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NONLINEAR GRAVITY AND CAPILLARY-GRAVITY WAVES

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ABSTRACT

This review deals primarily with the bifurcation, stability, and evolution of gravity and capillary-gravity waves. Recent results on the bifurcation of various types of capillary-gravity waves, including two-dimensional solitary waves at the minimum of the dispersion curve, are reviewed. A survey of various mechanisms (including the most recent ones) to explain the frequency downshift phenomenon is provided. Recent significant results are given on "horseshoe" patterns, which are three-dimensional structures observable on the sea surface under the action of wind or in wave tank experiments. The so-called short-crested waves are then discussed. Finally, the importance of surface tension effects on steep waves is studied.

1. INTRODUCTION

This review deals primarily with the bifurcation, stability, and evolution of gravity and capillary-gravity waves. The most recent reviews on water waves in the *Annual Review of Fluid Mechanics* are those of Hammack & Henderson (1993) on resonant interactions among surface water waves, Banner & Peregrine (1993) on wave breaking in deep water, Akylas (1994) on three-dimensional long water-wave phenomena, Melville (1996) on the role of surface-wave breaking in air-sea interaction, and Tsai & Yue (1996) on the numerical computation

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of free-surface flows. The present review is more in the spirit of the reviews by Yuen & Lake (1980, 1982) on instabilities of waves on deep water and Schwartz & Fenton (1982) on strongly nonlinear waves. Although the emphasis is on capillary-gravity waves, some results on gravity waves will be recalled as well.

In Section 2, some results are recalled on the water-wave problem, which will be discussed in later sections of this review. In particular, the stability and evolutionary properties of a weakly nonlinear wave train are considered, based on the Dysthe (1979) modulation equation, which is an equation describing the wave envelope in deep water. The main steps leading to the Dysthe equation are recalled, both by using the method of multiple scales and by starting from Zakharov's integral equation.

Section 3 is devoted to recent results on the bifurcation of capillary-gravity waves near the minimum of the dispersion curve, including the bifurcation of a new type of solitary wave. This discovery was made possible by using a spatial approach to study the water-wave problem. The wave evolution is then considered as a dynamical system in space. The application of this approach to the water-wave problem goes back to Kirchgässner (1988) and has been the subject of several papers since. For example, it can shed some light on the waves generated by a fishing-rod perturbing a stream flowing down at a speed close to the minimum of the dispersion curve. Another interesting application is the deformation of a sheet of ice under a load (for example when a vehicle moves on top of the ice). From a pedagogical point of view, a good starting point to explain the new results is the nonlinear Schrödinger (NLS) equation. Indeed, there are two speeds involved in the solutions of the NLS equation: the phase velocity of the oscillations of the carrier, and the group velocity at which the envelope propagates. Near the minimum of the dispersion curve, these two speeds are almost equal, and one can find steady solutions. The application of the spatial approach has also allowed the discovery of other types of waves, such as generalized solitary waves, which are solitary waves with ripples of small constant amplitude in their tails. The physical relevance of these new solutions will be discussed as well.

Section 4 is devoted to the frequency downshift phenomenon. The nonlinear evolution of water waves has been studied for several decades. The frequency downshift in the evolution of a uniformly traveling train of Stokes waves, reported initially by Lake et al (1977), remained a challenging problem for a long time. This phenomenon concerns the four-wave interaction of pure gravity waves as well as short gravity waves influenced by the effects of both viscosity and surface tension. In the framework of two-dimensional (2D) motion, modulational instability and dissipation were found to be the fundamental ingredients in the permanent subharmonic transition of the wave field. However, Trulsen & Dysthe (1997) showed that dissipation may not be necessary to produce a permanent downshift in three-dimensional (3D) wave trains in deep water. A review of various mechanisms (including the most recent ones) to explain the downshift is provided.

Recent results on "horseshoe" patterns are described in Section 5. These horseshoe patterns are quite common 3D structures, which are often observed on the sea surface under the action of wind or in wave tank experiments. Despite their common character and ease of observation, experimental information as well as theoretical explanations of their formation and persistence are almost nonexistent. The presence of these coherent structures in the wave field sheds some doubt on the assumption of gaussian probability distribution of sea surface for the systems under consideration. Another important feature of these 3D wave patterns is their role in sea roughness and consequently in air-sea momentum transfer. We report on recent significant results dealing with horseshoe patterns.

Section 6 is devoted to the bifurcation of 3D waves from a state of rest. These 3D waves are commonly called short-crested waves. Surprisingly, the literature indicates that these waves have not been studied much. A review of the existing results is provided, including some new results on the bifurcation of short-crested waves and on their stability. A few words are said on the connection between short-crested waves and the 3D waves studied by Saffman and his collaborators in the early 1980s. Their 3D waves appear through a dimension-breaking bifurcation in the water-wave problem. Such bifurcations were first pointed out by McLean et al (1981), who studied 3D instabilities of finite-amplitude water waves.

The importance of surface tension effects on gravity waves is discussed in Section 7. In particular, we review recent results on limiting profiles, crest instabilities, and wave breaking.

2. GENERALITIES ON THE WATER-WAVE PROBLEM

The notations used in this section and in later sections are summarized in Tables 1 and 2.

The classical water-wave problem consists of solving the Euler equations in the presence of a free surface. These equations allow for rotational motions, but usually one makes the assumption that the flow is irrotational (potential flow). Besides analytical reasons (the irrotational equations are easier to handle than the rotational equations), this fact may be explained by the following property. The linearization of the rotational Euler equations around the state of rest shows that the rotational component of the perturbation is steady. One can split the perturbation into two parts: a rotational one and a potential one. The potential one satisfies the linear equation obtained by linearizing the potential

Table 1 Physical parameters and their dimensions. m, mass; l, length; t, time

Symbol	Physical quantity	Dimension
c	Wave velocity	$[l][t]^{-1}$
T	Coefficient of surface tension divided by density	$[l]^3[t]^{-2}$
g	Acceleration due to gravity	$[l][t]^{-2}$
h	Mean water depth	$[l]$
ρ	Water density	$[m][l]^{-3}$
ρ_1	Density of the heavier fluid	$[m][l]^{-3}$
ρ_2	Density of the lighter fluid	$[m][l]^{-3}$
ν	Kinematic viscosity	$[l]^2[t]^{-1}$
$\mathbf{k}$	Wave number vector	$[l]^{-1}$
k_o	Wave number of the carrier	$[l]^{-1}$
L	Wave length	$[l]$
a or A	Wave amplitude	$[l]$
ω	Frequency	$[t]^{-1}$
ω_o	Frequency of the carrier	$[t]^{-1}$
$\mathbf{x} = (x, y)$	Horizontal physical coordinates	$[l]$
z	Vertical physical coordinate	$[l]$
t	Time	$[t]$
$\phi(x, y, z, t)$	Velocity potential	$[l]^2[t]^{-1}$
$\eta(x, y, t)$	Elevation of the free surface	$[l]$

Euler equations. Stability or instability is then decided independently of the rotational or the irrotational perturbation, but since the rotational perturbation does not evolve in time, one must deal only with the potential perturbation. This investigation is precisely the problem of stability of the state of rest in the context of potential flows. Concerning the modulational stability of a nonlinear wave, Colin et al (1995, 1996) showed that in the rotational case, one ends up with the same amplitude equation as in the potential case. Therefore stability does not depend on the irrotationality assumption.

Table 2 Dimensionless quantities. Note that the Bond number is often defined as the inverse of B in the literature

Symbol	Definition	Dimensionless quantity
B	Tk^2/g	Bond number
R	$g^{1/2}/\nu k^{3/2}$	Reynolds number
ϵ	ka	Dimensionless amplitude
F^2	c^2/gh	Square of Froude number
α	gT/c^4	Weber number divided by F^2
r	$(\rho_1 - \rho_2)/(\rho_1 + \rho_2)$	Density ratio

From now on, it will be assumed that the flow is irrotational. The bottom is described by $z = -h(x, y)$. We summarize the governing equations and boundary conditions and state the classical surface wave problem: solve for $\eta(x, y, t)$ and $\phi(x, y, z, t)$ the set of equations

$$\begin{aligned}
\nabla^2\phi \equiv \phi_{xx} + \phi_{yy} + \phi_{zz} = 0 &\quad \text{in the flow domain,}\\
\eta_t + \phi_x\eta_x + \phi_y\eta_y - \phi_z = 0 &\quad \text{at } z = \eta(x, y, t),\\
\phi_t + \frac{1}{2}\left(\phi_x^2 + \phi_y^2 + \phi_z^2\right) + g\eta - T(w_{1x} + w_{2y}) = 0 &\quad \text{at } z = \eta(x, y, t),\\
\phi_x h_x + \phi_y h_y + \phi_z = 0 &\quad \text{at } z = -h(x, y).
\end{aligned}$$

The expressions for the curvatures w_1 and w_2 are

$$w_1 = \left(\frac{\eta_x}{\sqrt{1 + \eta_x^2 + \eta_y^2}}\right), \quad w_2 = \left(\frac{\eta_y}{\sqrt{1 + \eta_x^2 + \eta_y^2}}\right). \tag{2.1}$$

The equation for conservation of momentum is not stated; it is used to find the pressure p once η and ϕ have been found. In water of infinite depth, the kinematic boundary condition on the bottom is replaced by $|\nabla\phi| \to 0$ as $z \to -\infty$.

It is well known that the above problem admits 2D periodic solutions in the form of traveling gravity waves ($T=0$), the so-called Stokes waves (Stokes 1847). It also admits 2D periodic solutions in the form of standing gravity waves. A rigorous mathematical proof of the existence of standing waves is still absent but some progress has been made (Amick & Toland 1987, Iooss 1997b). The water-wave problem also has solitary gravity wave solutions. When surface tension is added, there are more types of solutions (see for example Sect. 3). Of course, there are also 3D periodic solutions, the so-called short-crested waves. The proof of their existence was first given by Reeder & Shinbrot (1981) but only in some region of parameter space that does not include the case of pure gravity waves. The dispersion relation for linear periodic waves is given by

$$\omega^2 = (g|\mathbf{k}| + T|\mathbf{k}|^3)\tanh(|\mathbf{k}|h).$$

An essential part in the study of Stokes waves is their stability. There are two important dates: the late 1960s, when it was discovered that Stokes waves in deep water are unstable with respect to long wave perturbations, and the early 1980s, when the stability with respect to all kinds of 3D perturbations was studied numerically.

The so-called long wave instability (or Benjamin-Feir instability, or instability to side-band perturbations) dominates for small-amplitude waves. Lighthill (1965) provided a geometric condition for wave instability, which is valid when mean flow effects can be neglected, such as in deep water, and essentially obtained what is now called the Benjamin-Feir instability. Benjamin & Feir (1967) showed the result analytically. Whitham (1967) obtained the same result independently by using an average Lagrangian approach, which is explained in his book (1974). At the same time, Zakharov (1968), using a Hamiltonian formulation of the water-wave problem, obtained the same instability result and derived the cubic NLS equation in the context of the modulational stability of water waves. The extension to finite depth was provided by Benney & Roskes (1969), who obtained the equations that are now called the Davey-Stewartson equations. Both Zakharov (1968) for infinite depth and Benney & Roskes (1969) for finite depth considered the stability with respect to 3D disturbances. Hasimoto & Ono (1972) used the method of multiple scales in time and in space to rederive the cubic NLS equation. Later, Davey & Stewartson (1974) extended the results of Hasimoto & Ono (1972) to 3D perturbations. Recently, Bridges & Mielke (1995) formulated rigorously the existence and linear stability problem for the Stokes periodic wave train in finite depth in terms of the spatial and temporal Hamiltonian structure of the water-wave problem.

The cubic NLS equation does not involve the wave-induced mean flow. Dysthe (1979) pursued the perturbation analysis one step further, to fourth order in wave steepness. One of the main effects at this order in infinite depth is precisely the influence of the wave-induced mean flow.

Later, numerical computations were used to analyze the stability of water waves. The main advantage is that there exists in principle no restriction concerning the length of disturbances and the steepness of the wave. Early numerical work was limited to 2D instability (Longuet-Higgins 1978a,b). This author reported a new type of instability for finite-amplitude gravity waves, resulting from a quintet resonance (see below).

McLean et al (1981) and McLean (1982) studied 3D instabilities of finite-amplitude water waves. McLean et al (1981) consider the stability of a 2D periodic Stokes wave of the form

$$\overline{\eta}(x,t) = \sum_{n=0}^{\infty} A_n \cos[nk_o(x-ct)].$$

Let $\eta(x,y,t) = \overline{\eta}(x,t) + Z(x,y,t)$. The water-wave equations are linearized about $\overline{\eta}(x,t)$ and nontrivial solutions of the linearized problem are sought in the form

$$Z(x,y,t) = \exp\{\mathrm{i}[pk_o(x-ct) + qk_oy] + st\} \\ \times \sum_{n=-\infty}^{\infty} a_n \exp[\mathrm{i}\,nk_o(x-ct)] + \text{c.c.},$$

where p and q are arbitrary real numbers and c.c. denotes the complex conjugate. The eigenvalues of the linearized problem are the values of s such that there is a nontrivial solution with time-dependence exp(st). Instability corresponds to Re $s \neq 0$. The spectrum is easy to compute when $\overline{\eta}(x, t) = 0$. One finds that the eigenvalues are all imaginary:

$$s = -\mathrm{i}[\pm\omega(\kappa) - ck_o(p+n)],$$

where

$$\omega(\kappa) = \sqrt{g\kappa} \quad \text{and} \quad \kappa = k_o\sqrt{(p+n)^2 + q^2}.$$

As the amplitude of the Stokes wave increases, the eigenvalues move. MacKay & Saffman (1986) showed that a necessary condition for a Stokes wave to lose spectral stability is that, for the linearized problem about it, there is a collision of eigenvalues of opposite Krein signature (Krein 1955) or a collision of eigenvalues at zero. The signature is related to the sign of the second derivative of the energy. The loci of collision at zero amplitude can be easily obtained. The curves are separated into two classes: class I when the collisions occur between modes with $n = m$ and $n = -m$, and class II when the collisions occur between modes with $n = m$ and $n = -m - 1$. The corresponding instabilities are called class I and class II instabilities. For $m = 1$, the instability I band lies near the curve in the (p, q) plane defined by

$$p - 1 + [q^2 + (p-1)^2]^{1/4} = p + 1 - [q^2 + (p+1)^2]^{1/4}, \tag{2.2}$$

while the instability II band lies near the curve defined by

$$p - 2 + [q^2 + (p-2)^2]^{1/4} = p + 1 - [q^2 + (p+1)^2]^{1/4}. \tag{2.3}$$

This band is symmetrical about $q = 0$ and $p = \frac{1}{2}$. In fact, class I instabilities correspond to quartet interactions between the carrier $\mathbf{k}_o = k_o(1, 0)$ counted twice and the satellites $\mathbf{k}_1 = k_o(1 + p, q)$ and $\mathbf{k}_2 = k_o(1 - p, -q)$, while class II instabilities correspond to quintet interactions between the carrier $k_o(1, 0)$ counted three times and the satellites $k_o(1 + p, q)$ and $k_o(2 - p, -q)$. Another way to interpret these instabilities is in terms of resonance conditions. Equations (2.2) and (2.3) can be expressed as

$$n\mathbf{k}_o = \mathbf{k}_1 + \mathbf{k}_2, \tag{2.4}$$

$$n\omega_o = \omega(\mathbf{k}_1) + \omega(\mathbf{k}_2), \quad n = 2, 3, \ldots, \tag{2.5}$$

where $\omega_o = \omega(\mathbf{k}_o)$. $n = 2$ corresponds to quartet resonant interactions, $n = 3$ to quintet resonant interactions, etc. The class II instability for finite-amplitude gravity waves, which results from a quintet resonance, was in fact conjectured by Zakharov (1968).

McLean et al (1981) found that for $p = \frac{1}{2}$ the instability is copropagating with the unperturbed wave (i.e. Im $s = 0$). This implies that steady 3D waves bifurcate from 2D waves at the point of stability exchange with $p = \frac{1}{2}$. This phenomenon was first exhibited within the framework of the Zakharov equation (Saffman & Yuen 1980) and then of the full Euler equations (Meiron et al 1982). The bifurcated 3D waves are of two types: symmetric or asymmetric (another terminology is skew-symmetric). As pointed out by Meiron et al (1982), bifurcation can occur for any value of p and can be identified with the stationary states of the instability. There are, however, good physical reasons why only $p = \frac{1}{2}$ should occur, namely that only for this value is the state with stationary disturbances also a state of stability exchange. For $p \neq \frac{1}{2}$, all stability-exchange disturbances propagate with respect to the wave. Note the analogy with resonant triads in shear flows [Craik 1985 (Sect. 17)].

The extension of the long-wave instability to capillary-gravity waves was given independently by Kawahara (1975) (2D perturbations) and Djordjevic & Redekopp (1977) in finite depth, and by Hogan (1985) in infinite depth. The numerical computations were extended to capillary waves by Chen & Saffman (1985) and to capillary-gravity waves by Zhang & Melville (1987). They studied numerically the stability of gravity-capillary waves of finite amplitude including, besides quartet instabilities, triad and quintet instabilities.

In the next subsections, we review the main steps of the derivation of the modulation equations, first by using the method of multiple scales, then by starting from the Zakharov equations. We refer to the paper by Das (1997) for a similar description.

2.1 *Method of Multiple Scales*

Consider gravity-capillary waves on the surface of deep water. Accounting for nonlinear and dispersive effects correct to third order in the wave steepness, the envelope of a weakly nonlinear gravity-capillary wavepacket in deep water is governed by the NLS equation. A more accurate envelope equation, which includes effects up to fourth order in the wave steepness, was derived by Dysthe (1979) for pure gravity wavepackets. He used the method of multiple scales. Later, Stiassnie (1984) showed that the Dysthe equation is merely a particular case of the more general Zakharov equation that is free of the narrow spectral-width assumption. Hogan (1985) extended Stiassnie's results to deep-water gravity-capillary wavepackets. Apart from the leading-order nonlinear and dispersive terms present in the NLS equation, the fourth-order equation of Hogan features certain nonlinear modulation terms and a nonlocal term that describes the coupling of the envelope with the induced mean flow. In addition to playing a significant part in the stability of a uniform wavetrain, this mean flow turns out to be important at the tails of gravity-capillary solitary waves in deep water (Akylas et al 1998).

The common ansatz used in the derivation of the NLS equation (or of the Dysthe equation if one goes to higher order) is that the velocity potential ϕ and the free-surface elevation η have uniformly valid asymptotic expansions in terms of a small parameter ϵ (the dimensionless amplitude of the wave, $k_o A$, for example). One writes

$$\eta = \sum_{n=1}^{3} \epsilon^n \eta_n(x_0, x_1, x_2, y_1, y_2; t_0, t_1, t_2) + O(\epsilon^4), \tag{2.6}$$

$$\phi = \sum_{n=1}^{3} \epsilon^n \phi_n(x_0, x_1, x_2, y_1, y_2, z; t_0, t_1, t_2) + O(\epsilon^4), \tag{2.7}$$

where

$$x_0 = x, x_1 = \epsilon x, x_2 = \epsilon^2 x, y_1 = \epsilon y, y_2 = \epsilon^2 y, t_0 = t, t_1 = \epsilon t, t_2 = \epsilon^2 t. \tag{2.8}$$

The order one component of η is

$$\eta_1 = A e^{i(k_o x - \omega_o t)} + \text{c.c.} \tag{2.9}$$

Applying the method of multiple scales leads to a generalization of the Dysthe equation for the evolution of the complex amplitude A of the wave,

$$\begin{aligned} 2\mathrm{i}\frac{\partial A}{\partial t_2} + p\frac{\partial^2 A}{\partial X^2} + q\frac{\partial^2 A}{\partial y_1^2} + \gamma A|A|^2 = &- \mathrm{i}\epsilon\big(sA_{Xy_1y_1} + rA_{XXX} + uA^2A_X^* \\ &- v|A|^2 A_X\big) + \epsilon A\overline{\phi}_X\big|_{z_1=0}. \end{aligned} \tag{2.10}$$

The evolution equation has been written in dimensionless form, with all lengths nondimensionalized by k_o, time by ω_o, and potential by $2k_o^2/\omega_o$. Here $X = (x_1 - c_g t_1)$, $z_1 = \epsilon z$, are scaled variables that describe the wavepacket modulations in a frame of reference moving with the group velocity c_g. As expected, to leading order in the wave steepness $\epsilon \ll 1$, Equation 2.10 reduces to the familiar NLS equation, while the coupling with the induced mean flow mentioned earlier is reflected in the last term of Equation 2.10.

Specifically, the mean-flow velocity potential $\epsilon^2\bar{\phi}(X, z_1, t_2)$ satisfies the boundary-value problem

$$\begin{aligned} \bar{\phi}_{XX} + \bar{\phi}_{z_1 z_1} &= 0 \qquad (-\infty < z_1 < 0, -\infty < X < \infty), \\ \bar{\phi}_{z_1} &= (|A|^2)_X \qquad (z_1 = 0), \\ \bar{\phi} &\to 0 \qquad (z_1 \to -\infty), \end{aligned}$$

from which it follows that

$$\bar{\phi}_X|_{z_1=0} = -\int_{-\infty}^{\infty} |s| e^{isX} \mathcal{F}(|A|^2)\, \mathrm{d}s, \tag{2.11}$$

where

$$\mathcal{F}(\cdot) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-isX} (\cdot)\, \mathrm{d}X$$

denotes the Fourier transform. Hence, the coupling of the envelope with the induced mean flow enters via a nonlocal term in the fourth-order envelope equation. The coefficients of the rest of the terms in Equation 2.10 are given by the following expressions:

$$p = \frac{k_o^2}{\omega_o} \frac{\mathrm{d}^2\omega_o}{\mathrm{d}k_o^2} = \frac{3B^2 + 6B - 1}{4(1+B)^2}, \tag{2.12}$$

$$q = \frac{k_o}{\omega_o} \frac{\mathrm{d}\omega_o}{\mathrm{d}k_o} = \frac{1 + 3B}{2(1+B)}, \tag{2.13}$$

$$\gamma = -\frac{2B^2 + B + 8}{8(1-2B)(1+B)}, \tag{2.14}$$

$$r = -\frac{(1-B)(B^2 + 6B + 1)}{8(1+B)^3}, \tag{2.15}$$

$$s = \frac{3 + 2B + 3B^2}{4(1+B)^2}, \tag{2.16}$$

$$u = \frac{(1-B)(2B^2 + B + 8)}{16(1-2B)(1+B)^2}, \tag{2.17}$$

$$v = \frac{3(4B^4 + 4B^3 - 9B^2 + B - 8)}{8(1-2B)^2(1+B)^2}. \tag{2.18}$$

2.2 *Derivation From Zakharov's Integral Equation*

This derivation was first considered by Stiassnie (1984) for gravity waves and by Hogan (1985) for capillary-gravity waves.

Zakharov's integral equation for $B(\mathbf{k}, t)$, which is related to $\eta(\mathbf{x}, t)$ through Equation 2.22, is

$$\begin{aligned} \mathrm{i}\frac{\partial B}{\partial t}(\mathbf{k}, t) = \iiint_{-\infty}^{+\infty} & U(\mathbf{k}, \mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3) \\ & \times B^*(\mathbf{k}_1, t) B(\mathbf{k}_2, t) B(\mathbf{k}_3, t) \delta(\mathbf{k} + \mathbf{k}_1 - \mathbf{k}_2 - \mathbf{k}_3) \\ & \times \exp\{\mathrm{i}[\omega(\mathbf{k}) + \omega(\mathbf{k}_1) - \omega(\mathbf{k}_2) - \omega(\mathbf{k}_3)]t\}\, \mathrm{d}\mathbf{k}_1\, \mathrm{d}\mathbf{k}_2\, \mathrm{d}\mathbf{k}_3, \end{aligned} \tag{2.19}$$

where the wave vector $\mathbf{k}$ and the frequency ω are related through the linear dispersion relation

$$\omega(\mathbf{k}) = (g|\mathbf{k}| + T|\mathbf{k}|^3)^{\frac{1}{2}}.$$

$U(\mathbf{k}, \mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3)$ is a lengthy scalar function given for example in Krasitskii (1990, 1994). Of course this integral equation was derived earlier by Zakharov (1968), but we refer to the work of Krasitskii because he explains well why the so-called Zakharov's equation, which is commonly used, is not Hamiltonian despite the Hamiltonian structure of the exact water-wave equations. This is due to shortcomings of its derivation. In particular, the kernel $U(\mathbf{k}, \mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3)$ must satisfy certain symmetries. Apart from the resonance surface, determined from the equations

$$\mathbf{k} + \mathbf{k}_1 = \mathbf{k}_2 + \mathbf{k}_3, \tag{2.20}$$

$$\omega(\mathbf{k}) + \omega(\mathbf{k}_1) = \omega(\mathbf{k}_2) + \omega(\mathbf{k}_3), \tag{2.21}$$

the kernel can be changed in an arbitrary way. On the resonant surface, the kernel satisfies

$$U(\mathbf{k}, \mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3) = U(\mathbf{k}_1, \mathbf{k}, \mathbf{k}_2, \mathbf{k}_3) = U(\mathbf{k}, \mathbf{k}_1, \mathbf{k}_3, \mathbf{k}_2) = U(\mathbf{k}_2, \mathbf{k}_3, \mathbf{k}, \mathbf{k}_1),$$

which follows from the requirement that the equation is Hamiltonian. There are many ways to make a continuation of these symmetry conditions onto the whole eight-dimensional space $(\mathbf{k}, \mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3)$, the most natural of them being the canonical transformation, which leads automatically to reduced equations with Hamiltonian structure. The "new" equations have been used recently for computational purposes. The key properties of the Zakharov equation are summarized in Krasitskii (1994) and Badulin et al (1995). Concerning the Hamiltonian structure of the water-wave problem, it is worth pointing out the recent work of Craig & Worfolk (1995), who considered the Birkhoff normal form for the water-wave problem. They verified that in the fourth-order normal form, the coefficients vanish for all nongeneric resonant terms, and showed that the resulting truncated system is completely integrable. In contrast they showed that there are resonant fifth-order terms with nonvanishing coefficients, answering to the negative the conjecture of Dyachenko & Zakharov (1994) on the integrability of free-surface hydrodynamics. The same answer was provided independently by Dyachenko et al (1995). Although the present review does not deal with the statistical description of waves, it is worth noting that it was shown recently by Dyachenko & Lvov (1995) that the two approaches describing wave turbulence introduced by Hasselmann (1962, 1963) and by Zakharov (1968), respectively, result in the same kinetic equation for the second-order correlator.

As pointed out by Zakharov (1968), there are difficulties in applying Equation 2.19 to capillary-gravity waves. This is because, unlike gravity waves, these waves can satisfy triad resonances. The condition for triad resonance will give a zero denominator in one of the terms of $U(\mathbf{k}, \mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3)$ corresponding to the second-order interaction. But if the wave packet is sufficiently narrow, then the resonance condition cannot be satisfied.

$B(\mathbf{k}, t)$ is related to the free-surface elevation $\eta(\mathbf{x}, t)$ through the relation

$$\eta(\mathbf{x}, t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \left(\frac{|\mathbf{k}|}{2\omega(\mathbf{k})} \right)^{\frac{1}{2}} \{B(\mathbf{k}, t) \exp\{\mathrm{i}[\mathbf{k} \cdot \mathbf{x} - \omega(\mathbf{k})t]\} + \text{c.c.}\} \, \mathrm{d}\mathbf{k}, \tag{2.22}$$

where c.c. denotes the complex conjugate.

Let $\mathbf{k} = \mathbf{k}_o + \chi$ where $\mathbf{k}_o = (k_o, 0)$ and $\chi = (p, q)$. Let ϵ denote the order of the spectral width $|\chi|/k_o$. Let also $\omega(\mathbf{k}_o) = \omega_o$ and $\chi_i = (p_i, q_i)$, $i = 1, 2, 3$.

Introducing a new variable, $A(\chi, t)$, given by

$$A(\chi, t) = B(\mathbf{k}, t) \exp\{-\mathrm{i}[\omega(\mathbf{k}) - \omega(\mathbf{k}_o)]t\}, \tag{2.23}$$

in Equations 2.19 and 2.22 we get

$$\begin{aligned} \mathrm{i}\frac{\partial A}{\partial t}(\chi, t) - [\omega(\mathbf{k}) - \omega_o]A(\chi, t) \\ = \iiint_{-\infty}^{+\infty} U(\mathbf{k}_o + \chi, \mathbf{k}_o + \chi_1, \mathbf{k}_o + \chi_2, \mathbf{k}_o + \chi_3) \\ \times \delta(\chi + \chi_1 - \chi_2 - \chi_3) A^*(\chi_1) A(\chi_2) A(\chi_3) \, \mathrm{d}\chi_1 \, \mathrm{d}\chi_2 \, \mathrm{d}\chi_3, \end{aligned} \tag{2.24}$$

and

$$\eta(\mathbf{x}, t) = \exp\{\mathrm{i}(k_o x - \omega_o t)\} \frac{1}{2\pi} \int_{-\infty}^{+\infty} \left(\frac{|\mathbf{k}|}{2\omega(\mathbf{k})} \right)^{\frac{1}{2}} A(\chi, t) \mathrm{e}^{i\chi \cdot \mathbf{x}} \, \mathrm{d}\chi + \text{c.c.} \tag{2.25}$$

The Taylor expansion of $|\mathbf{k}_o + \chi|/2\omega(\mathbf{k}_o + \chi)$ in powers of $|\chi|/k_o$ is

$$\left(\frac{|\mathbf{k}_o + \chi|}{2\omega(\mathbf{k}_o + \chi)}\right)^{\frac{1}{2}} = \left(\frac{\omega_o}{2g(1+B)}\right)^{\frac{1}{2}} \left(1 + \frac{p}{4k_o}\frac{(1-B)}{(1+B)}\right), \tag{2.26}$$

in which terms up to order ϵ have been retained.

Substituting Equation 2.26 into Equation 2.25, $\eta(\mathbf{x}, t)$ can be expressed as

$$\eta(\mathbf{x}, t) = \mathrm{Re}\{a(\mathbf{x}, t)\mathrm{e}^{i(k_o x - \omega_o t)}\}, \tag{2.27}$$

where

$$a(\mathbf{x}, t) = \frac{1}{2\pi}\left(\frac{2\omega_o}{g(1+B)}\right)^{\frac{1}{2}} \int_{-\infty}^{+\infty} \left(1 + \frac{p}{4k_o}\frac{(1-B)}{(1+B)}\right) A(\chi, t)\mathrm{e}^{i\chi\cdot\mathbf{x}}\mathrm{d}\chi. \tag{2.28}$$

By Taylor expanding $\omega(\mathbf{k}) - \omega_o$ in powers of $|\chi|/k_o$ and keeping terms up to order ϵ, we get

$$\begin{aligned} \omega(\mathbf{k}) - \omega_o = \frac{1}{2}&\left(\frac{g}{k_o(1+B)}\right)^{\frac{1}{2}} \\ &\times \left\{ p(1+3B) + \frac{p^2}{4k_o}\left(\frac{-1+6B+3B^2}{1+B}\right) + \frac{q^2}{2k_o}(1+3B) \right. \\ &\left. + \frac{p^3}{8k_o^2}\left[\frac{(1-B)(1+6B+B^2)}{(1+B)^2}\right] - \frac{pq^2}{4k_o^2}\left(\frac{3+2B+3B^2}{1+B}\right)\right\}. \end{aligned} \tag{2.29}$$

In Equation 2.24, we substitute the expression 2.29 for $\omega(\mathbf{k}) - \omega_o$. By replacing $A(\chi, t)$ by a $(\mathbf{x}, t)$ and taking the inverse Fourier transform, one finds

$$\begin{aligned} \mathrm{i}\, a_t + \frac{1}{2}&\left(\frac{g}{k_o(1+B)}\right)^{\frac{1}{2}} \\ &\times \left\{ \mathrm{i}(1+3B)a_x + \left(\frac{-1+6B+3B^2}{4k_o(1+B)}\right)a_{xx} + \frac{(1+3B)}{2k_o}a_{yy} \right. \\ &\left. - \mathrm{i}\frac{(1-B)(1+6B+B^2)}{8k_o^2(1+B)^2}a_{xxx} + \mathrm{i}\frac{3+2B+3B^2}{4k_o^2(1+B)}a_{xyy}\right\} \\ &= \frac{1}{2\pi}\left(\frac{2\omega_o}{g(1+B)}\right)^{\frac{1}{2}} \iiint_{-\infty}^{+\infty} \left[1 + \frac{(p_2+p_3-p_1)}{4k_o}\frac{(1-B)}{(1+B)}\right] \\ &\quad \times U(\mathbf{k}_o + \chi_2 + \chi_3 - \chi_1, \mathbf{k}_o + \chi_1, \mathbf{k}_o + \chi_2, \mathbf{k}_o + \chi_3) \\ &\quad \times A^*(\chi_1)A(\chi_2)A(\chi_3)\exp\{\mathrm{i}(\chi_2 + \chi_3 - \chi_1)\cdot\mathbf{x}\}\,\mathrm{d}\chi_1\,\mathrm{d}\chi_2\,\mathrm{d}\chi_3. \end{aligned} \tag{2.30}$$

Now it can be shown that the Taylor expansion of U keeping terms up to order ϵ becomes

$$U(\mathbf{k}_o + \chi_2 + \chi_3 - \chi_1, \mathbf{k}_o + \chi_1, \mathbf{k}_o + \chi_2, \mathbf{k}_o + \chi_3) = \frac{k_o^3}{8\pi^2}$$
$$\times \left[\frac{(8 + B + 2B^2)}{4(1 + B)(1 - 2B)} + \frac{3(p_2 + p_3)}{8k_o} \frac{(8 - B + 9B^2 - 4B^3 - 4B^4)}{(1 + B)^2(1 - 2B)^2} \right.$$
$$\left. - \frac{(p_3 - p_1)^2}{k_o|\chi_1 - \chi_3|} - \frac{(p_2 - p_1)^2}{k_o|\chi_1 - \chi_2|} \right]. \tag{2.31}$$

By using this form of U, we find by using Equation 2.28 that the right side of Equation 2.30 becomes on integration

$$\frac{g}{16\omega_o} \left[\frac{k_o^3(8 + B + 2B^2)}{(1 - 2B)} a|a|^2 - \frac{1}{2}\,\mathrm{i}\,k_o^2 \frac{(1 - B)(8 + B + 2B^2)}{(1 + B)(1 - 2B)} a^2 a_x^* \right.$$
$$\left. - 3\mathrm{i}\,k_o^2 \frac{(8 - B + 9B^2 - 4B^3 - 4B^4)}{(1 + B)(1 - 2B)^2} |a|^2 a_x \right] - \frac{k_o^2}{4\pi^2} a I, \tag{2.32}$$

where

$$I = \iint_{-\infty}^{+\infty} \frac{(p_1 - p_2)^2}{|\chi_1 - \chi_2|} A^*(\chi_1) A(\chi_2) \mathrm{e}^{i(\chi_2 - \chi_1)\cdot\mathbf{x}} \,\mathrm{d}\chi_1 \mathrm{d}\chi_2. \tag{2.33}$$

It can be shown that

$$I = \left[\frac{g(1 + B)}{2\omega_o} \right] 2\pi \int_{-\infty}^{+\infty} \frac{\partial}{\partial \xi}(|a|^2) \frac{(x - \xi)}{|\mathbf{x} - \xi|^3} \,\mathrm{d}\xi. \tag{2.34}$$

The integral I can be related to the mean-flow velocity potential $\bar{\phi}$ (2.11). Collecting the results from Equations 2.30 and 2.34, together with those from the expression 2.32, and making the same scaling transformation as in the previous subsection leads to Equation 2.10.

3. BIFURCATIONS OF WATER WAVES WHEN THE PHASE AND GROUP VELOCITIES ARE NEARLY EQUAL

The dispersion relation for capillary-gravity waves on the surface of a deep layer of water is given by

$$c^2 = \frac{g}{k} + Tk, \tag{3.1}$$

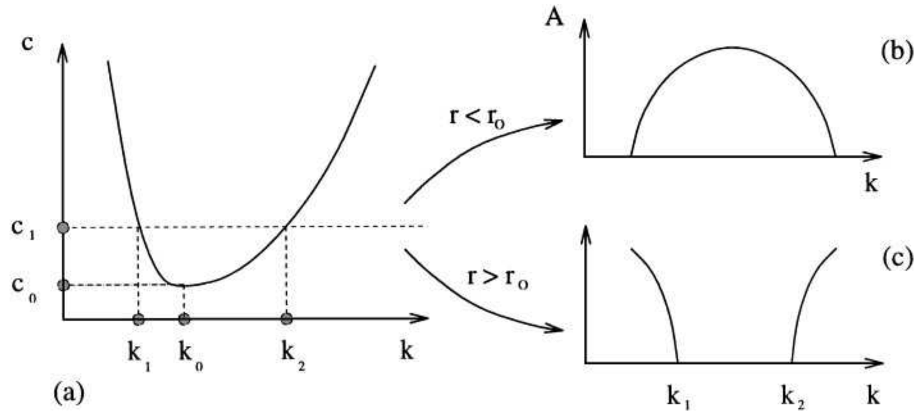

Figure 1 Local bifurcation of periodic traveling waves near the critical point (k_o, c_o): (*a*) dispersion curve for the linear problem, (*b*) global loop in the nonlinear problem when $r < r_o = \sqrt{5}/4$ and $c > c_o$, (*c*) branches of nonlinear traveling waves when $r > r_o$ and $c > c_o$ (from Bridges et al 1995).

and is plotted in Figure 1*a*. A trivial property of this dispersion relation is that it exhibits a minimum c_o. For water this minimum is reached at $k = 3.63$ cm^{-1} (or $L = 1.73$ cm). The corresponding speed and frequency are $c = 23.2$ cm/s and $f = 13.4$ Hz. Of course the presence of this minimum is obvious, but surprisingly it was only quite recently that some of its consequences were discovered. In the classic textbooks (Lamb 1932, pp. 462–68; Lighthill 1978, pp. 260–69; Whitham 1974, pp. 407–08, pp. 446–54; Milne-Thomson 1968, pp. 447–49; Stoker 1957), the presence of this minimum is of course mentioned. It represents a real difficulty for linearized versions of the water-wave problem because it leads to small denominators. For example, consider the fishing-rod problem, in which a uniform current is perturbed by an obstacle. Rayleigh (1883) investigated this problem. He assumed a distribution of pressure of small magnitude and linearized the equations around a uniform stream with constant velocity c. He solved the resulting linear equations in closed form. For $c = c_1(> c_o)$, the solutions are characterized by trains of waves in the far field of wavenumbers k_2 and $k_1 < k_2$. The waves corresponding to k_1 and k_2 appear behind and ahead of the obstacle, respectively. The asymptotic wave trains are given by

$$\eta \sim -\frac{2P}{(k_2 - k_1)T} \sin(k_1 x), \quad x > 0,$$

$$\eta \sim -\frac{2P}{(k_2 - k_1)T} \sin(k_2 x), \quad x < 0,$$

where P is the integral of pressure. For $c < c_o$, Rayleigh's solutions do not predict waves in the far field, and the flow approaches a uniform stream with constant velocity c at infinity. This is consistent with the fact that Equation 3.1 does not have real roots for k when $c < c_o$. Rayleigh's solution is accurate for $c \neq c_o$ in the limit as the magnitude of the pressure distribution approaches zero. However, it is not uniform as $c \rightarrow c_o$. It is clear that the linearized theory fails as one approaches c_o: The two wavenumbers k_1 and k_2 merge, the denominators approach zero, and the displacement of the free surface becomes unbounded. Therefore there was a clear need for a better understanding of the limiting process, but for several decades this problem was left untouched. In the late 1980s and early 1990s, several researchers worked on the nonlinear version of this problem independently. Longuet-Higgins (1989) indirectly touched on this problem with numerical computations. Iooss & Kirchgässner (1990) tackled the problem mathematically and realized that it was a 1:1 resonance problem. Vanden-Broeck & Dias (1992) made the link between the numerical computations of Longuet-Higgins (1989) and the mathematical analysis of Iooss & Kirchgässner (1990). Benjamin (1992, 1996) considered the same resonance for interfacial waves. One can say that the mathematical results shed some light on the difficulty: There is a difference between a temporal approach and a spatial approach. Roughly speaking, in temporal bifurcation theory the wavenumber k is treated as a given real parameter, whereas in spatial bifurcation theory the wavespeed c is treated as a given real parameter. Look at Figure 1, where the effect of stratification has been added to make our point clearer: It is not the same point of view if one fixes k or if one fixes c. Consider periodic waves. Cut at a fixed value of c (say c_1) above c_o. For water waves or for interfacial waves with $r > r_o = \sqrt{5}/4$, the branching behavior for periodic solutions is shown in Figure 1*c*. As r decreases (and in particular in the Boussinesq limit, where both densities are close to each other), the branching behavior becomes quite different and is shown in Figure 1*b*.

Now let c vary through the critical value c_o. In the case $r > r_o$, which includes water waves, one gets the sequence of branching behaviors shown at the bottom of Figure 2. In the case $r < r_o$, which includes the Boussinesq limit, one gets the sequence of branching behaviors shown at the top of Figure 2.

If one repeats the analysis at a fixed value of k, one will not be able to see any difference between interfacial and surface waves! In particular, the temporal approach does not allow one to find the detached branch of waves for $r > r_o$, $c < c_o$. Therefore, it may be useful to consider the water-wave problem as a dynamical system with space as the evolution variable (as opposed to time). Of course, one can work directly with the full water-wave equations, but the best way to understand what happens is to consider a model equation. The appropriate model equation is the nonlinear Schrödinger equation for the

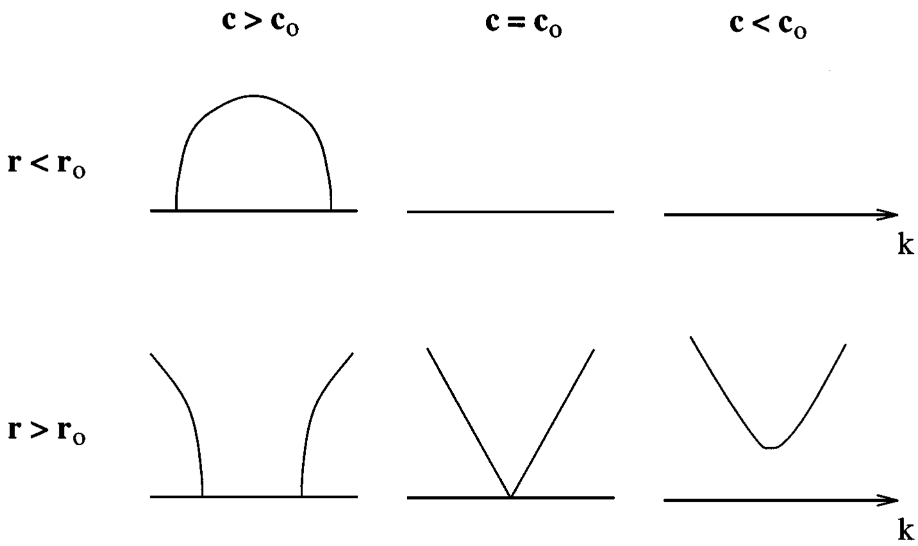

Figure 2 Bifurcation of periodic traveling waves near the critical point (k_o, c_o) when $r \neq r_o$. The case $c > c_o$ was explained in Figure 1. For $r < r_o$, there are no periodic solutions with $c < c_o$. For $r > r_o$, the branches of periodic solutions coalesce at $c = c_o$ and become a detached branch for $c < c_o$. These new periodic solutions exist only at finite amplitude.

amplitude A of a modulated wave train (see left-hand side of Equation 2.10):

$$2\mathrm{i}\frac{\partial A}{\partial t_2} + p\frac{\partial^2 A}{\partial X^2} + q\frac{\partial^2 A}{\partial y_1^2} + \gamma A|A|^2 = 0. \tag{3.2}$$

Akylas (1993) and Longuet-Higgins (1993) showed that, for values of c less than c_o, Equation 3.2 admits particular envelope-soliton solutions, such that the wave crests are stationary in the reference frame of the wave envelope. These solitary waves, which bifurcate from linear periodic waves at the minimum value of the phase speed, have decaying oscillatory tails and are sometimes called "bright" solitary waves. More generally, one can look for stationary solutions of Equation 3.2. Now using T/c^2 as unit length and T/c^3 as unit time, allowing for interfacial waves, considering waves without y_1-variations and evaluating the coefficients p and γ at $c = c_o$, one can show that these stationary solutions satisfy the equation

$$-\frac{2r}{1+r}\mu A + A_{XX} + \frac{16r^2 - 5}{2(1+r)^2}A|A|^2 = 0, \tag{3.3}$$

where the bifurcation parameter μ is defined by $\mu = \alpha - \alpha_o$. The parameter α was introduced in Table 2. At $c = c_o$, it is equal to $\alpha_o = 1/2r(1+r)$. The corresponding profile for the modulated wave is given by

$$\eta(X) = 2(1+r)\mathrm{Re}[A(X)\exp(\mathrm{i}X/(1+r))].$$

Introduce the scaling $|\mu|^{\frac{1}{2}}\tilde{A} = A$, $\tilde{X} = |\mu|^{\frac{1}{2}}(2r/(1+r))^{\frac{1}{2}}X$, and the coefficient

$$\tilde{\gamma} = \frac{16r^2-5}{4r(1+r)}.$$

The resulting equation is

$$\mathrm{sgn}(\mu)\tilde{A} - \tilde{A}_{\tilde{X}\tilde{X}} - \tilde{\gamma}\tilde{A}|\tilde{A}|^2 = 0. \tag{3.4}$$

Writing $\tilde{A} = s(\tilde{X})\mathrm{e}^{i\theta(\tilde{X})}$ leads to

$$s_{\tilde{X}\tilde{X}} - \mathrm{sgn}(\mu)s + \tilde{\gamma}s^3 - s(\theta_{\tilde{X}})^2 = 0, \tag{3.5}$$

$$2\theta_{\tilde{X}}s_{\tilde{X}} + s\theta_{\tilde{X}\tilde{X}} = 0. \tag{3.6}$$

The system has two first integrals, I_1 and I_2, defined as follows:

$$u\theta_{\tilde{X}} = I_1, \tag{3.7}$$

$$\frac{1}{4}(u_{\tilde{X}})^2 = \mathrm{sgn}(\mu)u^2 - \frac{1}{2}\tilde{\gamma}u^3 - I_1^2 + I_2u, \tag{3.8}$$

where $u \equiv s^2$. These two integrals are related to the energy flux and flow force, respectively, as shown by Bridges et al (1995).

For a full description of all the bounded solutions of Equation 3.4, one can refer to Iooss & Pérouème (1993) and Dias & Iooss (1993, 1996). There are four cases to consider:

- $r < r_o$, $c > c_o$: there are periodic solutions (see Figure 2), quasiperiodic solutions and solitary waves, homoclinic to the same periodic wave with a phase shift at $+\infty$ and $-\infty$ (these homoclinic solutions are sometimes called dark solitary waves if the amplitude vanishes at the origin and grey solitary waves if it does not). A dark solitary wave is plotted in Figure 4.
- $r < r_o$, $c < c_o$: there are no bounded solutions.
- $r > r_o$, $c > c_o$: there are periodic solutions (see Figure 2) and quasiperiodic solutions.
- $r > r_o$, $c < c_o$: there are periodic solutions (of finite amplitude only), quasi-periodic solutions and solitary waves, homoclinic to the rest state (such a solitary wave is plotted in Figure 3).

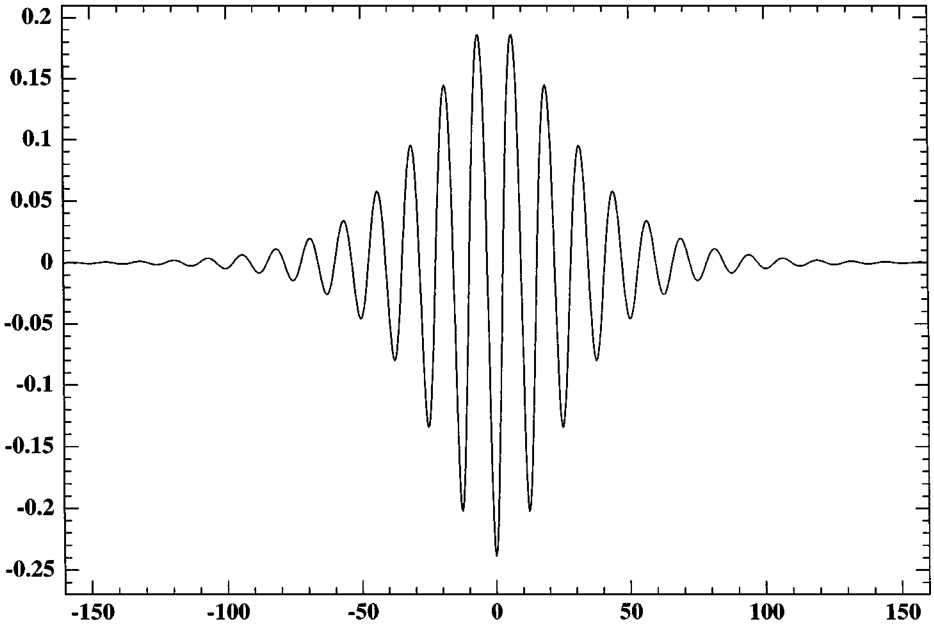

Figure 3 Profile of a solitary wave in deep water. The horizontal axis is X and the vertical axis is η.

When $r > r_o$, $c < c_o$ $(\mu > 0)$, there are bright solitary waves, the envelope of which is given by

$$\tilde{A} = \pm\frac{\sqrt{2}}{\sqrt{\tilde{\gamma}}\cosh\tilde{X}}.$$

This case includes water waves. The elevation of the solitary wave for water waves is given by

$$\eta(X) = \pm\frac{16\sqrt{\mu}}{\sqrt{11}}\frac{\cos(X/2)}{\cosh\sqrt{\mu}X}. \tag{3.9}$$

When $r < r_o$, $c > c_o$ $(\mu < 0)$, there is a one-parameter family of grey solitary waves (Dias & Iooss 1996, Laget & Dias 1997). The "darkest" one, which is such that the amplitude vanishes at the origin, has an envelope given by

$$s = |\tilde{\gamma}|^{-\frac{1}{2}}\tanh\left(\frac{|\tilde{X}|}{\sqrt{2}}\right).$$

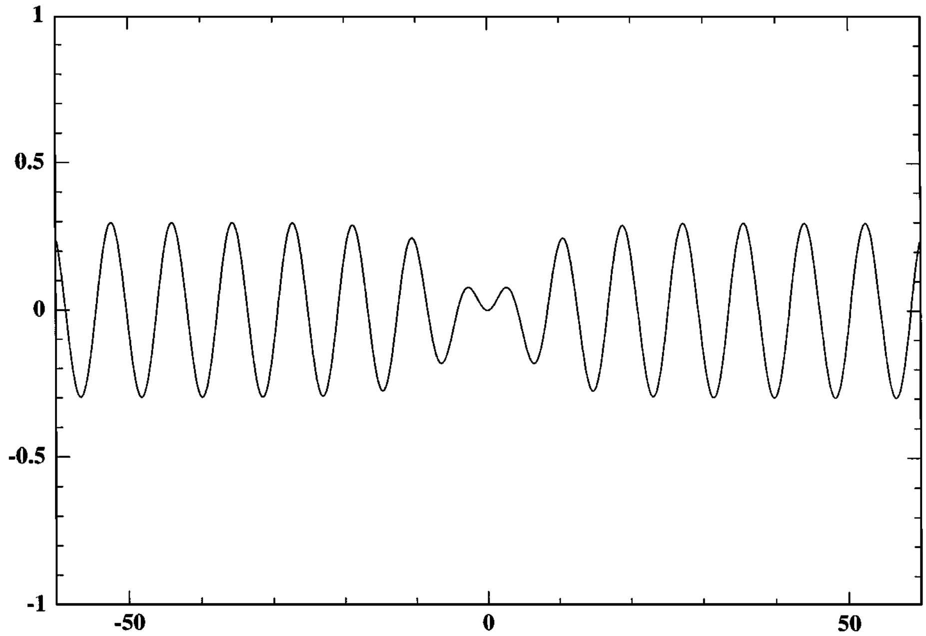

Figure 4 Profile of a dark solitary interfacial wave. The horizontal axis is *X* and the vertical axis is η.

The elevation of the dark solitary wave is given by

$$\eta(X) = \pm 4\sqrt{\frac{r(1+r)^3}{5-16r^2}}\tanh\left(\sqrt{\frac{r}{1+r}}|\mu|^{\frac{1}{2}}X\right)\sin\left(\frac{1}{1+r}X\right). \qquad (3.10)$$

The results obtained on the NLS equation also apply to the full interfacial wave problem. In particular, envelope-soliton solutions have been studied in detail by Longuet-Higgins (1989), Vanden-Broeck & Dias (1992), Dias et al (1996), and Dias & Iooss (1993).

Supporting the asymptotic and numerical studies cited above, Iooss & Kirchgässner (1990) provided a rigorous proof, based on center-manifold reduction, for the existence of small-amplitude symmetric solitary waves near the minimum phase speed in water of finite depth. The proof could not be extended to the infinite-depth case, however. Later, Iooss & Kirrmann (1996) managed to handle this difficulty by following a different reduction procedure, which also brought out the fact that the solitary-wave tails behave differently in water of infinite depth, their decay being slower than exponential, although the precise decay rate could not be determined. By assuming the presence of an algebraic decay, Sun (1997) was able to show that the profiles of interfacial solitary waves in deep fluids must decay like $1/x^2$ at the tails. Earlier, Longuet-Higgins (1989) had inferred such a decay on physical grounds for deep-water solitary waves. Akylas et al (1998) showed that the profile of these gravity-capillary solitary waves actually decays algebraically (like $1/x^2$) at infinity, owing to the induced mean flow that is not accounted for in the NLS equation. Moreover, the same behavior was found at the tails of solitary-wave solutions of the model equation proposed by Benjamin (1992) for interfacial waves in a two-fluid system. Another property of these waves in infinite depth is that the net mass is equal to zero (Longuet-Higgins 1989).

There are a few experiments available, showing that some of these waves can be observed in laboratory experiments (Zhang 1995, Longuet-Higgins & Zhang 1997). Note that experiments can be performed in ferromagnetic fluids as well (Browaeys et al, submitted).

One fact of importance is that these solitary waves can be found even where one does not expect them. Dias & Iooss (1996) performed the unfolding of the singularity $\tilde{\gamma} = 0$ (or $r = r_o$) and obtained a modified NLS equation similar to the equation obtained by Johnson (1977) for gravity waves near the critical depth where the Benjamin-Feir instability disappears. This modified equation admits nontrivial solitary-wave solutions, with algebraic decay at infinity (Iooss 1997a), in the region $r < r_o$. Laget & Dias (1997) computed numerically bright solitary waves in the region $r < r_o$ on the full Euler equations.

The NLS equation also admits asymmetric solitary waves, obtained by shifting the carrier oscillations relative to the envelope of a symmetric solitary wave. Yang & Akylas (1997) examined the fifth-order Korteweg–de Vries equation, a model equation for gravity-capillary waves on water of finite depth, and showed by using techniques of exponential asymptotics beyond all orders that asymmetric solitary waves of the form suggested by Equation 3.3 are not possible. On the other hand, an infinity of symmetric and asymmetric solitary waves, in the form of two or more NLS solitary wavepackets, exist at finite amplitude (see also Buffoni et al 1996).

The spatial approach described in this section has been used to study the water-wave problem in other parameter regimes. In particular, it was found that the well-known gravity solitary waves that exist when the Froude number is slightly larger than 1 become generalized as soon as surface tension is added (Iooss & Kirchgässner 1992, Sun 1991, Beale 1991, Vanden-Broeck 1991, Sun & Shen 1993, Yang & Akylas 1996, Lombardi 1997, 1998). Generalized means that ripples of small constant amplitude are superimposed on the solitary waves in their tails. When surface tension is large enough, the solitary waves exist when the Froude is slightly smaller than 1 and become classical again. Another interesting application of the spatial approach is the study of waves that are quasiperiodic in space (Bridges & Dias 1996).

The implications of the study of the 1:1 resonance in the context of water waves have gone far beyond the field of surface waves. Applications have been given to all sorts of problems in physics, mechanics, thermodynamics, and optics since these studies on water waves.

Near the minimum, there is also an interesting behavior for wave resistance: This time the linearized theory does not blow up as $c \to c_o$, but a jump occurs (Webster 1966, Raphaël & de Gennes 1996). To our knowledge, the effects of nonlinearity on the jump in wave resistance have not been studied yet.

In this section, the importance of a spatial approach was emphasized. Ideally, one would like to work on a formulation of the problem in which a temporal approach and a spatial approach are present simultaneously. This was accomplished recently by Bridges (1996, 1997), who introduced a multisymplectic formulation for the water-wave problem. This topic is outside the scope of the present review.

4. THE FREQUENCY DOWNSHIFT PHENOMENON

The Benjamin-Feir instability results from a quartet resonance, that is, a resonant interaction between four components of the wave field. According to the classification of McLean (1982), this instability belongs to class I ($m = 1$) interactions that are predominantly 2D. It is well known that the Benjamin-Feir instability is at the origin of the frequency downshift phenomenon observed by Lake et al (1977).

Their experimental and theoretical investigation showed that the evolution of a 2D nonlinear wave train on deep water, in the absence of dissipative effects, exhibits the Fermi-Pasta-Ulam (FPU) recurrence phenomenon. This phenomenon is characterized by a series of modulation-demodulation cycles in which initially uniform wave trains become modulated and then demodulated until they are again uniform. Modulation is caused by the growth of the two dominant sidebands of the Benjamin-Feir instability at the expense of the carrier. During the demodulation, the energy returns to the components of the original wave train (carrier, sidebands, harmonics). However, when the initial steepness is large enough, the long-time evolution of the wave train is different. The evolving wave trains experience strong modulations followed by demodulations, but a careful inspection of the new nearly uniform wave trains reveals that the dominant component is the component at the frequency of the lower sideband of the original carrier k_o (1, 0). This is the frequency downshift phenomenon. While the FPU recurrence process is well described by the classical NLS equation, the frequency downshift phenomenon is not predicted by solutions of this equation. Lake et al (1977) suggested that the shift to lower carrier frequencies they observed in their experiments might be attributed to the effects of dissipation. They mentioned two candidates: formation of capillary waves and breaking waves (spilling breaker) when wave trains become so strongly modulated that individual waves become steep enough to generate these violent but localized activities. Later on, Melville (1982), Su et al (1982), and Huang et al (1996) observed, in a wave tank, frequency downshift in the wave field evolution. The asymmetrical behavior of the two sidebands lies beyond the realm of applicability of the NLS equation. The Dysthe equation, which is Equation 2.10 with $B = 0$, exhibits an asymmetrical evolution of the sidebands when the modulation is the greatest and a symmetrical evolution during the demodulation of the wave train. The greatest difference between the two satellites occurs when both attain their maxima; the lower sideband is the dominant component of the wave train, and a temporary downshift is then observed. The time histories of the normalized amplitude of the carrier (*solid line*), lower sideband (*dashed line*), and upper sideband (*dashed-dotted line*) are plotted in Figure 5. The initial condition is a Stokes wave disturbed by its most unstable perturbation. The fundamental wavenumber of the Stokes wave is $k_o(1, 0)$, and the dominant

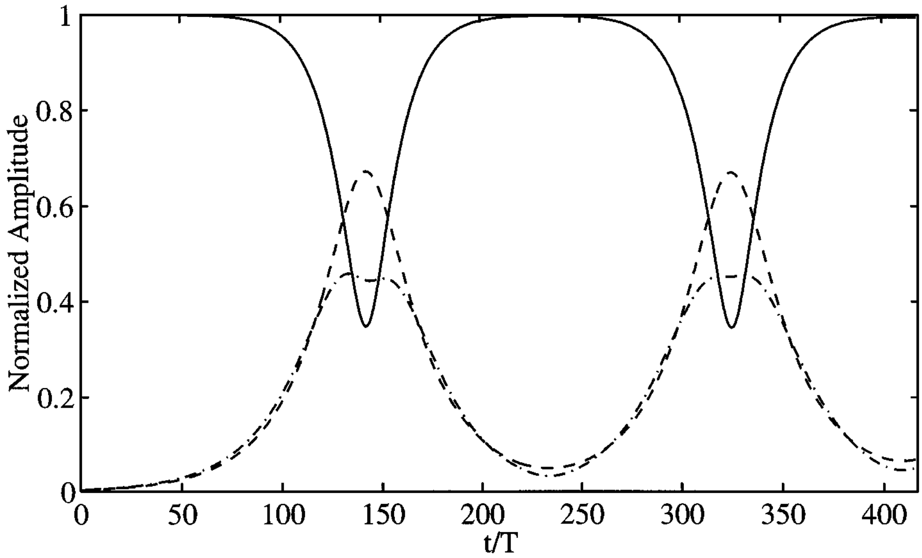

Figure 5 Time histories of the amplitude of the fundamental, subharmonic, and superharmonic modes ($p = \frac{2}{9}$) for an evolving perturbed Stokes wave of initial steepness $\epsilon = 0.13$ and fundamental period T.

sidebands are $k_o(1-p,\,0)$ and $k_o(1+p,\,0)$ for the subharmonic and superharmonic parts of the perturbation, respectively. There exist higher harmonics present in the interactions, which are not plotted in Figure 5.

Trulsen & Dysthe (1990) were the first to find numerically a permanent downshift. They added to the right-hand side of the Dysthe equation a source term of the form

$$S = \frac{1}{\tau} A \left[\left(\frac{|A|}{A_0} \right)^r - 1 \right] H(|A| - A_0), \tag{4.1}$$

where H is the Heaviside unit step function and A_0 is the critical value of the steepness at which breaking is first observed. The parameters r and τ define the relaxation time by which $|A|$ relaxes towards A_0. Dissipation is introduced heuristically by imposing that all waves exceeding a critical height lose their excess of energy. They pointed out the tendency toward spatial localization of the part of the wave train contributing to the upper satellite around the strongest modulated waves where breaking occurs. The breaking then damps the developing sidebands selectively, such that the lower satellite comes out of the modulation-breaking process. In the framework of conservative evolution of a train of Stokes waves, other authors, using either approximate equations (Dysthe equation or Zakharov equation) or the exact hydrodynamic equations, did not find permanent downshift. This emphasizes the key role of nonconservative effects in the subharmonic transition of nonlinear 2D water waves.

A well-known feature in the evolution of spectra of wind waves is that as the fetch increases, the peak of the spectrum shifts to lower frequencies and increases in energy, while the form of the spectrum remains quasisimilar. Hara & Mei (1991) developed a 2D model for the effect of moderate wind on the long-time evolution of a narrow-banded gravity wave train. Eddy viscosity models were chosen for turbulence in air and water. The weak nonconservative effects they introduced (wind input and dissipation caused by eddy viscosity in water) occur over the same time scale as the asymmetric evolution of the wave spectrum (fourth order in wave steepness). Using a boundary-layer correction, they derived a modified Dysthe equation taking into account wind forcing, wind-induced current, and turbulent dissipation. They found the amplification of the sidebands to be sensitive to the profile of the steady and horizontally uniform wind-induced current in water. Their results agreed with the experiments of Bliven et al (1986), when they used a log-linear basic shear flow in water (linear in height near the interface and logarithmic high above). Note that the observational conditions (wave steepness and wind stress) were beyond the domain of applicability of the theory. For different values of the friction velocity and wave steepness, numerical simulations of the nonlinear evolution of Benjamin-Feir instability of weakly nonlinear wave trains exhibit a permanent frequency downshift. To avoid breaking, the theory does not apply to strong wind. Hara & Mei (1994) extended their model to the nonlinear evolution of slowly varying gravity-capillary waves. They used the same assumptions except that the wave-induced flow in both media was assumed to be laminar. A fourth-order evolution equation was derived, including wind forcing and dissipation, similar to Equation 2.10 when these two effects are neglected. The nonlinear development of modulational instability gives rise to persistent frequency downshift for relatively long gravity-capillarity waves and frequency upshift for very short waves. It would be desirable to extend the model of Hara & Mei to larger wave steepnesses based on the exact hydrodynamics equations so that more quantitative comparison can be made with the laboratory experiments of Bliven et al (1986). More universal models of turbulence than the one used by Hara & Mei could be applied to turbulence near the interface. Trulsen & Dysthe (1992) introduced the effects of wind and breaking in an ad hoc manner by adding two source terms to the right-hand side of the Dysthe equation. For strong winds, they found the modulational instability to disappear altogether, as shown experimentally by Bliven et al (1986) and Li et al (1987). The latter authors observed that the sideband growth was enhanced by a weak wind.

A nonlinear Schrödinger equation with higher-order correction terms, derived in nonlinear optics, was used by Uchiyama & Kawahara (1994) to investigate the frequency downshift in a uniform wave train. It was found that the term responsible for the damping of induced mean flow causes the frequency downshift. Kato & Oikawa (1995) added a nonlinear damping term to the Dysthe equation. This term was not derived from the water-wave equations, but they believed that such a model equation could help to better understand the behavior of the spectral components of the modulated wave train, including water waves. Numerical results suggested that two factors are essential: The first is the level of nonlinearity of the uniform wave train to produce sufficient asymmetry in the spectrum, and the second is the nonlinear dissipation that affects the higher components when strong modulations prevail. A similar explanation was given by Poitevin & Kharif (1991) based on the exact boundary conditions when surface tension and viscous effects are considered simultaneously. Skandrani et al (1996) improved the model of Poitevin & Kharif by taking into account the vorticity generated by viscosity in the vicinity of the free surface. For free-surface problems in water, viscous effects are generally weak, producing a thin rotational layer adjacent to the potential flow. For low viscosity, Ruvinsky et al (1991) derived from the linearized vorticity equation a simple evolution equation for the vortical component of fluid velocity. The weak viscous effect is incorporated into the two boundary conditions, and the problem is formulated in a quasipotential approximation. Skandrani et al (1996) applied the numerical method developed by Dommermuth & Yue (1987) to nonlinear gravity waves, in the presence of weak viscous effects and surface tension. This method is based on the mode-coupling idea but is generalized to include interactions up to an arbitrary order in wave steepness. The long-time evolution of the wave train depends on three dimensionless parameters, the Reynolds number, the Bond number, and the initial steepness of the wave train, $\epsilon = ak$. The parameters R, B, and ϵ characterize the effects of viscosity, surface tension, and nonlinearity, respectively. For $\epsilon = 0.13$, numerical experiments performed with viscosity or surface tension or both do not exhibit the frequency downshift phenomenon. Unlike the results in Figure 5, typical of approximate equations, direct numerical simulations without viscosity and surface tension break down after a few modulation-demodulation cycles owing to possible local breaking phenomena. These results agree with those of Dold & Peregrine (1986) and Banner & Tian (1996). For a relatively large initial steepness of the Stokes wave, Skandrani et al (1996) observed a downshift when effects of both surface tension and viscosity are considered. They chose a uniform train of Stokes waves of initial steepness $\epsilon = 0.20$ and wavelength $L = 10$ cm, comparable to those same parameters in the experiments of Lake et al (1977), disturbed by its most unstable perturbation. Time histories of the normalized amplitude of the carrier, subharmonic, and superharmonic sidebands are shown in Figure 6. The

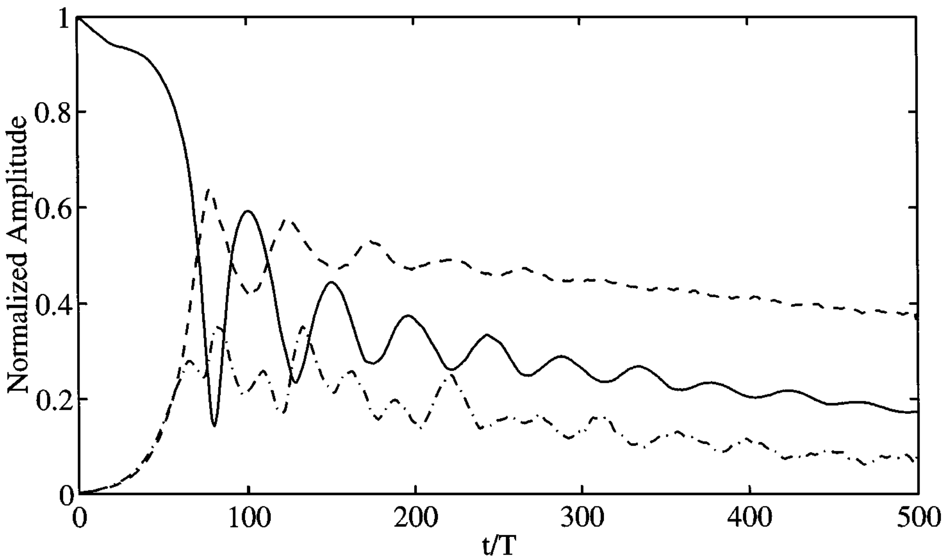

Figure 6 Time histories of the amplitude of the fundamental, subharmonic, and superharmonic modes ($p = \frac{1}{3}$) for an evolving perturbed train of Stokes waves of initial steepness $\epsilon = 0.20$, wavelength $L = 10$ cm, and fundamental period T.

fundamental wavenumber of the initial Stokes wave is $k_o(1, 0)$ (*solid line*), and the dominant sidebands are $k_o(1-p, 0)$ (*dashed line*) and $k_o(1+p, 0)$ (*dashed-dotted line*). The shift to the lower sideband component occurs after the first modulation-demodulation cycle and persists during the evolution of the train. A strong nonlinear damping is observed during the first and second modulation, owing to the generation of high harmonics on the steepest parts of the train by interaction between the fundamental and the upper sideband. The nonlinear damping of the energy is lower when capillarity is neglected. Generation of these small scales, which are strongly damped by viscosity, is enhanced by capillarity and prevents breaking of the waves.

In conclusion, all the previous analyses show that the frequency downshift of 2D trains of Stokes waves is mainly caused by the presence of nonconservative effects, namely damping effects. Surface tension and viscous effects are found to play a significant role in the frequency downshift of short gravity waves, whereas breaking appears to be the main mechanism responsible for the subharmonic transition of longer waves. Using a boundary-integral method for solving the exact hydrodynamics equations, Okamura (1996) confirmed this result for the long-time evolution of nonlinear 2D standing waves in deep water. However, the recent work of Trulsen & Dysthe (1997) on 3D wave trains seems to prove that such an effect is not necessary to produce the downshift. From the Dysthe equation modified for broader bandwidth, they observed the frequency downshift for conservative evolutions of 3D wave trains in deep basin. The study concerns confined motions in the transverse direction and raises the following question: Is the downshift observable for full 3D wave fields in the absence of dissipation? However, in their conclusion, Trulsen & Dysthe emphasize that the explanation of the downshift probably involves effects of both 3D nonlinear modulation and damping (wave breaking and dissipation). Before the study of Trulsen & Dysthe (1997), the NLS equation was extended to the evolution of 3D wave fields. Martin & Yuen (1980) found that leakage to high-frequency modes makes the 2D NLS equation inadequate for the description of the evolution of weakly nonlinear 3D deep-water waves. Unlike the cubic NLS equation, Dysthe's fourth-order equation was found by Lo & Mei (1987) to suppress leakage of energy even in the nonlinear stage. However, they did not observe any frequency downshift.

Melville (1983) pointed out experimentally that the crest-pairing phenomenon described by Ramamonjiarisoa & Mollo-Christensen (1979) corresponds to a phase reversal or phase jump. He emphasized the possible role of these phase jumps in the shift to lower frequency after wave breaking. Based on experimental observations, Huang et al (1996) found that the frequency downshift of mechanically generated 2D gravity waves was an accumulation of wave fusion events similar to the crest-pairing phenomenon. Based on the work of

Hara & Mei (1991), who excluded breaking to obtain frequency downshift, Huang et al suggested that breaking was not necessary for frequency downshifting. Nevertheless, Hara & Mei needed damping by dissipation to simulate permanent frequency downshift. Dold & Peregrine (1986) found numerically that at maximum modulation there is normally one wave crest lost. However, they did not observe a permanent downshift, and their computations support the hypothesis that breaking is essential for 2D frequency downshift. Very recently, Trulsen (1998) emphasized that crest pairing could happen without a concomitant frequency shift.

5. WATER-WAVE HORSESHOE PATTERNS

For sufficiently steep Stokes waves, it was shown by McLean et al (1981) and McLean (1982) that the dominant instability becomes a 3D perturbation of class II ($m = 1$). This instability results from a resonant interaction between five components of the wave field and was first discovered by Longuet-Higgins (1978b) for purely 2D perturbations. Su et al (1982) and Su (1982) performed a series of experiments on instabilities of gravity-wave trains of large steepness in deep water in a long tow tank and a wide basin. They observed 3D structures corresponding to the nonlinear evolution of the dominant instability discovered by McLean et al (1981). The initial 2D wave train of large steepness evolves into a series of 3D spilling breakers, followed by a transition to a more or less 2D wave train. These 3D patterns take the form of crescent-shaped perturbations riding on the basic waves. In the wide basin, obliquely propagating wave groups were generated during the transition from 3D breakers to 2D wave forms. These oblique wave groups were not seen in the tank. Further experimental investigations were described by Melville (1982). Shemer & Stiassnie (1985) used the modified Zakharov equation to study the long-time evolution of class II ($m = 1$) instability of surface gravity waves in deep water. They derived from this equation the long-time history of the amplitudes of the components of the wave field composed of a Stokes wave (carrier) and its most unstable, initially infinitesimal disturbance. They found that a kind of FPU recurrence phenomenon, similar to that reported for class I instability in the previous section, also exists for class II instability and speculated that the growth of the crescent-shaped waves and their disappearance were one cycle of the recurring phenomenon observed in experiments (Su et al 1982). A 3D perspective plot of the crescent structures obtained from direct numerical simulation of the exact hydrodynamics equations for inviscid deep-water waves is displayed in Figure 7 and gives an idea of how these patterns look. The contour plot of the 3D structures riding on the crests of the waves exhibits the front-back asymmetry. In the case of a conservative system, it is worth noting that the crescent-shaped

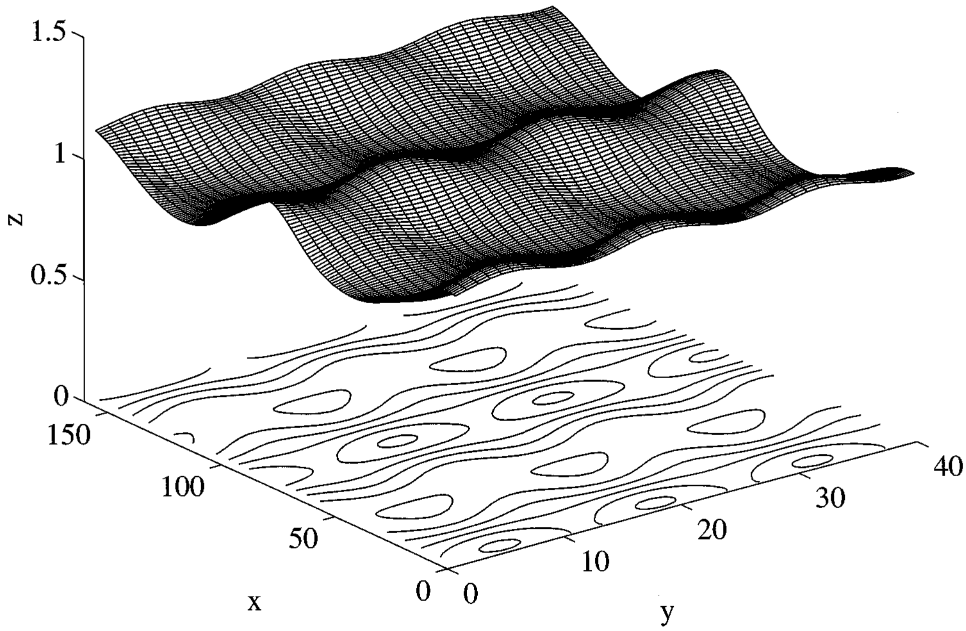

Figure 7 3D wave pattern that has evolved from a uniform Stokes wave train ($\epsilon = 0.23$) by instability of class II ($m = 1$) with $p = \frac{1}{2}$ and $q = \frac{3}{2}$. The waves are propagating from right to left. The horseshoe patterns are in their phase of growth and have their wave fronts oriented forward. Drawing by C Skandrani (1997).

patterns do not present permanently forward-oriented wave fronts during their time evolution.

Later, Stiassnie & Shemer (1987) examined the coupled evolution of class I and class II instabilities and found, in contrast to single class (I or II) evolution, that the coupled behavior was nonperiodic. In addition they observed, except for the very steep waves, a dominance of the class I interactions over those of the class II. The horseshoe or crescent-shaped patterns may also be produced in wave tank experiments in the presence of wind (Kusuba & Mitsuyasu 1986). They are 3D structures quite commonly observable on the sea surface under the effect of wind. Despite their common character and ease of observation, there are only a few experimental and theoretical studies of their formation and persistence. The horseshoe patterns present the following features: (*a*) they can be observed at an early stage of wave development when a fresh wind blows over the sea surface, (*b*) they occur in the range of short gravity waves and are relatively long-lived, i.e. their characteristic time is much greater than the wave period, (*c*) they are rather steep with sharpened crests and flattened troughs, (*d*) they have front-back asymmetry, i.e. the front slopes are steeper than the rear ones. The most notable feature is the specific horseshoe shape of the wave fronts, always oriented forward. Shrira et al (1996) developed a model that explains the persistent character of the patterns and their specific front-back asymmetry. The model was derived from the integro-differential formulation of water-wave equations, i.e. Zakharov's equation, modified by taking into account small, nonconservative effects. They showed that the main physical processes that participate in the generation and persistence of the 3D structures often seen on the sea surface are quintet resonant interactions, wind input, and dissipation, balancing each other. They demonstrated that the persistence of these patterns was due to the processes of dissipation and generation.

Using Zakharov's equation, Saffman & Yuen (1980) found a new class of 3D deep-water gravity waves of permanent form. The solutions were obtained as bifurcations from plane Stokes waves. It was pointed out that the bifurcation is degenerate since there are two families of 3D waves, one symmetric about the direction of propagation and the other skewed. The first type of bifurcation gives rise to a steady symmetric wave pattern propagating in the same direction as the Stokes waves, whereas the second type gives rise to the steady skew wave patterns that propagate obliquely from the direction of the Stokes waves. Later, Meiron et al (1982) computed steady 3D symmetric wave patterns from the full water-wave equations as well as from the approximate Zakharov equation. These 3D waves always have symmetric fronts for both weakly nonlinear and exact equations.

Su (1982) observed the formation of the bifurcated 3D symmetric waves when the wave steepness, ϵ, was in the range 0.25–0.33. It seems plausible to assume that the 3D patterns most likely to emerge are those that are closely connected with the most unstable perturbations of plane waves, i.e. transverse disturbances. These most unstable perturbations, which are phase-locked with the plane wave, ensure the possibility of permanent 3D forms.

There is some confusion related to the term *skew pattern*. Su also observed unsteady skew wave patterns in the range 0.16–0.18, while Bryant (1985) computed doubly periodic progressive permanent skew waves in deep water, in the framework of the exact inviscid equations. Despite a certain similarity in appearance between his wave patterns and those of Su, it should be emphasized that the skew wave patterns generated experimentally by Su were part of an evolving wave field, rather than the steady wave patterns he calculated. This point was also discussed in the detailed review of Saffman & Yuen (1985).

In this section, attention is drawn to steady 3D patterns that are symmetric about the direction of propagation with the front-back asymmetry, in contrast to the solutions of Meiron et al (1982). The main steps of the theory of water-wave horseshoe patterns set up by Shrira et al (1996) are briefly presented below.

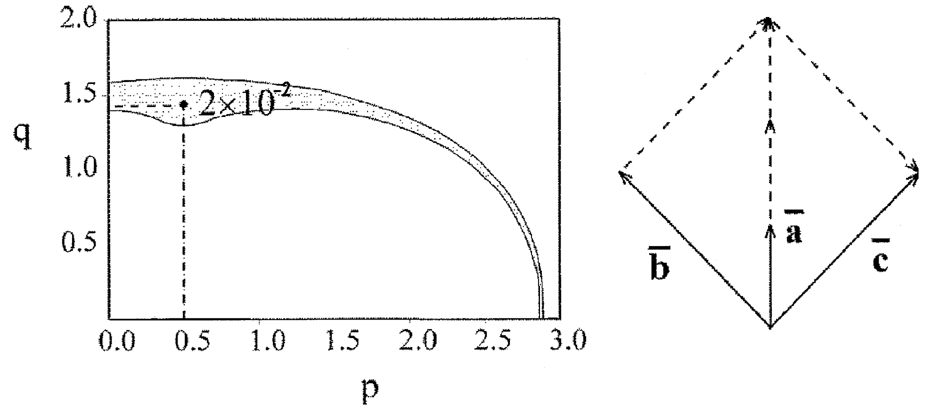

Figure 8 Selection of the dominant triad. Instability region for the Stokes wave of steepness $\epsilon = 0.3$ (the dot labels the point of maximum instability) and the dominant symmetric resonant triad in **k**-space.

SELECTION OF THE DOMINANT RESONANT TRIAD To get a low-dimensional system, only the basic wave ($\mathbf{k_a}$, ω_a) and the two modes corresponding to growing transversal perturbations ($\mathbf{k_b}$, ω_b) and ($\mathbf{k_c}$, ω_c) were considered. In Figure 8 are plotted the stability boundaries in the $p-q$ plane for the class II ($m = 1$) and the dominant resonant triad corresponding to the most unstable perturbation, phase-locked with the basic wave. Performing the transformation to real amplitudes and phases, $a = Ae^{-i\alpha}$, $b = Be^{-i\beta}$, $c = Ce^{-i\gamma}$, the Zakharov equation was reduced to a set of ordinary differential equations for the amplitudes A, B, C, and the phase $\Phi = 3\alpha - \beta - \gamma$.

WEAKLY NONCONSERVATIVE SYSTEM The dynamics of wind waves is not entirely Hamiltonian: The nonconservative effects caused by wind generation, viscous and turbulent dissipation, can strongly affect wave field evolution. Because of the complexity of the mechanisms of wave generation and wave dissipation, energy input and sink were introduced heuristically and assumed to be of the order of the quartic nonlinear terms, i.e. of order ϵ^4. For a transversally symmetric triad ($B = C, \beta = \gamma$), Shrira et al (1996) considered a system of first-order differential equations that describes the simplest model for nonlinear coupling of three waves in a gravity wave field when the main processes are the quintet resonant interactions, input, and dissipation caused by wind.

CONSERVATIVE 3D STRUCTURES Besides the trivial stationary solutions corresponding to the well-known short-crested waves ($A_0 = 0, B_0 \neq 0$) and plane Stokes wave ($A_0 \neq 0, B_0 = 0$), Shrira et al (1996) found nontrivial 3D stationary states: the in-phase states ($\Phi_0 = \pm 2m\pi$, m is an arbitrary integer) and the out-of-phase states ($\Phi_0 = \pm(2m+1)\pi$). The in-phase equilibria were identical to those calculated numerically by Meiron et al (1982) within the exact equations, whereas the out-of-phase equilibria were new. The existence of these new steady 3D patterns in deep water, within the exact equations, has not yet been proved.

NONCONSERVATIVE 3D STRUCTURES OR DISSIPATIVE STRUCTURES Meiron et al pointed out that it was not possible to observe persistent asymmetric patterns within the framework of purely Hamiltonian dynamics. For the existence of a nonconservative equilibrium it is necessary for the system to have a balance between energy input and dissipation. This means that either energy input into central harmonics is balanced by the dissipation in the satellites or vice versa. While the conservative stationary points could have only two fixed values (0 and π) of the phase Φ, the phases of the nonconservative equilibria fill up the whole interval $[0, 2\pi]$. It was shown how the presence of both dissipation and generation could promote the existence of attractive equilibria. Shrira and co-workers examined the most interesting limiting case: that of the maximal ratio of nonconservative to quartic nonlinear terms ($\sin \Phi_0 = \pm 1$). The corresponding equilibria, called saturated states ($\Phi_0 = \pm \frac{\pi}{2}$), were those exhibiting the most curved and asymmetric fronts. The minus sign corresponds to the realistic situation of generation of the central harmonic and dissipation of satellites, and the opposite sign means the opposite balance. They found that among the linearly stable patterns, the "most distinguished" were those in the vicinity of the saturated equilibrium, which display the forward-oriented crescents. An example of the trajectories in the phase space in the vicinity of a saturated equilibrium is given in Figure 9. The wave patterns and contour plot corresponding to the attractor displayed in Figure 9 are depicted in Figure 10. These patterns exhibit the front-back asymmetry typical of the experimentally observed horseshoe structures. For waves steeper than a certain threshold (about 0.20), it was shown that the first transition from 2D to 3D waves (i.e. the transition occurring at minimal amplitude of the plane wave) selects the vicinity of the saturated states ($\Phi_0 = -\frac{\pi}{2}$). This suggests that if a plane wave evolves to an equilibrium owing to transverse instability and generation-dissipation effects, it is more likely to reach the vicinity of the saturated steady states. Nevertheless, extensive numerical simulations are needed to support this hypothesis.

The model of Shrira et al reproduces qualitatively all the main features of the horseshoe structures observed experimentally. However, it does not explain the experiments of Su et al (1982) in which there was no wind input. Note that the patterns observed by Su and co-workers had a relatively short time existence (of order ϵ^{-3}). A comprehensive theory of nonsteady horseshoe

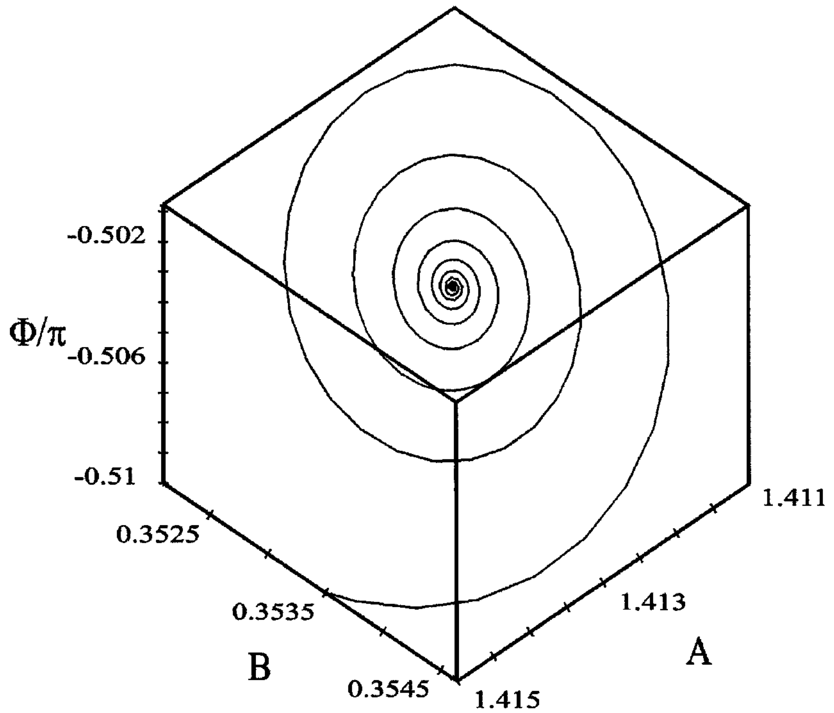

Figure 9 A typical scenario of the system evolution in the vicinity of the saturated-state equilibrium. The wave steepness is about 0.35. A trajectory is plotted in the 3D phase space.

patterns developed recently by SY Annenkov & VI Shrira (submitted), which does not require wind input to the basic wave as a prerequisite, explains these experiments.

The justification for selection of only three modes in Zakharov's variables and neglect of the class I instability may be found in the work of Annenkov & Shrira (submitted), in which a new nonlinear selection mechanism has been identified and the relatively weak effect of Benjamin-Feir instability was demonstrated. Additional arguments are provided in the papers of Bliven et al (1986) and Badulin et al (1995), respectively. The first authors observed experimentally that wind over regular waves reduced and even suppressed the modulational instability. For highly nonlinear Stokes waves, the second group of authors explained the reduction and disappearance of class I ($m = 1$) instabilities in terms of interactions between the classes of instabilities, i.e. the four-wave instability is suppressed by the five-wave processes. However, the removal of the class I instability remains questionable when class I and class II have

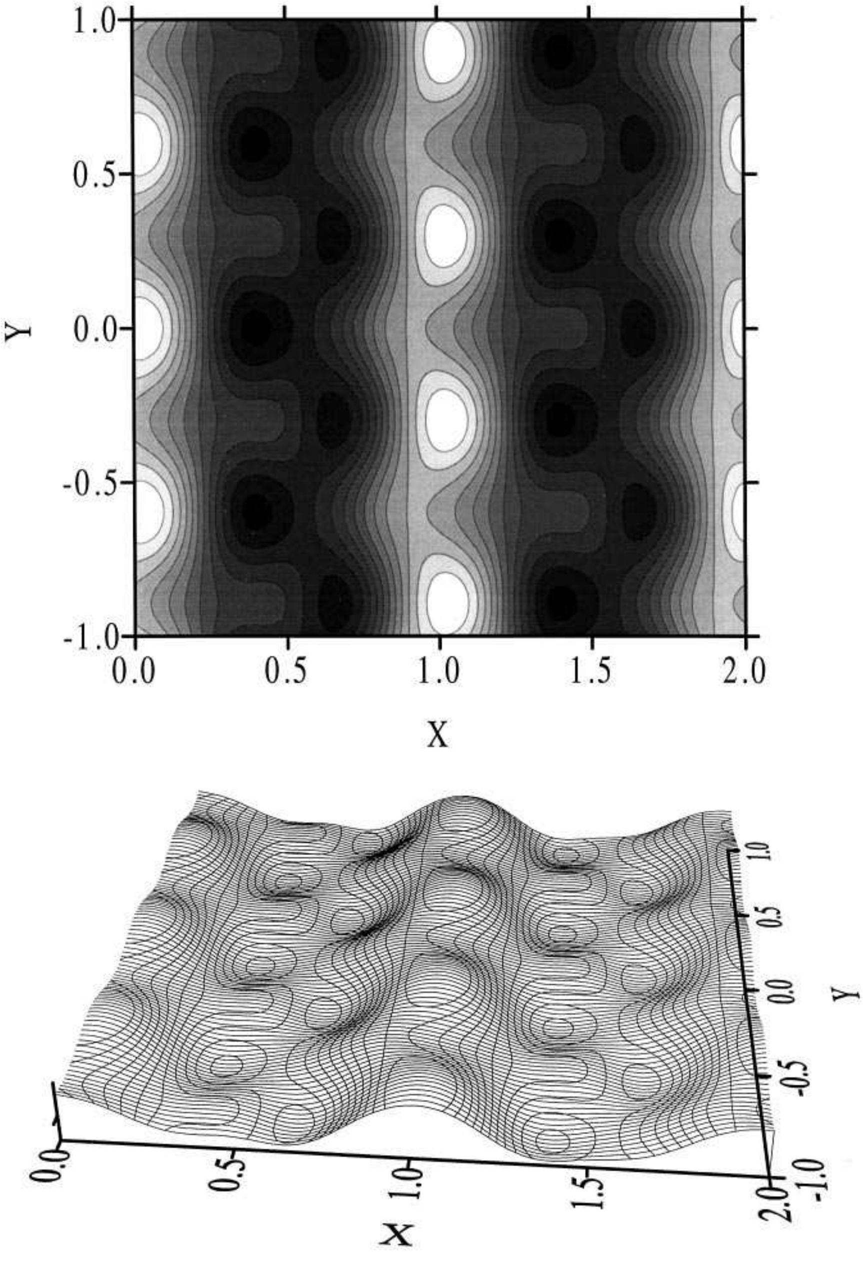

Figure 10 The saturated-state patterns corresponding to the attractor depicted in Figure 9 (maxima are white; minima are black). The waves are propagating from left to right.

comparable strengths. Su & Green (1984, 1985) reported the results of experimental investigations on the coupling of the two classes of instabilities and came to the conclusion that class I and II instabilities interact strongly during the evolution of wave trains with moderate initial steepness. They suggested that modulations produced by the essentially 2D instabilities (class I) were sufficient to trigger the predominantly 3D instabilities (class II) that consequently limit the growth of the class I. Stiassnie & Shemer (1987) carried out numerical simulations of the modified Zakharov equation. They found the suppression of class II instabilities whenever the initial level of class I perturbations was substantially higher than that of class II, and so they did not observe the trigger mechanism. However, they pointed out that significant class II intensity could accompany high levels of class I instabilities and stressed the importance of the phase shift between the central harmonic and the satellites in the behavior of the 3D instabilities. These conclusions have been confirmed by Annenkov & Shrira (submitted), where the analysis was carried out within the framework of the more general multimodal Zakharov equation with nonconservative effects also taken into account.

Wind action on regular waves leads eventually to external randomness in the system, transforming the equations into stochastic evolution equations. The survival of the attractors will depend on the relative strength of attraction and random forcing caused by all the other interactions neglected in the model. Shrira et al (1996) speculated that the system, under forcing, will evolve to the old equilibria provided that the domain of attraction is large enough.

6. SHORT-CRESTED WAVES

The simplest 3D waves one can consider are the so-called short-crested waves. These waves, which come from the superposition of two oblique traveling waves with the same amplitude, are symmetric doubly periodic waves. A special case of course occurs when the two waves travel in opposite directions, and the resulting wave is a standing wave. Short-crested waves are relevant from a physical point of view in important maritime situations, for example when a traveling wave is reflected by a seawall or in the open ocean (remote sensing). The study of 3D wave fields is essential to get a more realistic description of the sea surface: In particular, M Ioualalen et al (submitted) showed how three-dimensionality could play a crucial role in altimetry by introducing a substantial bias on the sea-level and wind modulus measurements. Quite different in nature are the spontaneous (here we use the terminology of Saffman & Yuen 1985) 3D waves, which result from the bifurcation of a 2D Stokes wave of finite amplitude and were discussed in Section 5. Recall that these 3D waves, first mentioned by McLean et al (1981), were computed numerically by Meiron et al (1982) and compared favorably with the waves observed by Su (1982). Quite surprisingly, the early 1980s saw a sequence of papers on these two types of 3D waves, which are periodic in two horizontal directions. In addition to the papers already quoted, there is the first paper to our knowledge with rigorous results on 3D waves: Reeder & Shinbrot (1981) proved the existence of small-amplitude short-crested waves in a certain region of parameter space. Sun (1993) provided an alternative construction of short-crested waves. In both papers, the proof works outside a forbidden set, which is given in Section 7 of Reeder & Shinbrot (1981). Roughly speaking, the forbidden set consists of the parameters that allow resonances among harmonics. Unfortunately, both gravity waves and capillary-gravity standing waves fall inside the forbidden set. Fuchs (1952), Chappelear (1961), Hsu et al (1979), Ioualalen (1993), Menasce (1994), and Kimmoun (1997) used perturbation expansions to compute from a formal point of view these small-amplitude 3D waves. Roberts & Peregrine (1983) described an analytic solution to fourth order in wave steepness, which matches short-crested waves on one hand and 2D progressive waves on the other. Numerical methods as well have been developed by Roberts (1983), Roberts & Schwartz (1983), Bryant (1985), and Marchant & Roberts (1987). Note that Bryant (1982) worked on a model equation, the Kadomtsev-Petviashvili (1970) equation. Results on 3D long waves have also been obtained by Segur & Finkel (1985), Hammack et al (1989), Hammack et al (1995), Dubrovin et al (1997), and Milewski & Keller (1996). A review on 3D long waves was recently written by Akylas (1994).

To prove rigorously the existence of short-crested waves is a difficult task, and so far only capillary-gravity waves with high enough surface tension have been shown to exist. The difficulties for gravity waves or for capillary-gravity waves with small surface tension are similar to the difficulties encountered in proving the existence of standing waves (Amick & Toland 1987, Iooss 1997b). However, there is numerical evidence that short-crested gravity waves exist (see, for example, the numerical results of Roberts 1983 and Marchant & Roberts 1987). Let $k_o h$, B, and $\tan\varphi = \frac{k_y}{k_x}$ be the three parameters for the description of short-crested waves, where k_x is the wave number in the first horizontal direction, k_y the wave number in the second horizontal direction, and $k_o = \sqrt{k_x^2 + k_y^2}$ the modulus of the wave number. For a given angle φ, the region where the proof of existence of short-crested waves has been given lies to the right of the corresponding curve shown in Figure 11 and given by

$$B(k_o h, \varphi) = \frac{2\tanh(k_o h) - \cos\varphi \tanh(2k_o h\cos\varphi)}{4\cos^3\varphi \tanh(2k_o h\cos\varphi) - 2\tanh(k_o h)}. \tag{6.1}$$

In this region, linear resonances between modes cannot occur. The curve for

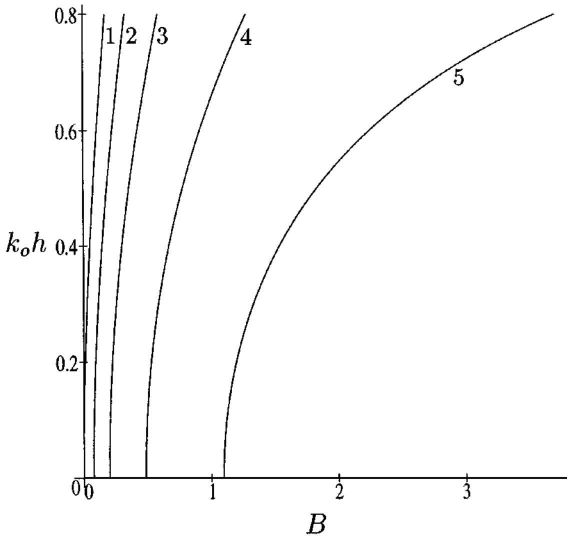

Figure 11 For a fixed angle φ, the existence of short-crested waves has been proved to the right of the curve corresponding to that angle. The curves represented here correspond to $\varphi = 0$ (*1*), $\varphi = \pi/8$ (*2*), $\varphi = \pi/6$ (*3*), $\varphi = \pi/5$ (*4*), $\varphi = 2\pi/9$ (*5*).

$\varphi = 0$ corresponds to the curve for Wilton ripples

$$B = \frac{\tanh^2(k_o h)}{3 - \tanh^2(k_o h)}. \tag{6.2}$$

If one takes the limit of (6.1) as $k_o h \to 0$ (resp. $k_o h \to \infty$), one finds

$$B(0, \varphi) = \frac{\sin^2 \varphi}{4\cos^4 \varphi - 1}, \quad B(\infty, \varphi) = \frac{2 - \cos\varphi}{4\cos^3 \varphi - 2}.$$

Regions without linear resonances can be found only for values of φ between 0 and $\pi/4$.

It is important to note that the limiting process in which both oblique waves become parallel is tricky and does not provide pure traveling waves. This is well explained in Roberts (1983), and this limit was in fact the subject of the paper by Roberts & Peregrine (1983). See also the work of T Bridges et al (submitted).

Some work has also been performed on the stability of short-crested waves. Ioualalen & Kharif (1993, 1994) investigated numerically the linear stability of short-crested waves on deep water to superharmonic and subharmonic disturbances. They showed that for moderate wave steepness, the dominant interactions are sideband-type instabilities. Instabilities associated with the harmonic resonances were found to be sporadic "bubbles" of instability caused by the collision of two superharmonic modes at zero frequency. The strongest superharmonic growth rates were at least two orders in wave steepness lower than the subharmonic ones. This result suggests that the resonances are unlikely to be significant because they will not have time to develop, as subharmonic instabilities have a much more rapid growth. It was also shown that short-crested waves are more stable than Stokes waves. Later, Badulin et al (1995) considered two approaches to study the stability of short-crested waves, one based on both classical Stokes expansion and Galerkin methods used previously by Ioualalen & Kharif (1994) and one analytic method based on the Zakharov equation. A comparison between the two approaches pointed out that the analytical results hold their validity for rather steep waves (up to $\epsilon = 0.4$) for a wide range of wave patterns. A generalization of the classical Phillips' concept of weakly nonlinear wave instabilities was given by describing the interaction between the elementary classes of instability. Kimmoun (1997) extended to larger amplitude the work of Ioualalen & Kharif (1994). Ioualalen et al (1996) carried out numerical computations of the superharmonic instabilities of short-crested waves in finite depth. They found that these instabilities can be significant for some particular geometries in contrast to deep-water waves. They suggested a critical value of the nondimensional depth parameter $kh = 1$ for which shallower water motion might become unobservable. However, they concluded that finite-depth short-crested waves are generally observable because the strong superharmonic instabilities are localized in very narrow bands in the parameter range. M Ioualalen et al (1998) investigated the stability regimes of finite-depth weakly nonlinear short-crested water waves and emphasized that these waves are quasi-observable, as pointed out by Hammack et al (1989).

Recently, a qualitative study was made of the two processes giving rise to 3D waves: the dimension-breaking bifurcation through which a 2D wave of finite amplitude $u(x)$ bifurcates into a 3D wave $u(x, y)$, and the bifurcation from the rest state $u = 0$. Here $u(x, y)$ satisfies a differential equation. These two processes were described from a mathematical point of view by Hărăguş & Kirchgässner (1995), who used tools of dynamical systems theory to illustrate these processes on two model equations: the Ginzburg-Landau equation for the dimension-breaking bifurcation and the Kadomtsev-Petviashvili (1970) equation for the bifurcation from rest. They also mentioned a third process: the generation of 3D waves through the appearance of turning- or fold-points along a branch of 2D solutions.

In the future, it would be interesting to deepen the link between the mathematical analysis of Hărăguş & Kirchgässner (1995) and the work of Saffman & Yuen (1980) on the dimension-breaking bifurcations. Of interest too is gaining as understanding of the link between short-crested waves and the other 3D waves.

7. IMPORTANCE OF SURFACE TENSION EFFECTS

In some cases, small surface tension effects may have important consequences on steep waves. Debiane & Kharif (1996) considered gravity waves on deep water that are weakly influenced by surface tension effects ($B \ll 1$)—in other words this means that the dominant restoring force is gravity—and discovered numerically a new family of limiting profiles of steady gravity waves, which exhibit two trapped bubbles, one on either side of the crest. This result can be viewed as an extension to small values of B of the work of Hogan (1980) on steep water waves. Debiane & Kharif (1996) showed that the highest wave is steeper than the highest pure gravity wave. For $B = 0.0003$, which corresponds to a wavelength of 1 m, the steepness of the highest computed wave is $\epsilon = 0.4439$. This value is slightly greater than 0.4432, which corresponds to the steepest pure gravity wave. The new profile could be computed thanks to the work of Debiane & Kharif (1997), who extended the method developed by Longuet-Higgins (1978c) to capillary-gravity waves. Schultz et al (1998) also emphasized the importance of surface tension on highly nonlinear standing waves. When a small surface tension ($B \ll 1$) is included, the crest form changes significantly. Surface tension effects increase the limiting wave height for standing waves and may explain the observation by Taylor (1953) of a limiting wave with a crest of nearly 90°.

In previous sections, the instabilities of Stokes wave trains have been divided into two classes (I and II). There exists another classification that separates instabilities into superharmonic and subharmonic. Superharmonic perturbations are associated with integer values of p; otherwise the perturbations are subharmonic. The dominant instabilities of Stokes waves are generally subharmonic. Nevertheless when the steepness becomes very large, near its limiting value, the difference between the growth rates of the two types of instabilities is weak (see Kharif & Ramamonjiarisoa 1988). Tanaka (1983) found numerically that superharmonic disturbances to periodic waves of permanent form become unstable at a wave steepness $\epsilon = 0.4292$ corresponding to the first maximum of the total wave energy. Later Saffman (1985) proved analytically that superharmonic perturbations exchange stability when the wave energy is an extremum as a function of wave height. Longuet-Higgins & Cleaver (1994) and Longuet-Higgins et al (1994) investigated crest instabilities of steep Stokes waves near their limiting form calculated from the theory of the almost-highest wave (see Longuet-Higgins & Fox 1977, 1978). Longuet-Higgins & Tanaka (1997) used the method of Tanaka (1983) to show that the superharmonic instabilities of Stokes waves are indeed "crest instabilities." These instabilities are essentially localized phenomena near the wave crests and may lead to the overturning of the waves (Jillians 1989, Longuet-Higgins & Dommermuth 1997). For short gravity waves, surface tension effects can have important consequences on the further evolution of crest instabilities. Longuet-Higgins (1996a) suggested that the introduction of surface tension in the nonlinear evolution of the crest instability will not lead to overturning but instead to a locally steep surface gradient called capillary jump, on the front face of the steep wave, with a few ripples or parasitic capillaries. This researcher came to the conclusion that the crest instability is a comparatively weak instability at short gravity wavelengths. The explanation of the formation of the jump can be found in Longuet-Higgins (1996b). The initial stage of breaking of a short gravity wave, according to the same author (1992, 1994), involves the formation of parasitic capillary waves, ahead of a bulge on the forward face of the wave near the crest, followed by a train of longer capillary waves, above the toe of the bulge, both of which collapse into turbulence. These two types of waves were observed by Duncan et al (1994). Lin & Rockwell (1995) investigated experimentally the stages of the evolution of a quasisteady breaker from the onset of a capillary pattern to a fully evolved breaking wave. They found the pattern observed by Duncan et al (1994) and depicted by Longuet-Higgins (1994) to occur over a rather narrow band of Froude numbers. The formation of capillary waves on the front face of steep short gravity waves was also investigated by others experimentally (Cox 1958, Chang et al 1978, Yermakov et al 1986, Ebuchi et al 1987, Perlin et al 1993, Zhang 1995, etc), theoretically and numerically (Longuet-Higgins 1963, 1995; Crapper 1970; Ruvinsky et al 1991; Watson & Buchsbaum 1996; Dommermuth 1994; Mui & Dommermuth 1995; Fedorov & Melville 1997; etc). These waves are known to generate a vorticity field enhancing surface currents and also to strongly enhance wave damping (Ruvinsky et al 1991, Fedorov & Melville 1997). One of the motivations for studying such small-scale structures is their importance for air-sea transfers of gas, momentum, and heat (Saylor & Handler 1997) and for microwave remote sensing of the ocean surface (Melville 1996).

8. POSSIBLE FUTURE DIRECTIONS

It is widely admitted that in two dimensions, without any dissipative effects added, there is no permanent frequency downshift. The situation seems to be different in three dimensions as reported by Trulsen & Dysthe (1997). It would be desirable to confirm this result by using other approaches such as the Zakharov equation or the full water-wave equations. Recent numerical simulations of the breaking of short gravity waves (including the splash-up phenomenon) were performed by G Chen et al (submitted) using the 2D Navier-Stokes equation and the volume-of-fluid method. This numerical treatment should allow a numerical investigation of the frequency downshift phenomenon during wave breaking as well as the vortical structures caused by the breaking of a short gravity wave when surface tension effects are taken into account. It is well known that coherent vortical structures and cascades to smaller scales are quite different in two and three dimensions, and for this reason an extension to 3D simulations is necessary for further research. Another area for further investigation is the effect of randomness on the 3D structures (horseshoe patterns) generated by wind in the sea. In the field of short-crested waves, future research includes the stability analysis of gravity-capillary waves on arbitrary depth. Finally, further developments are expected in the application to the water-wave problem of the spatial approach as well as of the multisymplectic approach.

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