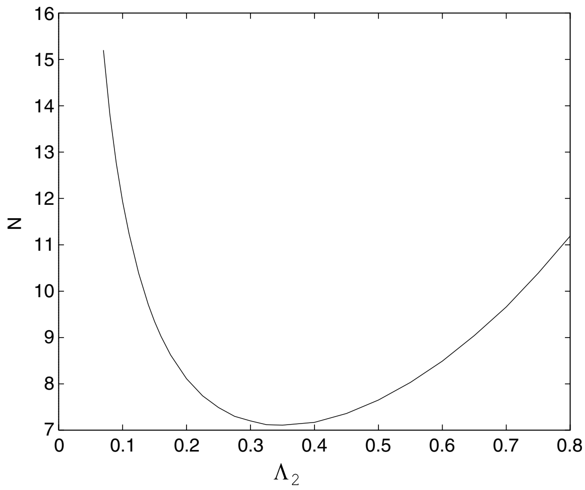

Fig. 13. The dependence N on Λ_2 for the case $\Lambda_1^2 = \Lambda_2^2 + 1$ and $k = 1$. $\Lambda_2 = 0$ corresponds to the subcritical bifurcation point.

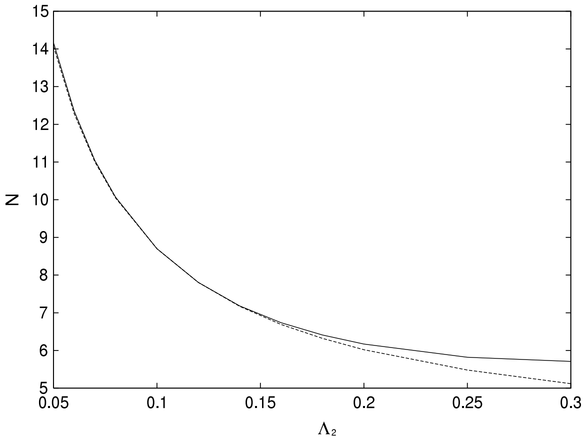

Fig. 14. The curve $N(\Lambda_2)$ for $\Lambda_1^2 = \Lambda_2^2 + 1$ and $k = 0$ near the subcritical bifurcation boundary ($\Lambda_2 = 0$); the dashed line displays the approximation $f(\Lambda_2)$ with $a = 0.537$ and $b = 3.333$.

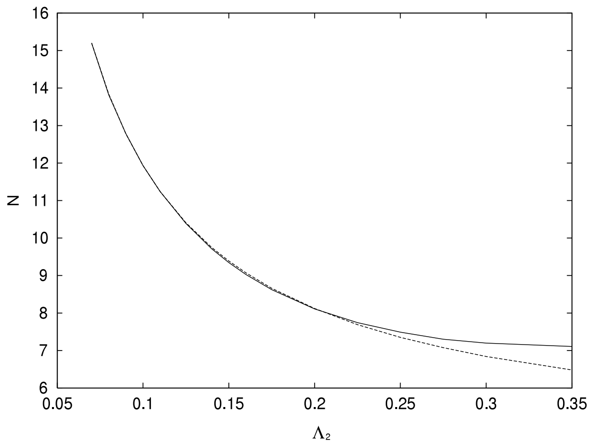

Fig. 15. The curve $N(\Lambda_2)$ for $\Lambda_1 = 1$ and $k = 1$ near the subcritical bifurcation boundary ($\Lambda_2 = 0$); the dashed line corresponds to the approximation $f(\Lambda_2)$ with $a = 0.7630$ and $b = 4.299$.

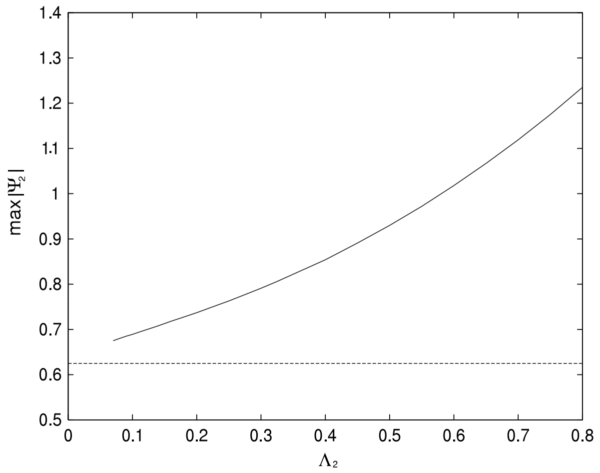

Fig. 16. The maximum value of $|\psi_2|$ near the subcritical bifurcation boundary versus Λ_2 ($\Lambda_1^2 = \Lambda_2^2 + 1$ and $k = 1$).

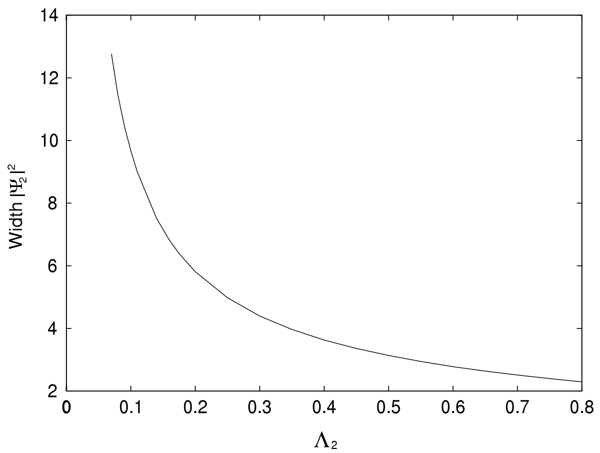

Fig. 17. The width of $|\psi_2|^2$ near the subcritical bifurcation boundary versus Λ_2 ($\Lambda_1^2 = \Lambda_2^2 + 1$ and $k = 1$).

can be proved rigorously for $k = 0$. For $\Lambda_2 > \Lambda_{2min}$ the sign of the derivative changes and therefore we should expect a stable soliton branch. However, the VK criterion cannot be applied to this case. It is connected with very fine details in the proof for the VK type criteria (for details see [47]). For the NLS case the proof is based on use of the oscillation theorem for scalar Schrödinger operators, which appear after linearization of the NLS on the background of the soliton. However, for the linear stability problem for the FF–SH solitons, instead of scalar operators, we have a matrix (2×2) Schrödinger operators for which the oscillation theorem does not hold. Recall that the oscillation theorem for the scalar Schrödinger operator establishes a correspondence between a number of nodes of the wave function and a level number. Therefore the VK type criterion, being a sufficient criterion for instability, cannot be used for stability. Thus we can expect instability for the region of the negative derivative of $N(\Lambda_2^2)$. In the region of positive derivative, nevertheless, one can make a conclusion about stability by using a combination of the (incomplete) VK criterion and the Lyapunov approach. As first shown in [26] (for details see also [47]) the Hamiltonian of the FF–SH system is bounded from below for a fixed Manley–Rowe integral. A key point for stability is to consider the dependence $H(N, P)$ for soliton solutions. If this dependence is monotonic and unique then the soliton solution will be stable. Indeed, the boundedness of the Hamiltonian from below means that solutions realizing its minimum will be stable in accordance with the Lyapunov theorem. In the case of a unique surface $H(N, P)$ this minimizer will belong to this surface. Strictly speaking, the latter needs also compactness of the considered functionals, which can be proved by standard methods for such systems. If the function $H(N, P)$ is not monotonic, then there exist several branches for fixed N and P, i.e. $H = H(N, P)$ represents a set of separate surfaces. Then the solitons from the lower branch, or from the lower surface, will be stable only in the Lyapunov sense.

As for the stability of solitons near a supercritical bifurcation, this problem has some peculiarities. As shown before the soliton family near the boundary $\Lambda_1 = 0$ transforms into NLS solitons, generally speaking, independently of the problem dimension. In this case the most dangerous perturbations will be disturbances of the modulation type. Their dynamics will be described by the time-dependent nonlinear Schrödinger equation. It we saw above (see, e.f., [38]) that only

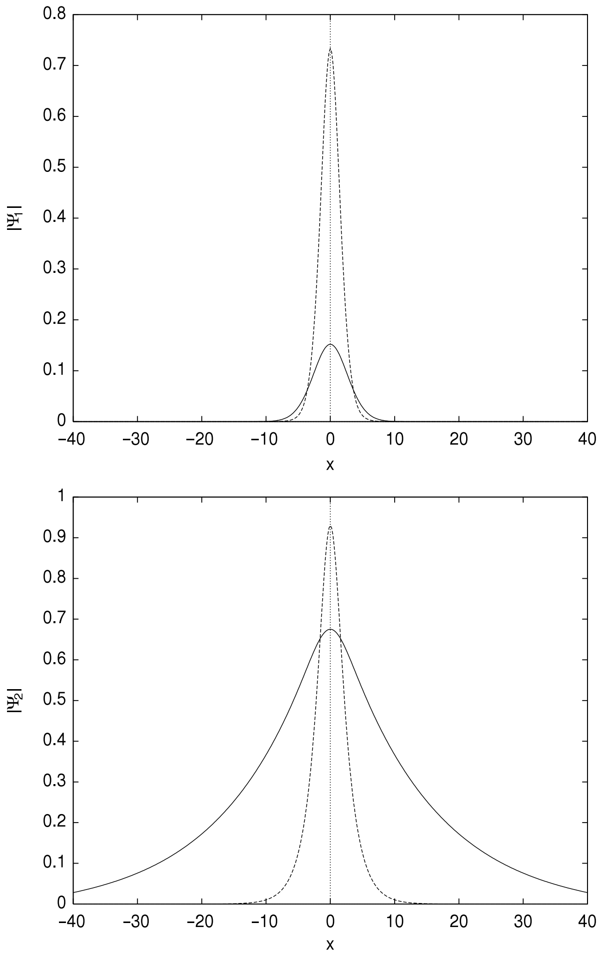

Fig. 18. The functions $|\psi_1(x)|$ and $|\psi_2(x)|$ near the subcritical bifurcation boundary ($\Lambda_1^2 = \Lambda_2^2 + 1 \text{and} k = 1$). Solid line is $\Lambda_2 = 0.1$ and the dashed line $\Lambda_2 = 0.5$.

one-dimensional solitons are stable with respect to perturbations with the same dimension. For instance, the solutions (6.22) will be stable with respect to one-dimensional modulation perturbations, guaranteed, in particular, by the criterion (4.33). Thus, in the one-dimensional case the soliton boundary in the parameter space coincides with the stability boundary. However, for dimensions $D \geq 2$ the NLS solitons are unstable (see Section 4). From another perspective, the boundedness of the Hamiltonian for fixed N provides stability of solitons realizing the minimum of H. This means that for $D \geq 2$, in the parameter space the stability region is separated from the soliton boundary.

6.5. Concluding remarks

It is necessary to emphasize that the results presented in this section about the behavior of solitons near both the supercritical and subcritical bifurcations are in complete agreement with a general theory given in Sections 4 and 5. It is interesting to note that if the phase mismatch parameter Ω is negative for stationary solitons ($V = 0$) only the supercritical bifurcation is possible; the subcritical bifurcation is forbidden. For propagating solitons we have another possibility. For instance, if

$$\left(1 - \frac{\kappa}{2}\right) V^2 + \Omega < 0$$

which can be satisfied if

$$\kappa < 0, \quad \text{or} \quad p_2 > 2p_1 \tag{6.35}$$

and then necessarily $\Omega < 0$, the only possible bifurcation is the subcritical one. Note, that equality in (6.35) recovers the Galilean invariance of the FF–SH system. Thus, the soliton family significantly depends on the phase mismatch Ω and κ (i.e. $p_2 - 2p_1$) with a large asymmetry with respect to these parameters. The latter will be interesting to observe in experimentally, for instance, in nonlinear optics.

We would like to underline once more that the results of this section can be easily extended to the multi-dimensional case. In particular, the main conclusion about the character of solitons near subcritical and supercritical bifurcations holds in the multi-dimensional case. For a dimension $D \geq 2$ solitons near a supercritical bifurcation for the FF amplitude will coincide with D-dimensional NLS solitons, and the SH amplitude will be proportional to square of the FF amplitude as it is in 1D case. When approaching a subcritical bifurcation point in the multi-dimensional case, the Manley–Rowe integral diverges which results in instability of solitons.

7. Stability of the NLS type solitons

The main aim of this section is to demonstrate how two methods discussed in the previous sections, i.e. the Lyapunov approach and linear stability analysis, can be applied to investigating the stability of solitons in the three-wave system.

As was mentioned, the three-wave system describes spatial solitons as well as spatial–temporal solitons in $\chi^{(2)}$ media [24,25]. This system couples amplitudes of three quasi-monochromatic waves due to quadratic nonlinearity. As we saw in the previous section in a special case this system describes the interaction of fundamental and second harmonics. When the difference in group velocities of three wave packets is large enough this system coincides with the Bloembergen equations [24] which can be integrated by the inverse scattering transform [98,12]. For close group velocities in the system one needs to take into account both dispersion and diffraction terms [25,26]. In this case this model can be considered as a vector NLS system but with quadratic nonlinearity. Solitons in this system are possible as a result of a balance between nonlinear interaction and dispersive effects. For this system soliton solutions are stationary points of the Hamiltonian, for fixed momentum and the Manley–Rowe integrals. In this section we give the proof of the Lyapunov stability of such ground solitons based on the Sobolev embedding theorem. This proof [47] generalizes all results about application of this approach to the three-wave system starting from the first results obtained by Kanashev and Rubenchik [26] and those obtained later by Turitsyn [102] and Berge et al. [103]. The linear stability criterion given in this section is also the generalization but of the VK criterion. We would like to remind that the crucial point in its derivation for the NLSE solitons is based on the oscillation theorem for the stationary Schrödinger operator. This theorem establishes the one-to-one correspondence between a level number and a number of nodes of the eigenfunction. As well known, this theorem is valid only for scalar (one-component) Schrödinger operators and cannot be extended, for example, to the analogous matrix operators. This means that the Vakhitov–Kolokolov type of criteria, as a rule, define only sufficient conditions for soliton instability and cannot necessarily determine the stability of solitons. For three-wave system the linearized operator represents a product of two (3×3)-matrix Schrödinger operators to which the oscillation theorem cannot be applied. We discuss this situation in detail for solitons describing a bound state of the fundamental frequency and its second harmonics.

7.1. The three-wave system

Consider the three-wave system, written in the form (see, for instance, [26,33]):

$$i\frac{\partial \psi_1}{\partial t} - \omega_1\psi_1 + i(\mathbf{v}_1\nabla)\psi_1 + \frac{1}{2}\omega_1^{\alpha\beta}\partial^2_{\alpha\beta}\psi_1 = V\psi_2\psi_3, \tag{7.1}$$

$$i\frac{\partial \psi_2}{\partial t} - \omega_2\psi_2 + i(\mathbf{v}_2\nabla)\psi_2 + \frac{1}{2}\omega_2^{\alpha\beta}\partial^2_{\alpha\beta}\psi_2 = V\psi_1\psi_3^*, \tag{7.2}$$

$$i\frac{\partial \psi_3}{\partial t} - \omega_3\psi_3 + i(\mathbf{v}_3\nabla)\psi_3 + \frac{1}{2}\omega_3^{\alpha\beta}\partial^2_{\alpha\beta}\psi_3 = V\psi_1\psi_2^*. \tag{7.3}$$

Here $\psi_l(\mathbf{x}, t)$ $(l = 1, 2, 3)$ are amplitudes of three wave packets, slowly varying with respect to $\mathbf{x}$, so that values of their wave vectors $\mathbf{k}_l$ satisfy restrictions $\mathbf{k}_l L_l \gg 1$ where L_l is a characteristic size of lth packet. The frequencies $\omega_l = \omega_l(\mathbf{k}_l)$ are supposed to be close to the resonance condition:

$$\omega_1(\mathbf{k}_1) = \omega_2(\mathbf{k}_2) + \omega_3(\mathbf{k}_3), \tag{7.4}$$

$$\mathbf{k}_1 = \mathbf{k}_2 + \mathbf{k}_3. \tag{7.5}$$

$\mathbf{v}_l = \partial\omega_l(\mathbf{k}_l)/\partial\mathbf{k}_l$ is the group velocity for the lth packet,

$$\omega_l^{\alpha\beta} = \frac{\partial^2\omega_l(\mathbf{k}_l)}{\partial k_{l\alpha}\partial k_{l\beta}}$$

is the dispersion tensor, and V is a three-wave matrix element which can be taken as real, without loss of generality. Like the NLSE, this system is Hamiltonian

$$i\frac{\partial \psi_l}{\partial t} = \frac{\delta H}{\delta \psi_l^*},$$

where

$$H = H_0 + H_1,$$

$$H_0 = \sum_{l=1}^{3} \left[\int \omega_l |\psi_l|^2 d\mathbf{r} - i \int \psi_l^*(v_l \nabla)\psi_l d\mathbf{r} + \frac{1}{2} \int \nabla_\alpha \psi_l^* \omega_l^{\alpha\beta} \nabla_\beta \psi_l d\mathbf{r} \right], \tag{7.6}$$

$$H_1 = V \int (\psi_1 \psi_2^* \psi_3^* + \psi_1^* \psi_2 \psi_3) d\mathbf{r}. \tag{7.7}$$

Here the Hamiltonian H_0 relates to the linear characteristics of the three wave packets: the first term in (7.6) is the main contribution to their energy, the next term describes propagation of the wave packets with the group velocities $\mathbf{v_l}$, while the last term accounts for the effect of the finite width Δk_l of each wave packet, responsible for their dispersion broadening. It is worth to noting that the dispersion term is small compared with respect to the second one through the parameter $\Delta k_l / k_l$. Therefore it is necessary to take this term into account only if the difference in the group velocities is small enough.

If the carrier frequencies ω_l and wave vectors $\mathbf{k}_l$ satisfy exactly the resonant conditions (7.4) then the terms proportional to ω_l can be excluded by the transformation: $\psi_l \to \psi_l \exp(-i\omega_l t)$.

In the 1D case the general system (7.1)–(7.3) can be simplified. By introducing new variables

$$\psi_l = \tilde{\psi}_l(x - vt, t) e^{i\kappa_l x}; \qquad \kappa_1 = \kappa_2 + \kappa_3, \tag{7.8}$$

one can exclude the first derivatives in the equations if one chooses the velocity v and wave numbers κ_l as follows[2]

$$v = \frac{v_1 d_1 - v_2 d_2 - v_3 d_3}{d_1 - d_2 - d_3}, \qquad \kappa_l = d_l (v - v_l), \quad d_l = \frac{1}{\omega_l''}. \tag{7.9}$$

As a result, in the 1D case the system (7.1)–(7.3) takes the form (tilde above ψ_l is omitted and V is put equal to -1):

$$i \frac{\partial \psi_1}{\partial t} - \Omega_1 \psi_1 + \frac{1}{2} \omega_1'' \psi_{1xx} = -\psi_2 \psi_3, \tag{7.10}$$

$$i \frac{\partial \psi_2}{\partial t} - \Omega_2 \psi_2 + \frac{1}{2} \omega_2'' \psi_{2xx} = -\psi_1 \psi_3^*, \tag{7.11}$$

$$i \frac{\partial \psi_3}{\partial t} - \Omega_3 \psi_3 + \frac{1}{2} \omega_3'' \psi_{3xx} = -\psi_1 \psi_2^*. \tag{7.12}$$

where

$$\Omega_l = \omega_l + \kappa_l v_l + \omega_l'' \kappa_l^2 / 2. \tag{7.13}$$

Here, as before, the new frequencies are close to resonance:

$$\Omega_1 \approx \Omega_2 + \Omega_3$$

or, in other words, the mismatch is assumed to be small enough.

At the next step Ω_2 and Ω_3 are excluded with the help of the transformation:

$$\psi_1(x, t) \to \psi_1(x, t) e^{-i(\Omega_2 + \Omega_3)t}, \qquad \psi_2(x, t) \to \psi_2(x, t) e^{-i\Omega_2 t}, \qquad \psi_3(x, t) \to \psi_3(x, t) e^{-i\Omega_3 t}.$$

As a result, the Eqs. (7.10)–(7.12) take the form

$$i \frac{\partial \psi_1}{\partial t} - \Omega \psi_1 + \frac{1}{2} \omega_1'' \psi_{1xx} = -\psi_2 \psi_3, \tag{7.14}$$

$$i \frac{\partial \psi_2}{\partial t} + \frac{1}{2} \omega_2'' \psi_{2xx} = -\psi_1 \psi_3^*, \tag{7.15}$$

$$i \frac{\partial \psi_3}{\partial t} + \frac{1}{2} \omega_3'' \psi_{3xx} = -\psi_1 \psi_2^*. \tag{7.16}$$

where $\Omega = \Omega_1 - \Omega_2 - \Omega_3$ is the phase mismatch characterizing how far the carrying frequencies of the three wave packets are from the resonance (7.4).

The resulting system (7.14)–(7.16) keeps a Hamiltonian structure. A new Hamiltonian is of the form

$$H = \int \Omega |\psi_1|^2 dx + \sum_l \int \frac{1}{2} \omega_l'' |\psi_{lx}|^2 dx - \int (\psi_1^* \psi_2 \psi_3 + c.c.) dx. \tag{7.17}$$

2 When $d_1 = d_2 + d_3$ the equations become invariant relative to the Galilean transformations.

Besides H, the system (as well as the old one) has two other conservative quantities, the so-called Manley–Row integrals,

$$N_1 = \int (|\psi_1|^2 + |\psi_2|^2)dx, \qquad N_2 = \int (|\psi_1|^2 + |\psi_3|^2)dx. \tag{7.18}$$

These invariants appear as a sequence of the averaging procedure excluding all other non-resonant processes except three-wave interaction.

The three-wave system (7.14)–(7.16) in the partial case of the interaction of fundamental and second harmonics transforms into the system (6.2), (6.3) and, respectively, the Hamiltonian (6.5) transforms into (7.17)

For an arbitrary dimension d all the above transformations for system (7.10)–(7.12) can be performed. In this case $\omega_l^{''}\psi_{lxx}$ for both systems transforms into $\omega_l^{\alpha\beta}\partial^2_{\alpha\beta}\psi_l$, with a familiar change in all integrals of motion. In the transformations (7.9) instead of d_l it is necessary to substitute the matrix inverse to $\omega_l^{\alpha\beta}$, the velocities v_l there have meaning of vectors.

These changes allow us to restrict our consideration to the one-dimensional three-wave system only. Some difference between one-dimensional and multi-dimensional cases is not too basic for this system, at least as far as soliton stability concerns.

7.2. Soliton solutions of the three-wave system

Consider soliton-like solutions of the system (7.14)–(7.16) in the form,

$$\begin{aligned} \psi_1(x,t) &= \psi_{1s}(x)e^{i(\lambda_1+\lambda_2)t}, \\ \psi_2(x,t) &= \psi_{2s}(x)e^{i\lambda_1 t}, \\ \psi_3(x,t) &= \psi_{3s}(x)e^{-i\lambda_2 t}, \end{aligned} \tag{7.19}$$

where functions $\psi_{1s}(x)$, $\psi_{2s}(x)$, $\psi_{3s}(x)$ are supposed to be real and decay at the infinity. They obey the equations:

$$L_1\psi_{1s} = -\psi_{2s}\psi_{3s}, \qquad L_1 = -(\lambda_1+\lambda_2+\Omega) + \frac{1}{2}\omega_1^{''}\partial_x^2; \tag{7.20}$$

$$L_2\psi_{2s} = -\psi_{1s}\psi_{3s}, \qquad L_2 = -\lambda_1 + \frac{1}{2}\omega_2^{''}\partial_x^2; \tag{7.21}$$

$$L_3\psi_{3s} = \psi_{1s}\psi_{2s}, \qquad L_2 = -\lambda_2 + \frac{1}{2}\omega_3^{''}\partial_x^2. \tag{7.22}$$

The solutions to this system yield extremums of the Hamiltonian for two fixed Manley–Row invariants:

$$\delta(H + \lambda_1 N_1 + \lambda_2 N_2) = 0, \tag{7.23}$$

that is soliton solutions are stationary points of the Hamiltonian for fixed $N_{1,2}$.

Soliton solutions of this system will be exponentially decreasing at the infinity if simultaneously three inequalities are satisfied:

$$\mu_1^2 \equiv d_1(\lambda_1+\lambda_2+\Omega) > 0, \qquad \mu_2^2 \equiv d_2\lambda_1 > 0, \qquad \mu_3^2 \equiv d_3\lambda_2 > 0. \tag{7.24}$$

where $d_l = 1/\omega_l^{''}$.

It is possible to come to the this result also if one analyzes integrals consisting of the Hamiltonian and $N_{1,2}$ on the soliton solution (for more details see the review [104]). For this purpose multiply Eq. (7.20) by ψ_{1s} and then integrate over x. As a result, one gets

$$-(\lambda_1+\lambda_2+\Omega)n_1 + D_1 = -I. \tag{7.25}$$

An analogous procedure applied to Eqs. (7.21) and (7.22) gives

$$-\lambda_1 n_2 + D_2 = -I, \tag{7.26}$$

$$-\lambda_2 n_3 + D_3 = -I \tag{7.27}$$

where

$$n_l = \int |\psi_{ls}|^2 dx, \qquad D_l = \frac{1}{2}\int \omega_l^{''}\left|\frac{d\psi_{ls}}{dx}\right|^2 dx, \qquad I = \int \psi_{1s}\psi_{2s}\psi_{3s}dx.$$

These integral relations should be completed by the condition following from the variational problem (7.23) after applying scaling transformation to be remaining $N_{1,2}$:

$$\psi_{ls}(x) \to a^{-1/2}\psi_{ls}(x/a).$$

Under this transformation the Hamiltonian becomes a function of the scaling parameter a:

$$H(a)=\sum_l\int\left(\Omega_l|\psi_{ls}|^2+\frac{1}{2a^2}\omega_l''|\psi_{lsx}|^2\right)dx-\frac{2}{a^{1/2}}\int\psi_{1s}\psi_{2s}\psi_{3s}dx.$$

By using both this relation and Eq. (7.23), one gets

$$\left.\frac{dH}{da}\right|_{a=1}=0\quad\text{or}\quad\sum_l\int\omega_l''|\psi_{lsx}|^2dx-\int\psi_{1s}\psi_{2s}\psi_{3s}dx=0. \tag{7.28}$$

Combining the integral relations (7.25)–(7.28) and using the positive nature of n_l (by definition) we get to the conditions (7.24). Thus, it turns out that all operators L_l must simultaneously have the same sign definiteness. Note, that this latter requirement holds for all physical dimensions $D\le 3$, and so the matrices $\omega_l^{\alpha\beta}$ must also have the same sign definiteness.

7.3. Nonlinear stability

Next we demonstrate how Lyapunov stability can be established for solitons in this three-wave system. As for the NLS solitons, for demonstration of the Hamiltonian boundedness we used the embedding theorems. The difference with the NLSE case, consists, first of all, in that, instead of one complex field ψ, now we have three fields ψ_1, ψ_2 and ψ_3, while, instead of cubic nonlinearity, we have a quadratic one. This means that one should consider two spaces $L_{3,3}$ and W_2^1 with the norms

$$\|u\|_{L_{3,3}}=\left[\int(|\psi_1|^3+|\psi_2|^3+|\psi_3|^3)d^Dx\right]^{1/3},$$

$$\|u\|_{W_2^1}=\left[\tilde{\lambda}_1\int(|\psi_1|^2+|\psi_2|^2)d^Dx+\tilde{\lambda}_2\int(|\psi_1|^2+|\psi_3|^2)d^Dx+\frac{1}{2}\sum_l\int\partial_\alpha\psi_l^*\omega_l^{\alpha\beta}\partial_\beta\psi_ld^Dx\right]^{1/2}$$

where $\tilde{\lambda}_{1,2}>0$ and the tensors $\omega_l^{\alpha\beta}$ are assumed positive definite. Then the Sobolev inequality reads as follows:

$$\|u\|_{L_{3,3}}<M\|u\|_{W_2^1}. \tag{7.29}$$

It is worth noting that between the norm $\|u\|_{L_{3,3}}$ and the interaction Hamiltonian for the three-wave system, here is the following simple inequality:

$$\|u\|_{L_{3,3}}^3\ge\frac{3}{2}\int(\psi_1^*\psi_2\psi_3+c.c.)d^Dx. \tag{7.30}$$

A multiplicative variant for the Sobolev inequality (7.29) can be obtained by the same manner as for the NLSE. We present here only the analog of (6.28) for the one-dimensional case:

$$J\le M_1[\tilde{\lambda}_1N_1+\tilde{\lambda}_2N_2]^{5/4}I^{1/4}$$

where

$$J=\int(\psi_1^*\psi_2\psi_3+c.c.)d^Dx,\qquad I=\frac{1}{2}\sum_l\int\partial\psi_l^*\omega_l''\partial\psi_ldx.$$

In this expression $\tilde{\lambda}_{1,2}$ are still arbitrary positive constants. By minimization over both these parameters we have

$$J\le C(N_1N_2)^{5/8}I^{1/4}. \tag{7.31}$$

The next step is to find the best constant C as a minimal value of the corresponding functional

$$C_{best}=\min_{[\psi]}F[\psi],\qquad F=\frac{J}{(N_1N_2)^{5/8}I^{1/4}}.$$

It is easy to check that this minimum is attained on the ground soliton solution, namely, on the solution of the system (7.20)–(7.22) without nodes[3]:

$$C_{best}=F[\psi_s]. \tag{7.32}$$

Analogously to (2.31), with the help of the inequality (7.31) where instead of C stands the best constant (7.32), it is easy to get the estimate for the Hamiltonian (7.17) for the case when the phase mismatch is absent ($\Omega=0$):

$$H\ge I-2I_s^{3/4}I^{1/4}\ge H_s(\Omega=0).$$

[3] It is possible to show that for the ground soliton all functions ψ_{ls} can be considered as positive quantities.

This proves stability of the ground-state soliton in one dimension for a zero phase mismatch. Again, this inequality becomes precise for the soliton solution. By the same scheme stability of the ground-state solitons for all other physical dimensions can be proved.

Now consider how a finite phase mismatch effects soliton stability. The answer which will be obtained is that the Hamiltonian is always bounded from below, independently of both the value and the sign of Ω. Consider the Hamiltonian (7.17) in which it is convenient to separate the phase mismatch term:

$$H = \Omega \int |\psi_1|^2 dx + \tilde{H}$$

so that the remainder coincides with the Hamiltonian with $\Omega = 0$:

$$\tilde{H} = \sum_l \int \frac{1}{2}\omega_l'' |\psi_{lx}|^2 dx - \int (\psi_1^* \psi_2 \psi_3 + c.c.) dx.$$

There are two possibilities: $\Omega > 0$ and $\Omega < 0$. In the first case we have the following evident estimate:

$$H \geq H_s(\Omega = 0). \tag{7.33}$$

Thus, the Hamiltonian is bounded from below by its value on the ground soliton solution with zero phase mismatch.

The second case gives

$$H \geq H_s(\Omega = 0) - |\Omega| \int |\psi_1|^2 dx. \tag{7.34}$$

The integral $\int |\psi_1|^2 dx$ is always bounded from above by $\min(N_1, N_2)$. Therefore the estimate (7.34) can be written as follows

$$H \geq H_s(\Omega = 0) - |\Omega| \min(N_1, N_2). \tag{7.35}$$

The resulting inequality completes the proof of Hamiltonian boundedness for an arbitrary phase mismatch. Importantly, this result takes place for all physical dimensions $D \leq 3$. In the general situation the Hamiltonian is majorized by its value with zeroth phase mismatch taken on the ground soliton solution and some additional term proportional to the minimal Manley–Row integral. The constant of proportionality is 0 or $|\Omega|$.

The fact of the Hamiltonian boundedness for the three-wave system (7.1)–(7.3) was first demonstrated by Kanashev and Rubenchik [26] when the dispersive operators were there for an isotropic media. Later, Turitsyn [102] showed for the partial case – interaction between fundamental frequency (FF) and second harmonics (SH) – under assumption that the second-order operators in the system are Laplacians in each equation that for zero phase mismatch the Hamiltonian reaches its minimum at the ground-state soliton. Then, in the paper [103] the boundedness of the Hamiltonian for the FF–SH interaction was demonstrated in the presence of a nonzero phase mismatch. Here we gave the stability proof for ground solitons in the three-wave system in the general case for arbitrary dispersion tensors $\omega_l^{\alpha\beta}$ having the same sign definiteness, for instance, being positive. It should be noted also that both inequalities (7.33) and (7.35) demonstrate different criteria at $\Omega > 0$ and $\Omega < 0$, in the correspondence with the conditions (7.24) for soliton existence which are not symmetric under the change $\Omega \to -\Omega$.

7.4. Linear stability for the system with FF–SH interaction

Let us turn to the linear stability problem for the solitons in the three-wave system. For the sake of simplicity we shall consider the partial case of the three-wave system, namely, the FF–SH interacting system (6.2), (6.3) and show how the VK type criterion can be obtained in this case.

Soliton solutions for the FF–SH interacting system are defined from the Eqs. (6.2), (6.3). They are of the form

$$\psi_1(x,t) = \psi_{1s}(x) e^{i\lambda^2 t}, \qquad \psi_2(x,t) = \psi_{2s}(x) e^{2i\lambda^2 t},$$

where the amplitudes ψ_{1s} and ψ_{2s} satisfy the equations:

$$-\lambda^2 \psi_1 + \frac{1}{2}\omega_1'' \psi_1 = -2\psi_2 \psi_1, \tag{7.36}$$

$$-2\lambda^2 \psi_2 - \Omega \psi_2 + \frac{1}{2}\omega_2'' \partial_x^2 \psi_2 = -\psi_1^2. \tag{7.37}$$

Here the solution ψ_{1s} and ψ_{2s} are assumed to be real and without nodes, i.e. they can be regarded as a ground-state soliton, where the index s for ψ_{1s} and ψ_{2s} is temporarily omitted.

Consider small perturbations on the background of this soliton solution, putting

$$\psi_1(x,t) = (\psi_{1s} + u_1 + iv_1) e^{i\lambda^2 t}, \qquad \psi_2(x,t) = (\psi_{2s} + u_2 + iv_2) e^{2i\lambda^2 t}.$$

Linearization of the system (6.2), (6.3) leads to the linear (Hamiltonian) equations:

$$u_t = \frac{1}{2}\frac{\delta \tilde{H}}{\delta v}, \qquad v_t = -\frac{1}{2}\frac{\delta \tilde{H}}{\delta u} \tag{7.38}$$

where $\tilde{H}$ is also the second variation of $F = H + \lambda^2 N$,

$$\tilde{H} = \langle v|L_0|v\rangle + \langle u|L_1|u\rangle \,, \tag{7.39}$$

u and v are vectors with two components u_1, u_2 and v_1, v_2, respectively, and N is given by the expression (6.6). Now the second-order differential operators L_0 and L_1 are the (2×2)-matrix operators:

$$L_{0,1} = \begin{bmatrix} \lambda^2 - \frac{1}{2}\omega_1''\partial_x^2 \mp 2\psi_2 & -2\psi_1 \\ -2\psi_1 & 2\lambda^2 - \frac{1}{2}\omega_2''\partial_x^2 - \Omega \end{bmatrix}.$$

Both operators remain self-adjoint. From the quantum mechanics point of view such operators correspond to the Schrödinger operators for a nonrelativistic particle with spin $S = 1/2$ moving in an inhomogeneous magnetic field. For such operators, as known, the oscillation theorem is not completely valid. The ground eigenfunction has no node, but correspondence between node number and level number is already absent. The lack of such correspondence, as will be seen below, does not allow us to make certain conclusions about soliton stability. The VK type criteria for this matrix system can give only sufficient conditions for instability. The same statement holds for the three-wave system.

As far as properties of the operators L_0 and L_1 are concerned, they are similar to those for the NLSE case. The operator L_0 is nonnegative. This follows if instead of v_1 and v_2 one introduces new functions χ_1 and χ_2 by means of the formulas

$$v_1 = \psi_1\chi_1, \qquad v_2 = \psi_2\chi_2.$$

As a result of such changes $\langle v|L_0|v\rangle$ can be rewritten as follows:

$$\langle v|L_0|v\rangle = \frac{1}{2}\int [\omega_1''\psi_1^2\chi_1^2 + \omega_2''\psi_2^2\chi_2^2]dx + \int \psi_1^2\psi_2(2\chi_1 - \chi_2)^2 dx.$$

Hence the nonnegativeness of L_0 becomes evident and the ground-state eigenvector is simply defined:

$$\chi_1 = c_1, \qquad \chi_2 = c_2, \qquad 2c_1 = c_2$$

or

$$v_0 = \begin{bmatrix} \psi_1 \\ 2\psi_2 \end{bmatrix}.$$

This eigenfunction, as for the NLSE case, provides conservation of N and consequently

$$\delta N = 2\int (\psi_1 u_1 + 2\psi_2 u_2)dx \equiv 2\,\langle v_0|u\rangle = 0. \tag{7.40}$$

As for the NLSE this relation represents a solvability condition for the linear system (7.38).

The next analysis is closely analogous to that for the NLSE. It is necessary to consider the eigenvalue problem for the operator L_1:

$$L_1|\varphi\rangle = E|\varphi\rangle + C|\psi_0\rangle \tag{7.41}$$

and then to expand $|\varphi\rangle$ over the complete set of eigenfunctions $\{|\varphi_n\rangle\}$ of the operator L_1. As a result, the solution of Eq. (7.41) is given by the expression:

$$|\phi\rangle = C\sum_n \frac{|\phi_n\rangle\langle\phi_n|v_0\rangle}{E_n - E}. \tag{7.42}$$

Analogously to (4.32), applying the solvability condition (7.40)–(7.42) leads us to the dispersion relation

$$f(E) \equiv {\sum_n}' \frac{\langle v_o|\phi_n\rangle\,\langle\phi_n|v_0\rangle}{E_n - E} = 0. \tag{7.43}$$

A prime here means, as before, absence in the sum of the state with $E = 0$ because $\langle\Psi_x|v_0\rangle = 0$, $L_1|\Psi_x\rangle = 0$ where

$$\langle\Psi_x| = (\psi_{1x}, \psi_{2x}).$$

Up to this point, everything looks similar to the NLSE case. The difference appears when we begin to analyze the function $f(E)$. Now the oscillation theorem does not hold. This means that below the level $E = 0$ a few levels are possible. Therefore the dispersion relation may have negative roots $E < 0$ independently on whether the derivative $\partial N_s/\partial\lambda^2$ is positive or negative. As a result, we can formulate only sufficient criterion for instability which has the same form as for the NLS solitons:

$$\frac{\partial N_s}{\partial \lambda^2} < 0. \tag{7.44}$$

But we cannot say anything in general about stability. The stability criterion

$$\frac{\partial N_s}{\partial \lambda^2} > 0$$

holds only if below $E_0 = 0$ the operator L_1 has the only (ground) level, but it is not a generic case. Thus, the Vakhitov–Kolokolov type of stability criteria when applied to the vector models provides only sufficient conditions for instability of solitons.

Nevertheless, a combination of the (incomplete) Vakhitov–Kolokolov criterion and the Lyapunov approach can give a complete answer to the stability problem. At the end of this section we discuss such an example when based on this combined method, and with the help of numerical integration of the FF–SH interacting system (7.36), (7.37), it is possible to make more or less certain conclusion about soliton stability. The dependences of H and N (on 1D soliton solutions) found numerically in [27] as functions of λ for $\Omega < 0$ show monotonic behavior: N grows with increasing λ but H decreases. As a result, it was found that there is only a soliton branch with one-to-one dependence $H(N)$. For $\Omega > 0$, however, N contains two branches. The first branch lies in the region $0 < \lambda < \lambda_{min}$. $N(\lambda)$ increases monotonically as λ approaches origin. At $\lambda = \lambda_{min}$ this function has minimum. All of this branch will be unstable in accordance with the criterion (7.44). For $\lambda > \lambda_{min}$ N grows monotonically but the linear criterion cannot be applied to this branch. However, the dependence of $H(\lambda)$ helps us to get a conclusion. This function has maximum at the point $\lambda = \lambda_{min}$, so that H as a function of N has at this point a cusp which separates two branches. The upper branch has larger values of H than the lower branch. If one assumes that for this given interval of N there are no other soliton solutions (numerically this is not too simple a task) then one can say that the lower branch represents a stable soliton family.

To conclude this section, it is worth noting that the linear stability criterion of the Vakhitov–Kolokolov type for these vector NLS systems can be considered only as a sufficient criterion for soliton instability. However, its combination with the Lyapunov approach represents a powerful tool for investigation of soliton stability. Another important point of this section is the embedding theorems which play a very essential role in proof of the Lyapunov stability for solitons. This method allows to demonstrate stability of solitons for the three-wave system. Although we have applied this approach to the 1D solitons, it can be successfully used in the general multi-dimensional case also. Let us recall, that, when we speak about χ^2 media, we mean, first of all, crystals without reflection symmetry. Electromagnetic waves propagating in such crystals have anisotropic dispersion relation. This means, for instance, that in the general situation the dispersion tensors $\omega_{\alpha\beta}$ for each wave packet coupled by three-wave interaction cannot be reduced simultaneously to diagonal form. However, the method presented here does not require any diagonalization of the dispersion tensors, and the only condition needed is a sign definiteness of all dispersion tensors (simultaneous positiveness or negativeness). Only in such a case do solitons exist. Sign definiteness of dispersion tensors, in turn, permits to introduce the corresponding Sobolev space and then to get the desired integral estimates for the Hamiltonian. It is also important that solitons realizing minimum of the Hamiltonian establish their stability not only with respect to small perturbations, but also against the finite ones. In this sense the Lyapunov stability criterion is equivalent to an energy principle.

Acknowledgments

The authors thank V.E. Zakharov, G. Iooss, R. Grimshaw, E.G. Shapiro, and D.S. Agafontsev for valuable discussions and fruitful collaborations while performing common works devoted to the reviewed subject. We acknowledge the support from the CNRS under the framework of PICS no. 4251 and the RFBR under grant 07-01-92165. The work of E.K. was supported by the Russian Federation Government Grant (the contract no. 0035 with the Ministry of Education and Science of RF, November 25, 2010), RFBR (grant no. 09-01-00631), the Program of RAS "Fundamental problems in nonlinear dynamics" and Grant NSh 7550.2006.2. E.K. is grateful to the Kavli Institute for Theoretical Physics (UCSB) where this work was finalized within the Program "'The Nature of Turbulence"' (spring, 2011), for its kind hospitality and the support during the visit.

References

[1] A.C. Scott, Introduction to nonlinear waves, in: R. Enus, et al. (Eds.), Nonlinear Phenomena in Physics and Biology, Plenum Press, 1980.
[2] R.Z. Sagdeev, Cooperative phenomena and shock waves in collisionless plasmas, in: M.A. Leontovich (Ed.), Reviews of Plasma Physics, Vol. 4, Consultants Bureau, New York, 1966, p.52.
[3] C.S. Gardner, O.K. Morikawa, Similarity in the asymptotic behaviour of collision-free hydromagnetic waves and water waves, Courant Inst. of Math. Sc. Rep. NYO 9082 (1960).
[4] C.S. Gardner, J.M. Green, M.D. Kruskal, R.B. Miura, Method for solving the Korteweg–de Vries equation, Phys. Rev. Lett. 19 (1967) 1095–1097.
[5] V.E. Zakharov, A.B. Shabat, Exact theory of two-dimensional self-focusing and one-dimensional self-modulation of waves in nonlinear media, Zh. Éksper. Teoret. Fiz. 61 (1971) 118–134. [Sov. Phys. JETP 34, 62-69 (1972)].
[6] A. Hasegawa, F. Tappet, Transmission of stationary nonlinear optical pulses in fibers I. Anomalous dispersion, Appl. Phys. Lett. 23 (1973) 142–170.

[7] L.F. Mollenauer, R.H. Stolen, M.N. Islam, Experimental demonstration of soliton propagation in long fibers: loss compensated by Raman gain, Opt. Lett. 10 (1985) 229–231.
[8] L.F. Mollenauer, E. Lichtman, M.J. Neubelt, G.T. Harvey, Demonstration using frequency-sliding guiding filters, of error-free soliton transmission over more than 20 mm at 10 Gbit/s, single channel, and over more than 13 mm at 20 Gbit/s in a two-channel WDM, Electron. Lett. 29 (1993) 910–911.
[9] Yu.S. Kivshar, G.P. Agrawal, Optical Solitons: From Fibers to Photonic Crystals, Academic Press, Amsterdam, 2003;
G.P. Agrawal, Nonlinear Fiber optics, Academic Press, Inc, Boston, 1989.
[10] J.M. Dudley, L.P. Barry, J.D. Harvey, M.D. Thomson, B.C. Thomsen, P.G. Bollond, R. Leonhardt, Complete characterisation of ultrashort pulse sources at 1550 nm, IEEE J. Quantum Electron. 35 (1999) 441–450.
[11] J.M. Dudley, J.R. Taylor, Ten years of nonlinear optics in photonic crystal fibre: progress and perspectives, Nat. Photon. 3 (2009) 85–90.
[12] V.E. Zakharov, S.V. Manakov, S.P. Novikov, L.P. Pitaevskii, Teoriya solitonov, Nauka, Moscow, 1980, (Russian) [in English: Soliton Theory, Plenum Press, New York, London, 1984].
[13] A.C. Newell, Solitons in Mathematics and Physics, SIAM, Philadelphia, 1985.
[14] M.J. Ablowitz, H. Segur, Solitons and the Inverse Scattering Transform, SIAM, Philadelphia, 1981.
[15] M.S. Longuet-Higgins, Capillary-gravity waves of solitary type on deep water, J. Fluid Mech. 200 (1989) 451–470.
[16] G. Iooss, K. Kirchgässner, Bifurcation d'ondes solitaires en présence d'une faible tension superficielle, C. R. Acad. Sci. Paris, Série I 311 (1990) 265–268.
[17] T.R. Akylas, Envelope solitons with stationary crests, Phys. Fluids 5 (1993) 789–791.
[18] L.D. Landau, E.M. Lifshitz, Statistical Physics, Part 1, Pergamon Press, New York, 1980 [Russian original: Nauka, Moscow, 1995, p. 521].
[19] F. Dias, G. Iooss, Capillary-gravity solitary waves with damped oscillations, Physica D 65 (1993) 399–442.
[20] F. Dias, G. Iooss, Water waves as a spatial dynamical system, in: Serre Friedlander (Ed.), Handbook of Mathematical Fluid Dynamics, vol. 2, Elsevier, 2003, pp. 443–499.
[21] M.S. Longuet-Higgins, Capillary-gravity waves of solitary type and envelope solitons on deep water, J. Fluid Mech. 252 (1993) 703–711.
[22] V.E. Zakharov, E.A. Kuznetsov, Optical solitons and quasisolitons, Zh. Éksper. Teoret. Fiz. 113 (1998) 1892. [JETP 86, 1035-1046 (1998)].
[23] E.A. Kuznetsov, Hard soliton excitation: stability investigation, Zh. Éksper. Teoret. Fiz. 116 (1999) 299. [JETP 89, 163 (1999)].
[24] N. Bloembergen, Nonlinear Optics, Benjamin, Reading, Mass, 1977.
[25] Yu.N. Karamzin, A.P. Sukhorukov, Mutual focusing of power light beams in media with quadratic nonlinearity, Zh. Eksp. Teor. Fiz. 68 (1975) 834–847. [Sov. Phys. JETP 41, 414–420 (1976)]; Nonlinear interaction of diffracted light beams in medium with quadratic nonlinearity; mutual focusing of beams and the efficiency limitation for optical frequency converters, Pis'ma Zh. Eksp. Teor. Fiz. 20 734–739 [JETP Lett. 20, 339-342 (1974)].
[26] A.A. Kanashev, A.M. Rubenchik, On diffraction and dispersion effect on three wave interaction, Physica 4D (1981) 122–134.
[27] R.H.J. Grimshaw, E.A. Kuznetsov, E.G. Shapiro, The two-parameter soliton family for the interaction of a fundamental and its second harmonic, Physica D 152–153 (2001) 325–339.
[28] L.D. Landau, Dokl. Akad. Nauk SSSR 44 (1944) 339;
L.D. Landau, E.M. Lifshitz, Fluid mechanics, 3rd ed., Pergamon Press, New York, 1986 (in English), [Russian original, 3rd ed., Nauka, Moscow, 1986].
[29] F. Dias, G. Iooss, Capillary-gravity interfacial waves in deep water, Eur. J. Mech. B/Fluids 15 (1996) 367–390.
[30] J.-M. Vanden-Broeck, F. Dias, Gravity-capillary solitary waves in water of infinite depth and related free-surface flows, J. Fluid Mech. 240 (1992) 549–557.
[31] D.S. Agafontsev, F. Dias, E.A. Kuznetsov, Bifurcations and stability of internal solitary waves, Pis'ma v ZhETF 85 (2006) 241. [JETP Lett., 83 201 (2006)].
[32] D.S. Agafontsev, F. Dias, E.A. Kuznetsov, Deep-water internal solitary waves near critical density ratio, Physica D 225 (2007) 153.
[33] V.E. Zakharov, E.A. Kuznetsov, Hamiltonian formalism for nonlinear waves, Uspekhi Fiz. Nauk 167 (1997) 1137–1168. [Phys. Uspekhi 40, 1087-1116 (1997)].
[34] M. Segev, G. Stegeman, Phys. Today 51 (8) (1998) 42;
G. Stegeman, M. Segev, Science 286 (1998) 1518;
Yu.S. Kivshar, Opt. Quantum Electron. 30 (1998) 571.
[35] G. Derrick, Comments on nonlinear wave equations as a model for elementary particles, J. Math. Phys. 5 (1966) 1252.
[36] T.B. Benjamin, The stability of solitary waves, Proc. R. Soc. Lond. Ser. A 328 (1972) 153–183.
[37] V.E. Zakharov, E.A. Kuznetsov, Three-dimensional solitons, ZhETF (JETP) 39 (1974) 285.
[38] E.A. Kuznetsov, A.M. Rubenchik, V.E. Zakharov, Soliton stability in plasmas and fluids, Phys. Rep. 142 (1986) 103–165.
[39] N.G. Vakhitov, A.A. Kolokolov, Stationary solutions of the wave equation in media with saturated nonlinearity, Izv. VUZ Radiofizika 16 (1973) 1020–1028. [Radiophys. Quantum Electron. 16, 783-789 (1973)].
[40] L. Nirenberg, An extended interpolation inequality, Ann. Sci. Norm. Sup. Pisa 20 (4) (1966) 733–737.
[41] O.A. Ladyzhenskaya, Mathematical Problems of the Dynamics of an Incompressible Viscous Fluid, Nauka, Moscow, 1970, (in Russian) [English translation: O.A. Ladyzhenskaya, The mathematical theory of viscous incompressible flow, Gordon and Breach, 1969].
[42] L.D. Landau, E.M. Lifshitz, Electrodynamics of Continuous Media, 2nd ed., Pergamon, London, 1984.
[43] Andonowati, W.M. Kusumawinahyu, E. van Groesen, A numerical study of the breaking of modulated waves generated at a wave maker, Appl. Ocean Res. 28 (2006) 9–17.
[44] N. Karjanto, E. van Groesen, Qualitative comparisons of experimental results on deterministic freak wave generation based on modulational instability, J. Hydro-environ. Res. 3 (2010) 186–192.
[45] G. Iooss, Existence d'orbites homoclines à un équilibre elliptique, pour un système réversible, C. R. Acad. Sci. Paris 324 (1997) 993–997.
[46] K.B. Dysthe, Note on a modification to the nonlinear Schrödinger equation for application to deep water waves, Proc. R. Soc. Lond. A 369 (1979) 105–114.
[47] E.A. Kuznetsov, Stability of solitons, in: V.E. Zakharov, S. Wabnitz (Eds.), Optical Solitons: Theoretical Challenges and Industrial Perspectives, Editions de Physique Springer, 1999, pp. 31–50.
[48] A.T. Il'ichev, A.Yu. Semenov, Stability of solitary waves in dispersive media described by a fifth-order evolution equation, Theoret. Comput. Fluid Dyn. 3 (1992) 307–326.
[49] A.R. Champneys, J.F. Toland, Bifurcation of a plethora of multi-modal homoclinic orbits for autonomous Hamiltonian systems, Nonlinearity 6 (1993) 665–722.
[50] G. Iooss, M.-C. Pérouème, Perturbed homoclinic solutions in reversible 1:1 resonance vector fields, J. Differential Equations 102 (1993) 62–88.
[51] V.E. Zakharov, E.A. Kuznetsov, Multi-scale expansions in the theory of systems integrable by the inverse scattering transform, Physica D 18 (1986) 455–463.
[52] R. Grimshaw, B. Malomed, E. Benilov, Solitary waves with damped oscillatory tails: an analysis of the fifth-order Korteweg–de Vries equation, Physica D 77 (1994) 473–485.
[53] T.-S. Yang, T.R. Akylas, On asymmetric gravity–capillary solitary waves, J. Fluid Mech. 330 (1997) 215–232.
[54] M.I. Weinstein, Nonlinear Schrödinger equations and sharp interpolation estimates, Commun. Math. Phys. 87 (1983) 567–576.
[55] F. Dias, E.A. Kuznetsov, On the nonlinear stability of solitary waves for the fifth-order Korteweg–de Vries equation, Phys. Lett. A 263 (1999) 98–104.
[56] S.P. Levandosky, A stability analysis of fifth-order water wave models, Physica D 125 (1999) 222–240.
[57] D.C. Calvo, T.-S. Yang, T.R. Akylas, On the stability of solitary waves with decaying oscillatory tails, Proc. R. Soc. Lond. Ser. A 456 (2000) 469–487.
[58] B. Buffoni, A.R. Champneys, J.F. Toland, Bifurcation and coalescence of a plethora of homoclinic orbits for a Hamiltonian system, J. Dynam. Differential Equations 8 (1995) 221–281.
[59] G.P. Agrawal, Nonlinear Fiber Optics, Academic Press, Inc., Boston, 1989.
[60] M.S. Longuet-Higgins, Bifurcation in gravity waves, J. Fluid Mech. 151 (1985) 457–475; Capillary-gravity waves on solitary type and envelope solitons on deep water 252 (1993) 703–711.

[61] J. Nycander, Steady vortices in plasmas and geophysical flows, Chaos 4 (1994) 253.
[62] E.M. Gromov, V.I. Talanov, Short envelope solitons combined nonlinear equation, Izv. Vyssh. Uchebn. Zaved., Radiofiz. 39 (1996) 735. [Radiophys. Quantum Electron. 39, 486-501 (1996)]; Nonlinear dynamics of short pulses in dispersive media, Zh. Éksper. Teoret. Fiz., 110 (1996) 137 [JETP 83, 73 (1996)]; Short optical solitons in fibers, Chaos 10 (2000) 551–558.
[63] R. Hirota, Exact solution of the Korteweg–de Vries equation for multiple collisions of solitons, J. Phys. Soc. Japan 33 (1973) 1456–1458. Exact envelope-soliton solutions of a nonlinear wave equation, J. Math. Phys. 14 (1973) 805–809.
[64] V.E. Zakharov, Collapse and self-focusing of Langmuir waves, in: A.A. Galeev, R.N. Sudan (Eds.), Handbook of Plasma Physics, in: Basic plasma physics, 2, Elsevier, North-Holland, 1984, pp. 81–121.
[65] N.N. Bogolyubov, D.V. Shirkov, Introduction to the Theory of Quantized Fields, Nauka, Moscow, 1976, (in Russian).
[66] E.A. Kuznetsov, S.K. Turitsyn, Talanov transformations for selffocusing problems and instability of waveguides, Phys. Lett. 112A (1985) 273–275.
[67] V.E. Zakharov, E.A. Kuznetsov, Quasi-classical theory of three-dimensional wave collapse, Sov. Phys. JETP 64 (1986) 773–780.
[68] E.A. Kuznetsov, S.L. Musher, effect of the collapse of sound waves on the structure of the collisionless shocks in a magnetized plasma, Sov. Phys. JETP 64 (1986) 947–955.
[69] V.E. Zakharov, Collapse of Langmuir waves, Zh. Eksp. Teor. Fiz. 62 (1972) 1745–1751. [Sov. Phys. JETP 35, 908-914 (1972)].
[70] E.A. Kuznetsov, Wave collapse in plasmas and fluids, Chaos 6 (1996) 381–390.
[71] S.N. Vlasov, V.A. Petrishchev, V.I. Talanov, Averaged description of wave beams in linear and nonlinear media (the method of moments), Izv. VUZ Radiofiz. 14 (1971) 1353. [Radiophys. Quantum Electron. 14, 1062 (1974)].
[72] V.I. Talanov, About selffocusing of light in cubic media, ZhETF Pis'ma 11 (1970) 303–305. [JETP Letters 11, 199-201 (1970)].
[73] S.N. Vlasov, V.I. Talanov, Private Communication, (1984).
[74] E.A. Kuznetsov, S.K. Turitsyn, Talanov transformations for selffocusing problems and instability of waveguides, Phys. Lett. A 112 (1985) 273–275.
[75] E.A. Kuznetsov, J.J. Rasmussen, K. Rypdal, S.K. Turitsyn, Sharper criteria of the wave collapse, Physica D 87 (1995) 273–284.
[76] G.B. Whitham, Non-linear dispersive waves, Proc. R. Soc. Lond. Ser. A 283 (1965) 238–261. A general approach to linear and non-linear dispersive waves using a Lagrangian, J. Fluid Mech. 22 (1965) 273–283; Linear and Nonlinear Waves, Wiley Interscience, New York 1974.
[77] D.S. Agafontsev, Deep-water internal solitary waves near critical density ratio, Pis'ma ZhETF 87 (2008) 225. [JETP Letters, 87, 195-198 (2008)].
[78] E.I. Kats, V.V. Lebedev, A.R. Muratov, Weak crystallization theory, Phys. Rep. 228 (1993) 1–91.
[79] V.E. Zakharov, in: A. Galeev, R. Sudan (Eds.), Handbook of Plasma Physics, vol. 2, Basic Plasma Physics II, North–Holland, Amsterdam, 1984, p. 81.
[80] E.A. Kuznetsov, Wave collapse in plasmas and fluids, Chaos 6 (1996) 381–390.
[81] Yu.A. Sinitsyn, V.M. Kontorovich, in: Ozmidov (Ed.), Studies of Turbulent Structure of Ocean, Izd. MGI, Sevastopol, 1975, First Symposium Issledovanie melkomaschtabnoi okeanicheskoi turbulentnosti, Kaliningrad, 1974;
V.M. Kontorovich, A possible role of internal waves in occurence of small-scale turbulence in stratified ocean., Izv. Vuzov, Radiofizika 19 (1976) 872–879.
[82] E.A. Kuznetsov, M.D. Spector, Existence of hexagon relief on the surface of the liquid dielectrics in an external electric field, Zh. Éksper. Teoret. Fiz. 71 (1976) 262. [Sov. Phys. JETP 44, 136 (1976)].
[83] E.A. Kuznetsov, M.D. Spector, V.E. Zakharov, Surface singularities of ideal fluid, Phys. Lett. 182A (1993) 387–393.
[84] F. Dias, T. Bridges, Geometric aspects of spatially periodic interfacial waves, Stud. Appl. Math. 93 (1994) 93–132.
[85] V.E. Zakharov, Stability of periodic waves of finite amplitude on the surface of a deep fluid, Zh. Prikl. Mekh. Tekh. Fiz. 9 (1968) 86–94. [J. Appl. Mech. Tech. Phys. 9, 190-194 (1968)];
V.E. Zakharov, N.N. Filonenko, The energy spectrum for stochastic oscillations of a fluid surface, Docl. Akad. Nauk SSSR 170 (1966) 1292–1295. [Sov. Phys. Docl. 11, 881-884 (1967)].
[86] T.B. Benjamin, T.J. Bridges, Reappraisal of the Kelvin–Helmholtz problem. I. Hamiltonian structure, J. Fluid Mech. 333 (1997) 301–325.
[87] W. Craig, M.D. Groves, Normal forms for wave motion in fluid interfaces, Wave Motion 31 (2000) 21–41.
[88] W. Craig, P. Guyenne, H. Kalisch, Hamiltonian long wave expansions for free surfaces and interfaces, Commun. Pure Appl. Math. 58 (2005) 1587–1641.
[89] V.I. Petviashvili, On the equation for unusual soliton, Fizika Plasmy 2 (1976) 469. [Sov. J. Plasma Phys., 2, 247 (1976)].
[90] D.S. Agafontsev, F. Dias, E.A. Kuznetsov, Collapse of solitary waves near transition from supercritical to subcritical bifurcations, Pis'ma ZhETF 87 (2008) 767–771. [JETP Letters 87, 667-671 (2008)].
[91] D.J. Kaup, A.C. Newell, Stationary solutions of the wave equation in the medium with saturated nonlinearity, J. Math. Phys. 19 (4) (1978) 798–801.
[92] V.E. Zakharov, Collapse of Langmure waves, Sov. Phys. JETP 35 (1972) 908.
[93] S.K. Turitsyn, Nonstable solitons and sharp criteria for wave collapse, Phys. Rev. E 47 (1993) R13–16.
[94] N.A. Zharova, A.G. Litvak, V.A. Mironov, Self-focusing of wave packets and envelope shock formation in nonlinear dispersive media, Zh. Éksper. Teoret. Fiz. 130 (2006) 21. [JETP 103 15 (2006)].
[95] A.A. Balakin, A.G. Litvak, V.A. Mironov, S.A. Skobelev, Structural features of the self-action dynamics of ultrashort electromagnetic pulses, Zh. Éksper. Teoret. Fiz. 131 (2007) 408. [JETP 104, 363 (2007)].
[96] G.M. Fraiman, Asymptotic stability of manifold of self-similar solutions in self-focusing, Zh. Éksper. Teoret. Fiz. 88 (1985) 390. [Sov. Phys. JETP 61, 228 (1985)].
[97] P. Chossat, F. Dias, The 1:2 resonance with O(2) symmetry and its applications in hydrodynamics, J. Nonlinear Sci. 5 (1995) 105–129.
[98] V.E. Zakharov, S.V. Manakov, Resonance interaction of wave packets, Pis'ma ZhETF 18 (1973) 413–417. [JETP Letters, 18, 243-245 (1973)]; The theory of resonance interaction of wave packets in nonlinear media, Zh. Éksper. Teoret. Fiz. 69 (1975) 1654–1673 [Sov. Phys. JETP, 42, 842-850 (1975)].
[99] L. Berge, V.K. Mezentsev, J.J. Rasmussen, J. Wyller, Formation of stable solitons in quadratic nonlinear media, Phys. Rev. A 52 (1995) R28.
[100] Yu.S. Kivshar, et al., in: A.D. Boardman (Ed.), Advanced Photonics with Second-order Optically Nonlinear Processes, Kluver, Netherlands, 1999, pp. 451–475.
[101] A.V. Buryak, Yu.S. Kivshar, Solitons due to second-harmonic generation, Phys. Lett. A 197 (1995) 407–412.
[102] S.K. Turitsyn, Stability of two- and three-dimensional optical solitons in a media with quadratic nonlinearity, Pis'ma ZHETF 61 (1995) 458–460. [JETP Letters, 61, 499-472 (1995)].
[103] L. Berge, O. Bang, J.J. Rasmussen, V.K. Mezentsev, Self-focusing and solitonlike structures in materials with competing quadratic and cubic nonlinearities, Phys. Rev. E 55 (1997) 3555–3570.
[104] E.A. Kuznetsov, V.E. Zakharov, Nonlinear coherent phenomena in continuous media, in: P.L. Christiansen, M.P. Soerensen, A.C. Scott (Eds.), Nonlinear science at the down of 21th centure, in: Lecture Notes in Physics, vol. 542, Springer, 2000, pp. 3–45.