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The numerical computation of freely propagating time-dependent irrotational water waves

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Abstract

Methods for the numerical computation of freely propagating irrotational water waves are reviewed. The emphasis is on the methods, not on the results. The primary focus is on methods for time-dependent fully nonlinear water waves, but aspects of steady waves are also discussed. For time-dependent waves, a range of topics from two-dimensional time-periodic waves over a flat bottom to unsteady three-dimensional waves over an arbitrary topography, including the statistical description of water waves, are discussed.

Keywords: Numerical methods; Free-surface flows; Three-dimensional water waves

1. Introduction

Water waves have long been studied because of their practical importance and because they offer an ideal setting for a variety of phenomena of nonlinear wave motion.

Compared to the analytical study of the water-wave problem, which was initiated at the beginning of the nineteenth century, the numerical study of water waves is relatively recent for obvious reasons. The first numerical computations were performed in the 1970s, with a few exceptions in the 1960s. Since then,

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the steady progress in the power of computers combined with the development of more and more efficient numerical methods has allowed researchers to tackle problems closer and closer to real-life situations.

One can distinguish four major classes of numerical computations dealing with water waves : (i) computations of progressive waves that propagate without changing form (time dependence can be removed from the equations by working in a frame of reference moving with the wave), (ii) computations of standing waves (these waves are periodic in time and in space, but time dependence cannot be removed from the equations), (iii) computations aimed at a statistical description of water waves (such computations require integration over a long time), and (iv) numerical wave tanks that are designed to mimic laboratory wave tanks or even the ocean in the sense that arbitrary wave motion can be described. For each case, one can make a further distinction between two-dimensional waves and three-dimensional waves.

The emphasis of the review is on the numerical computation of freely propagating time-dependent waves. This is a rich subject and bias is unavoidable. The mathematical model that is used in the present review is that of inviscid irrotational flow. The governing equations are the incompressible, irrotational Euler equations in the presence of a free surface. Among the numerical methods and equations based on this mathematical model, some are closer to the model than others. In particular, approximate equations such as the Korteweg–de Vries equation, the Boussinesq equation, the nonlinear Schrödinger equation and their variants represent a further approximation. Although they have been shown to provide very good results in a variety of applications, they are not considered here.[1] Moreover, by restricting ourselves to freely propagating waves, we do not discuss the interactions of water waves with structures, a topic of great industrial interest. For earlier reviews on water waves or on the numerical simulation of free-surface flows, one can refer to Schwartz and Fenton (1982), Tsai and Yue (1996), Dias and Kharif (1999), Scardovelli and Zaleski (1999), and Peregrine (2003).

2. Formulations of the water-wave problem used for numerical computations

A brief description of the common mathematical model used to study water waves is given first. The three-dimensional flow of an inviscid and incompressible fluid is governed by the conservation of mass

$$\nabla \cdot \mathbf{u} = 0 \tag{1}$$

and by the conservation of momentum

$$\rho \frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} = \rho \mathbf{g} - \nabla p, \tag{2}$$

where

$$\frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} = \frac{\partial \mathbf{u}}{\partial t} + \nabla(\tfrac{1}{2}\,\mathbf{u}\cdot\mathbf{u}) - \boldsymbol{\omega} \times \mathbf{u}.$$

In (2), $\boldsymbol{\omega} = \nabla \times \mathbf{u}$ is the vorticity vector. The horizontal coordinates are denoted by x and y, and the vertical coordinate by z. The vector $\mathbf{u}(x, y, z, t) = (u, v, w)$ is the velocity field, ρ is the fluid density (assumed to be constant throughout the fluid domain), $\mathbf{g}$ is the acceleration due to gravity and $p(x, y, z, t)$ the pressure field.

[1] In some cases, these approximate equations have been shown to be valuable even outside their range of validity!

The assumption that the flow is irrotational ($\boldsymbol{\omega} = \mathbf{0}$) is commonly made to analyze surface waves. Then there exists a scalar function $\phi(x, y, z, t)$ (the velocity potential) such that $\mathbf{u} = \nabla\phi$. The continuity Eq. (1) becomes

$$\nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} = 0. \tag{3}$$

With $\boldsymbol{\omega} = \mathbf{0}$, the equation of momentum conservation (2) can be integrated into Bernoulli's equation

$$\frac{\partial\phi}{\partial t} + \frac{1}{2}|\nabla\phi|^2 + gz + \frac{p - p_0}{\rho} = 0, \tag{4}$$

which is valid everywhere in the fluid. The constant p_0 is a pressure of reference, for example the atmospheric pressure.

2.1. Classical formulation

The surface wave problem consists in solving Laplace's equation (3) in a domain $\Omega(t)$ bounded above by a moving free surface (the interface between air and water) and below by a fixed solid boundary (the bottom).[2] The free surface is represented by $F(x, y, z, t) = \eta(x, y, t) - z = 0$. The bottom can have an arbitrary shape given by $z = -h(x, y)$. The main driving force is gravity, but the effects of surface tension may be equally important in some physical situations.

The free surface must be found as part of the solution. Two boundary conditions are required. The first one is the kinematic condition. It can be stated as $\mathrm{D}F/\mathrm{D}t = 0$ (the material derivative of F vanishes), which leads to

$$\eta_t + \phi_x\eta_x + \phi_y\eta_y - \phi_z = 0 \quad \text{at } z = \eta(x, y, t), \tag{5}$$

where subscripts denote the derivatives. The second boundary condition is the dynamic condition which states that the normal stresses must be in balance at the free surface. The normal stress at the free surface is given by the difference in pressure and is balanced by the effect of surface tension. If σ denotes the surface tension coefficient and C the curvature of the free surface, $p - p_0 = -\sigma C$. For a clean air/water interface, $\sigma = 0.074\,\mathrm{N/m}$. The expression for C is

$$C = \frac{\partial}{\partial x}\left(\frac{\eta_x}{(1 + \eta_x^2 + \eta_y^2)^{1/2}}\right) + \frac{\partial}{\partial y}\left(\frac{\eta_y}{(1 + \eta_x^2 + \eta_y^2)^{1/2}}\right). \tag{6}$$

Bernoulli's equation (4) evaluated on the free surface $z = \eta$ gives

$$\phi_t + \frac{1}{2}(\phi_x^2 + \phi_y^2 + \phi_z^2) + g\eta - \frac{\sigma}{\rho}C = 0 \quad \text{at } z = \eta(x, y, t). \tag{7}$$

Finally, the boundary condition at the bottom is

$$\phi_x h_x + \phi_y h_y + \phi_z = 0 \quad \text{at } z = -h(x, y). \tag{8}$$

[2] The surface wave problem can be easily extended to the case of a moving bottom. This extension is needed for example to model tsunami generation.

To summarize, one solves the set of equations (3), (5), (7) and (8) for $\eta(x, y, t)$ and $\phi(x, y, z, t)$. If the initial value problem is integrated, then the fields $\eta(x, y, 0)$ and $\phi(x, y, z, 0)$ must be specified at $t = 0$. The conservation of momentum equation (2) is not required in the solution procedure; it is used a posteriori to find the pressure p once η and ϕ have been found. In water of infinite depth, the kinematic boundary condition on the bottom (8) is replaced by

$$|\nabla\phi| \to 0 \quad \text{as } z \to -\infty.$$

The above formulation is the classical formulation of the water-wave problem. Variations of this formulation that are better suited for numerical computations have been proposed.

Before these alternative formulations are presented, some notation is introduced. The potential evaluated along the free surface is denoted by $\Phi(x, y, t) = \phi(x, y, \eta, t)$. The derivatives of the velocity potential evaluated on the free surface are denoted by $\Phi_{(*)}(x, y, t) = \phi_*(x, y, \eta, t)$, where the star stands for x, y, z or t. Consequently, Φ_* (defined for $* \neq z$) and $\Phi_{(*)}$ have different meanings. They are however related since

$$\Phi_* = \Phi_{(*)} + \Phi_{(z)}\eta_*.$$

Two-dimensional (2D) waves are waves without y-dependence. The stream function along the free surface is denoted by $\Psi(x, t) = \psi(x, \eta, t)$. When complex notation is introduced for 2D waves, the classical notation $z = x + \mathrm{i}y$ is used. In that case, the vertical coordinate is y while z is the complex coordinate. The complex potential $f(z) = \phi + \mathrm{i}\psi$ is also introduced. Permanent waves are waves with shapes that are invariant with respect to time, in a suitable frame of reference moving with a constant speed.

For 2D spatially periodic waves of permanent form in water of infinite depth, it is worth mentioning the formulation due to Levi-Civita (1925). Let λ denote the wavelength and c the wave speed. The independent variable

$$\zeta = \exp\left(-\frac{2\pi \mathrm{i} f}{c\lambda}\right) = r\exp(\mathrm{i}s) \quad \text{with } r = \exp\left(\frac{2\pi\psi}{c\lambda}\right), \quad s = -\frac{2\pi\phi}{c\lambda}$$

and the dependent variable

$$W = \mathrm{i}\log\left(\frac{1}{c}\frac{\mathrm{d}f}{\mathrm{d}z}\right) \ (=\theta + \mathrm{i}\tau, \text{ say})$$

are introduced, and W is considered as a function of ζ. The bottom $\psi \to -\infty$ corresponds to $\zeta = 0$ while the free surface $\psi = 0$ corresponds to $|\zeta| = 1$. Levi-Civita showed that $W = \theta + \mathrm{i}\tau$ satisfies the following equation on $r = 1$:

$$\mathrm{e}^{2\tau}\frac{\partial\tau}{\partial s} - \frac{g\lambda}{2\pi c^2}\mathrm{e}^{-\tau}\sin\theta + \frac{2\pi\sigma}{\rho c^2\lambda}\frac{\partial}{\partial s}\left(\mathrm{e}^{\tau}\frac{\partial\theta}{\partial s}\right) = 0. \tag{9}$$

2.2. Boundary integral formulation

The notation $\mathbf{x} = (x, y, z)$ is introduced. By using Green's functions, one can reduce the three-dimensional (3D) water-wave problem to a 2D boundary integral problem. In the case of the 2D water-wave problem, the problem is reduced to a simple one-dimensional integro-differential equation on the boundary.

The moving surface can be described by markers which are treated as Lagrangian points. In other words, they are moving with the fluid velocity. The velocity at the surface is determined from the velocity potential field, which can be followed on the surface using Bernoulli's Eq. (7). From the surface potential one can determine the tangential velocity, but the normal velocity must be found indirectly, using the fact that the potential is harmonic in the fluid interior.

The three-dimensional free space Green's function is defined as

$$G(\mathbf{x}, \mathbf{x}_l) = \frac{1}{4\pi|\mathbf{r}|}, \qquad \frac{\partial G}{\partial n}(\mathbf{x}, \mathbf{x}_l) = -\frac{1}{4\pi}\frac{\mathbf{r}\cdot\mathbf{n}}{|\mathbf{r}|^3}, \tag{10}$$

where $|\mathbf{r}| = |\mathbf{x} - \mathbf{x}_l|$ is the distance from the source point $\mathbf{x}$ to the collocation point $\mathbf{x}_l$ (both are on the boundary) and $\mathbf{n}$ is the normal vector pointing out of the fluid. The notation $\partial G/\partial n$ means the normal derivative, that is $\partial G/\partial n = \nabla G \cdot \mathbf{n}$. Green's second identity transforms Laplace's equation (3) into a boundary integral equation (BIE) on the boundary Γ of the fluid domain Ω

$$\alpha(\mathbf{x}_l)\,\phi(\mathbf{x}_l) = \int_{\Gamma(t)} \left\{ \frac{\partial \phi}{\partial n}(\mathbf{x})\, G(\mathbf{x}, \mathbf{x}_l) - \phi(\mathbf{x})\frac{\partial G}{\partial n}(\mathbf{x}, \mathbf{x}_l) \right\} \mathrm{d}\Gamma, \tag{11}$$

where $\alpha(\mathbf{x}_l)$ is proportional to the solid exterior angle made by the boundary at the collocation point $\mathbf{x}_l$.

The kinematic and dynamic boundary conditions on the free surface (5) and (7) are rewritten in a mixed Eulerian–Lagrangian form

$$\frac{\mathrm{D}\mathbf{x}}{\mathrm{D}t} = \mathbf{u} = \nabla\phi, \tag{12}$$

$$\frac{\mathrm{D}\phi}{\mathrm{D}t} = -g\eta + \frac{\sigma}{\rho}C + \frac{1}{2}\nabla\phi\cdot\nabla\phi, \tag{13}$$

where $\mathbf{x}$ is the position vector of a fluid particle on the free surface. Lateral boundaries can be fixed or moving boundaries (wave-maker for example).

For 2D periodic waves of permanent form, there is another integral formulation, known as Nekrasov's integral equation, that has nice properties for numerical computations,

$$\gamma(s) = -\frac{1}{3\pi}\int_{-\pi}^{\pi} \log\left|\sin\frac{s-t}{2}\right| \frac{\mu \sin\gamma(t)}{1 + \mu\int_0^t \sin\gamma(\tau)\mathrm{d}\tau}\,\mathrm{d}t, \qquad s \in [-\pi, \pi], \tag{14}$$

where $\gamma(s)$ is the unknown angle between the horizontal and the tangent to the free surface. Eq. (14) is valid for waves in water of infinite depth and follows directly from Levi-Civita's equation (9). The variable s has the same meaning as in equation (9). The parameter μ is related to the wave speed and is greater than 3. The wave of extreme form is characterized by $\mu \to \infty$. Integrating (14) by parts yields

$$\gamma = -\tfrac{1}{3}\mathscr{H}\mathscr{F}(\gamma, \mu), \tag{15}$$

where

$$\mathscr{H}g(s) = \frac{1}{2\pi}\int_{-\pi}^{\pi} [g(t) - g(s)]\cot[(s-t)/2]\,\mathrm{d}t$$

is the Hilbert transform and $\mathscr{F}$ is the nonlinear operator defined by

$$\mathscr{F}(g,\mu)(t) = \log\left|1 + \mu\int_0^t \sin g(\tau)\,\mathrm{d}\tau\right|.$$

More details can be found for example in the book by Okamoto and Shōji (2001).

2.3. Formulations based on Cauchy's theorem

When the flow field is two-dimensional, the powerful methods of complex analysis lead to a range of useful formulations. In particular, the use of Cauchy's theorem leads to a special case of a boundary integral formulation. Let $z = x + \mathrm{i}y$ and let $f(z)$ with $f = \phi + \mathrm{i}\psi$ be the complex potential. Then the use of Cauchy's theorem to formulate the equations proceeds as follows. First, details for stationary solitary waves in water of finite depth in a frame of reference moving at the wave speed c are given. The case of time-dependent conformal mappings is developed later.

For steady flows, the free surface can be represented by the streamline $\Psi = 0$ and it is parameterized by the parameter Φ. The complex velocity on the free surface is given by

$$\Phi_x - \mathrm{i}\Phi_y = \frac{\mathrm{d}f}{\mathrm{d}z} = \frac{1}{x'(\Phi) + \mathrm{i}\eta'(\Phi)}.$$

Bernoulli's equation (7) becomes

$$\frac{1}{2}\left(\frac{1}{x'^2(\Phi) + \eta'^2(\Phi)} - c^2\right) + g\eta + \frac{\sigma}{\rho}\,\frac{\eta'(\Phi)x''(\Phi) - x'(\Phi)\eta''(\Phi)}{(x'^2(\Phi) + \eta'^2(\Phi))^{3/2}} = 0. \tag{16}$$

One looks for $f(z)$, or equivalently $\mathrm{d}f/\mathrm{d}z$, in the domain $-\psi_b \leqslant \psi \leqslant 0$, where ψ_b is the value of the stream function on the bottom. Note that $x'(\phi) + \mathrm{i}y'(\phi) \to 1/c$ as $|\phi| \to \infty$. Since $y'(\phi)$ vanishes on the flat bottom $\psi = -\psi_b$, the function $x'(\phi) + \mathrm{i}y'(\phi) - 1/c$ can be extended by symmetry with respect to the line $\psi = -\psi_b$ as an analytic function in the domain $-2\psi_b \leqslant \psi \leqslant 0$. One has the property

$$(x' + \mathrm{i}y')(\phi - 2\mathrm{i}\psi_b) = (x' - \mathrm{i}y')(\phi + \mathrm{i}0).$$

In order to calculate $x'(\Phi_0) + \mathrm{i}\eta'(\Phi_0) - 1/c$ at a given point Φ_0 of the free surface, one applies Cauchy's integral formula on a rectangular contour going counterclockwise along the $\psi = 0$ axis from $\phi = \phi_\infty$ to $\phi = -\phi_\infty$, down vertically at $\phi = -\phi_\infty$ to $\psi = -2\psi_b$, horizontally along the $\psi = -2\psi_b$ line from $\phi = -\phi_\infty$ to $\phi = \phi_\infty$, and up vertically at $\phi = \phi_\infty$ to $\psi = 0$. Letting $\phi_\infty \to \infty$ and taking the real part of the integral yields

$$x'(\Phi_0) - \frac{1}{c} = -\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\eta'(s)}{s - \Phi_0}\,\mathrm{d}s + \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{2\psi_b(x'(s) - 1) - (s - \Phi_0)\eta'(s)}{(s - \Phi_0)^2 + 4\psi_b^2}\,\mathrm{d}s. \tag{17}$$

Eqs. (16) and (17) define a nonlinear integro-differential system where the unknown is $x'(\Phi_0) + \mathrm{i}\eta'(\Phi_0)$ on the free surface. For steady periodic waves, the formulation is similar with the kernels in the integral terms replaced by cotangent kernels.

Formulations based on Cauchy's theorem can be easily extended to unsteady waves. The first recorded instance where Cauchy's theorem and conformal mapping were used for numerical simulation of unsteady water waves is the work of Whitney (1971). Tanveer (1991) suggested to use conformal mapping for the time dependent problem directly on Bernoulli's equation. For periodic waves in deep water, he mapped the fluid region into the unit disk (see also Fornberg, 1980). Recently, the implementation of time-dependent conformal mapping has been improved by Dyachenko et al. (1996) and Zakharov et al. (2002).

Cauchy's integral theorem can also be used iteratively to solve Laplace's equation for successive time derivatives of the surface motion. Baker et al. (1982) derived the following equations

$$\Phi(x,t) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{D[\Phi(\xi) - \eta'(\xi)\Psi(\xi)] - \Psi(\xi) - \eta'(\xi)\Phi(\xi)}{1+D^2}\,\frac{\mathrm{d}\xi}{\xi - x}, \tag{18}$$

$$\Psi(x,t) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\Phi(\xi) - \eta'(\xi)\Psi(\xi) + D[\Psi(\xi) + \eta'(\xi)\Phi(\xi)]}{1+D^2}\,\frac{\mathrm{d}\xi}{\xi - x}, \tag{19}$$

which are valid for gravity waves in deep water. The function $D = (\eta(\xi) - \eta)/(\xi - x)$ decays as $1/|\xi - x|$ as $|\xi - x| \to \infty$ and $D \to \eta'$ as $\xi \to x$. Eqs. (18) and (19) were reformulated by Clamond and Grue (2001). They applied the Hilbert transform and determined Ψ from (18).

2.4. *Hamiltonian formulation*

In celestial mechanics and molecular dynamics the systems of equations are Hamiltonian, and this has led to widespread use of *symplectic integrators*, which have excellent properties for conservation of energy over long time scales. The equations of water waves are also Hamiltonian and so one would expect symplectic integrators to be used. However, the Hamiltonian structure is more complicated in the partial differential equation setting, and so the use of symplectic integrators is still an open problem. In this section, we review the Hamiltonian structures of water waves.

The Hamiltonian theory of surface waves goes back to the late 1960s and early 1970s. Zakharov (1968) first pointed out the Hamiltonian structure for water waves when the free surface is a graph.

For capillary-gravity waves in water of finite depth with a flat bottom at $z = -h$, the kinetic energy K and the potential energy V are defined as

$$K = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-h}^{\eta} \frac{1}{2}|\nabla\phi|^2\,\mathrm{d}z\,\mathrm{d}x\,\mathrm{d}y,$$

$$V = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left(\frac{1}{2}g\eta^2 + \frac{\sigma}{\rho}((1+\eta_x^2+\eta_y^2)^{1/2} - 1)\right)\mathrm{d}x\,\mathrm{d}y.$$

The kinetic energy in the variables $\Phi(x,y,t)$ and $\eta(x,y,t)$ can be characterized as the minimal energy for all flows in the given geometry that satisfy the condition $\phi(x,y,z,t) = \Phi(x,y,t)$ on the free surface

$$K(\Phi,\eta) = \mathrm{Min}\left\{\int \frac{1}{2}(\phi_x^2 + \phi_y^2 + \phi_z^2) \mid \phi(x,y,z,t) = \Phi(x,y,t) \ \text{ at } \ z = \eta(x,y,t)\right\}.$$

Indeed, the solution of this minimization problem satisfies

$$\nabla^2\phi = 0 \text{ in } \Omega, \quad \phi_z = 0 \ \text{ at } z = -h, \quad \phi(x,y,z,t) = \Phi(x,y,t) \ \text{ at } z = \eta(x,y,t),$$

and conversely.

Following Zakharov (1968), one can show that the surface wave problem can be written as an infinite dimensional Hamiltonian evolution system in the canonically conjugate variables Φ and η

$$\frac{\partial \Phi}{\partial t} = -\frac{\delta H}{\delta \eta}, \quad \frac{\partial \eta}{\partial t} = \frac{\delta H}{\delta \Phi}, \tag{20}$$

where the total energy $H = K + V$ is the Hamiltonian. Eq. (20) is a temporal Hamiltonian system for the water-wave problem. In Section 5, this Hamiltonian formulation is used to derive Zakharov's equations, which are commonly used for numerical computations.

2.5. Hamiltonian formulation for overhanging waves

Zakharov's formulation (20) has the disadvantage that the free surface must be a single-valued function of the horizontal position (x, y). It fails for the case of breaking waves and other waves with a multi-valued free surface. However, Benjamin and Olver (1982) have given a general Hamiltonian formulation valid for any parametric form of the free surface. For simplicity, attention is restricted to 2D water waves and surface tension is neglected.

Consider the two-dimensional water-wave problem with the free surface defined parametrically as

$$\Gamma(t) = \{ (x, z) \in \mathbb{R}^2 : x = X(\mu, t), \ z = Z(\mu, t), \ \mu \in \mathbb{R}\}.$$

The velocity potential at the free surface is defined by $\Phi(\mu, t) = \phi(X, Z, t)$. Benjamin and Olver (1982) (see their equation (A6) in Appendix 1) show that the governing equations for this problem are Hamiltonian,

$$\mathbf{K}(\mathbf{U})\mathbf{U}_t = (\delta H/\delta X, \delta H/\delta Z, \delta H/\delta \Phi)^{\mathrm{T}}, \quad \mathbf{U} = (X, Z, \Phi)^{\mathrm{T}}. \tag{21}$$

In (21) the operator $\mathbf{K}(\mathbf{U})$ takes the form

$$\mathbf{K}(\mathbf{U}) = \begin{bmatrix} 0 & -\Phi_\mu & Z_\mu \\ \Phi_\mu & 0 & -X_\mu \\ -Z_\mu & X_\mu & 0 \end{bmatrix}, \tag{22}$$

and, with $J = (X_\mu^2 + Z_\mu^2)^{1/2}$, and

$$H(\mathbf{U}) = \frac{1}{2} \int_{\mathbb{R}} \Phi_{(n)} \Phi J \, \mathrm{d}\mu + \frac{1}{2} g \int_{\mathbb{R}} Z^2 X_\mu \, \mathrm{d}\mu$$

is the total energy for the case of a parametrically defined surface. The symplectic structure for this system is obtained by differentiating the functional

$$\mathscr{A}(\mathbf{U}) = \int_{t_1}^{t_2} \int_{\mathbb{R}} \Phi(X_\mu Z_t - Z_\mu X_t) \, \mathrm{d}\mu \, \mathrm{d}t. \tag{23}$$

The proof that this form generates the symplectic structure is given in Benjamin and Bridges (1997). The first variation of $\mathscr{A}(\mathbf{U})$ recovers the left-hand side of (21),

$$\operatorname{grad} \mathscr{A}(\mathbf{U}) = \mathbf{K}(\mathbf{U})\mathbf{U}_t. \tag{24}$$

Using (24), a least action principle for (21) can be formulated. The system (21) is generated by the first variation of the Lagrangian

$$\mathscr{L}(\mathbf{U}) = \mathscr{A}(\mathbf{U}) - \int_{t_1}^{t_2} H(\mathbf{U})\,\mathrm{d}t. \tag{25}$$

Writing out the first variation of $\mathscr{L}$ yields the system

$$\begin{aligned} -\Phi_\mu Z_t + Z_\mu \Phi_t &= \delta H/\delta X,\\ \Phi_\mu X_t - X_\mu \Phi_t &= \delta H/\delta Z,\\ -Z_\mu X_t + X_\mu Z_t &= \delta H/\delta \Phi, \end{aligned} \tag{26}$$

which is equivalent to (21) with

$$\begin{aligned} \delta H/\delta X &= -gZZ_\mu - \frac{1}{2}(\Phi_{(x)}^2 + \Phi_{(z)}^2)Z_\mu - J\Phi_{(x)}\Phi_{(n)},\\ \delta H/\delta Z &= gZX_\mu + \frac{1}{2}(\Phi_{(x)}^2 + \Phi_{(z)}^2)Z_\mu - J\Phi_{(z)}\Phi_{(n)},\\ \delta H/\delta \Phi &= J\Phi_{(n)}, \end{aligned} \tag{27}$$

where $\Phi_{(x)} = \phi_x|\Gamma$, $\Phi_{(z)} = \phi_z|\Gamma$ and $\phi_{(n)} = \nabla\phi \cdot \mathbf{n}|\Gamma$.

A special case of this Hamiltonian structure arises when one takes the parameterized surface to be the boundary value of a conformal mapping. Indeed, Eqs. (26) were discovered independently some fifteen years after Benjamin and Olver by Dyachenko et al. (1996). However, the conformal mapping formulation does lead to a numerically efficient algorithm. A short description follows. Using our convention, z is now $x + \mathrm{i}y$, with y the vertical coordinate. First the fluid domain in the z-plane

$$(x, y) \in \mathbb{R} \times [-\infty, \eta]$$

is transformed through a conformal mapping $Z(w, t)$ into the lower half-plane $w = u + \mathrm{i}v$

$$(u, v) \in \mathbb{R} \times [-\infty, 0].$$

The profile of the free surface is written in the parametric form

$$y = y(u, t), \quad x = x(u, t) = u + \tilde{x}(u, t).$$

Here $\tilde{x}(u, t)$ and $y(u, t)$ are related through the Hilbert transform

$$y = \mathscr{H}\tilde{x}, \quad \tilde{x} = -\mathscr{H}y, \quad \mathscr{H}^2 = -1 \quad \text{and} \quad \mathscr{H}f(u) = \mathrm{PV}\left(\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{f(u')\,\mathrm{d}u'}{u - u'}\right),$$

where PV stands for principal value. For periodic boundary conditions, the Hilbert transform is replaced by

$$\mathscr{H}f(u) = \mathrm{PV}\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} f(u')\cot[(u - u')/2]\,\mathrm{d}u'\right).$$

The function $Z = x + \mathrm{i}y$ and the complex potential $F(w, t)$ given in the lower-half plane are analytic there. The equations satisfied by F and Z are

$$Z_t = \mathrm{i}UZ', \tag{28}$$

$$F_t = \mathrm{i}UF' - B + \mathrm{i}g(Z - w), \tag{29}$$

where U and B are functions that are also analytic in the lower-half plane and are obtained from Z and F by using the projection operator $\widehat{P}(f) = \frac{1}{2}(1 + \mathrm{i}\mathscr{H})f$.[3]

The system (28)–(29) takes the following form if written in terms of $R = 1/Z'$ and $V = \mathrm{i}F'/Z'$:

$$R_t = \mathrm{i}(UR' - U'R), \tag{30}$$

$$V_t = \mathrm{i}(UV' - RB') + g(R - 1). \tag{31}$$

The system (30)–(31) appears to be particularly well-suited for numerical computations (see Zakharov et al., 2002), while the system (28)–(29) gives rise to numerical instabilities.

3. Progressive waves

The analytical study as well as the numerical study of progressive waves are far more advanced than those of other types of waves. The reason is that time can be removed from the equations by studying progressive waves in a frame of reference moving with the wave. Note that it is not necessarily easier to compute steady solutions. It is well-known for example that stationary solutions of the compressible Euler equations are easier to obtain as the limit of time-dependent solutions. Here we review some aspects of the numerical methods for travelling waves which are relevant to the discussion of methods for time-dependent waves.

3.1. Two-dimensional progressive waves

Among progressive waves, 2D waves have been particularly well studied. The main reasons are that all methods work in 2D and that the computing cost is reasonable.

3.1.1. Periodic waves

Let us consider periodic waves of wavenumber k and frequency ω. The dispersion relation is $\omega^2 = gk \tanh kh$. In order to put the equations into non-dimensional form, it is natural to choose $1/k$ as reference length (the length of the computational domain is then equal to 2π) and $(gk)^{-1/2}$ as reference time (the speed of linearized gravity waves in deep water is then equal to 1). Because of the assumption that the wave is propagating without change of shape one may then solve for the functions $\phi(X, z)$ and $\eta(X)$, periodic in X with period 2π, where

$$X = kx - \omega t. \tag{32}$$

[3] Equations similar to (28) and (29) were obtained by Ovsyannikov (1973). More precisely, he obtained the imaginary part of Eq. (28) and the real part of Eq. (29). Moreover, Eq. (19) in Whitney (1971) is exactly the same as (4.6) and (4.5) in Dyachenko et al. (1996).

Normal practice is to superimpose upon the flow a velocity equal and opposite to the wave speed $c=\omega/k$ so that the motion becomes steady. Traditionally the main approach to the nonlinear problem of a train of waves of constant form propagating steadily over a flat bottom has been based on expansions in a small parameter. The best known is that of Stokes in which this parameter is the leading coefficient of a Fourier expansion, appearing in dimensionless form as ka_1. This expansion has been taken to very high order by Schwartz (1974) and Cokelet (1977). Bloor (1978) combined truncated Fourier series with conformal transformation to compute large amplitude surface waves. Tanaka (1983, 1985) used a method based on Levi-Civita's formulation with an additional conformal transformation to compute steep gravity waves as well as their stability. Rienecker and Fenton (1981) developed a method based on Fourier approximation techniques, having as its only approximation the truncation of the Fourier series and valid for deep and shallow water (but not for the solitary wave limit).

Several valuable solutions for maximum waves have been obtained by imposing a crest of the correct form and building other standard terms around it to account for the remainder of the profile. The maximum wave means the wave of largest amplitude. The highest point is a stagnation point in a frame of reference moving with the wave and has a discontinuity in slope. The enclosed crest angle is 120°. Yamada (1957a) was probably first to solve the deep-water limiting wave (see the book by Okamoto and Shōji, 2001, for a detailed description of Yamada's method). Williams (1981) improved the accuracy of this type of method and extended it to all ratios between depth and wavelength and to waves of less than maximum height. It was found that two terms rather than one were needed to describe the crest singularity. The second term was suggested by the work of Grant (1973).[4] The typical number of terms needed to obtain the required accuracy ranges from 80 at the shallow-water end of the computing range to 20 at the deep-water end. For example, the maximum deep-water wave is characterized by a ratio height over wavelength equal to 0.141063 and a speed $c/(g/k)^{1/2}$ equal to 1.092282. The wave height is the distance from crest to trough. Williams (1981) provides an impressive set of Tables for heights, speeds and integral properties of various waves.

The most accurate solutions for large amplitude periodic Stokes waves were obtained by Chandler and Graham (1993) by solving Nekrasov's integral equation (15) numerically. The method consists of applying a simple quadrature rule to that rearranged version of the original equation (14). Strongly graded meshes are used to resolve a boundary layer in the solution due to Gibbs phenomena. Chandler and Graham were able to compute waves very close to the limiting periodic Stokes wave.

The equations originally derived by Stokes are cubic in the Stokes coefficients. They have been used by many authors to study various properties of water waves. Longuet-Higgins (1978) discovered a system of quadratic equations for the Stokes coefficients which is equivalent to the original Stokes system of cubic equations. This system is simpler and makes it easier to get higher accuracy in numerical calculation of Stokes coefficients. New families of steady waves were discovered. Longuet-Higgins' derivation was somewhat mysterious and Balk (1996) later found a general principle leading directly to the quadratic system. At the same time, Balk found canonical variables which allow for overturning waves (the surface elevation is no longer restricted to be a single-valued function of the horizontal coordinates). The Lagrangian is a fourth-order polynomial in the Stokes coefficients, and their first time derivatives. For waves of permanent form (when the time derivatives of the Stokes coefficients are given by the speed of the wave and the Stokes coefficients themselves) this Lagrangian becomes a cubic polynomial and the equations for the Stokes coefficients (given by the variation of the Lagrangian) turn out to be quadratic,

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[4] An increased understanding of the almost-highest waves was provided by Longuet-Higgins and Fox (1977, 1978), who combined the method of matched asymptotics expansions with some numerical approximation.

equivalent to Longuet-Higgins' equations. Once the Lagrangian is found, the problem of computing the evolution of the water surface is reduced to the extremization of a functional of independent generalized coordinates (the principle of least action). A numerical scheme can be designed to preserve important properties of the original continuous system.

As said above, the focus of this review is not on steady periodic waves. However, it should be mentioned that there have been a lot of work on various bifurcations of periodic waves, following the seminal paper by Chen and Saffman (1980a) stating that gravity waves in water of infinite depth are not unique. Chen and Saffman used the method of Fourier series truncation. Vanden-Broeck (1983) generalized these results to finite depth by using an integro-differential formulation of the water-wave problem (see Section 2.3).

Capillary-gravity periodic waves have been studied numerically by various authors. The first paper is that of Schwartz and Vanden-Broeck (1979). They used an integral formulation based on Cauchy's theorem (with the cotangent kernel) and used a finite-difference method for the discretization. Bifurcations of steady gravity-capillary waves on deep water have been studied by Chen and Saffman (1980b) and various other authors (see for example the book by Okamoto and Shōji, 2001).

3.1.2. Solitary waves

A review of the most important investigations on solitary waves up to 1980 can be found in Miles (1980). Yamada (1957b) is the first known author to have solved for the limiting solitary wave numerically (see again the book by Okamoto and Shōji, 2001, for a detailed description of Yamada's method). Lenau (1966) used a series truncation method to compute the same wave. More precisely the complex velocity potential is expressed in terms of known singularities and an infinite power series with unknown coefficients. Approximate solutions are obtained by truncating the power series after N terms. Williams (1981) improved the accuracy. The maximum solitary wave is characterized by a ratio height over undisturbed depth equal to 0.8332 and a Froude number $c/(gh)^{1/2}$ equal to 1.290889.[5] It is interesting to note that Lenau's value for the Froude number is 1.2793 with only 1 term and 1.2862 with 9 terms! Given the fact that methods based on truncated power series are easy to implement, they deserve more attention. See also Hunter and Vanden-Broeck (1983b) for numerical results based on an integral-equation formulation.

Even though the focus is not on solitary waves, it is useful to mention some papers on the numerical computation of capillary-gravity solitary waves. Most results are based on an integro-differential equation formulation: Hunter and Vanden-Broeck (1983a), Vanden-Broeck (1991), Vanden-Broeck and Dias (1992), Dias et al. (1996), Champneys et al. (2002). The stability of solitary waves has been studied numerically by Tanaka (1986) and by Calvo and Akylas (2002).

3.2. Three-dimensional progressive waves

The simplest 3D waves one can consider are the so-called short-crested waves. These waves, which come from the superposition of two oblique travelling waves with the same amplitude, are symmetric doubly-periodic waves. The dispersion relation is given by

$$\omega^2 = g|\mathbf{k}|\tanh(|\mathbf{k}|h)\left(1+\frac{\sigma|\mathbf{k}|^2}{\rho g}\right),$$

where $|\mathbf{k}| = \sqrt{k^2+l^2}$ and k and l are the x- and y-direction wavenumbers, respectively.

[5] An increased understanding of the almost-highest solitary waves was provided by Longuet-Higgins and Fox (1996).

Quite different in nature are the spontaneous (here we use the terminology of Saffman and Yuen, 1985) 3D waves that result from the bifurcation of a 2D Stokes wave of finite amplitude. The origin of these 3D waves can be understood theoretically from the results of McLean et al. (1981) on the stability of 3D perturbations of a 2D progressive wave.

A third type is the general class of 3D progressive waves of which the above two are special cases. Rigorous and numerical results on these latter waves have been reported by Craig and Nicholls (2002).

The first numerical method to compute short-crested waves was introduced by Roberts and Schwartz (1983). They used a truncated Fourier-series method based on collocation to compute short-crested waves in water of arbitrary depth and gave some results for moderately steep waves in deep water. Because of the assumption that the wave is propagating without change of shape one may then solve for the functions $\phi(X, Y, z)$ and $\eta(X, Y)$, periodic in X and Y with period 2π, where

$$X = kx - \omega t, \quad Y = ly. \tag{33}$$

Roberts (1983) used high-order perturbation expansions in the spirit of those of Schwartz (1974) and Cokelet (1977) for 2D progressive waves and those of Schwartz and Whitney (1981) for standing waves (see below). Double-precision coefficients were calculated up to the 27th order in just over two minutes of computing time (at the time). Marchant and Roberts (1987) and Okamura et al. (2003) extended the computations of Roberts (1983) to finite depth. The stability of short-crested waves has been studied numerically by Ioualalen and Kharif (1993), Badulin et al. (1995), Ioualalen and Okamura (2002) and others. To our knowledge, the 3D high-order solutions with long crests computed analytically by Roberts and Peregrine (1983) have not yet been computed by 3D numerical codes based on the full water-wave equations.

Using Zakharov's equation (see Section 5), Saffman and Yuen (1980) found a new class of 3D deep-water gravity waves of permanent form. The solutions were obtained as bifurcations from plane Stokes waves. It was pointed out that the bifurcation is degenerate since there are two families of 3D waves, one symmetric about the direction of propagation and the other skewed. The first type of bifurcation gives rise to a steady symmetric wave pattern propagating in the same direction as the Stokes waves, whereas the second type gives rise to steady skew wave patterns that propagate obliquely from the direction of the Stokes waves. Later, Meiron et al. (1982) and Bryant (1985) computed steady 3D symmetric wave patterns from the full water-wave equations. These 3D waves always have symmetric fronts. The methods are similar, except that Meiron et al.'s method is based on collocation while Bryant's method is based on Fourier transforms.

The dominant instability of steep Stokes waves is due to 3D perturbations. This instability results from a resonant interaction between five components of the wave field. These 3D patterns take the form of crescent-shaped perturbations riding on the basic waves. Shemer and Stiassnie (1985) used the modified Zakharov equation (see Section 5) to study the long-time evolution of 3D instabilities of surface gravity waves in deep water. They derived from this equation the long-time history of the amplitudes of the components of the wave field consisting of a Stokes wave (carrier) and its most unstable, initially infinitesimal, disturbance. The 3D structures riding on the crests of the waves exhibit a front-back asymmetry. With large-scale simulation, Xue et al. (2001), Fuhrman et al. (2004)[6] and Fructus et al.

[6] Although Boussinesq-type models are not considered in the present review, an exception is made here because the computational results of Fuhrman et al. (2004) are the first examples of highly nonlinear (to the breaking point) deep-water wave modelling in two horizontal dimensions with a high-order Boussinesq model.

(2005) have confirmed the importance of these waves, and shown that they can be generated without dissipation.

There has been a recent interest in the numerical computation of 3D solitary waves. For example, Părău and Vanden-Broeck (2002) and Părău et al. (2005) used a numerical BIE method based on the algorithm of Forbes (1989).

4. Standing waves

The numerical computation of standing waves is much less developed than the computation of progressive waves. In addition to difficulties with numerics, standing waves have inherent analytical difficulties.

4.1. Two-dimensional standing waves

It is usual to introduce dimensionless quantities using the wavenumber k and the wave frequency ω of the standing wave. Then one can treat standing waves with 2π period for space and time without loss of generality. The wave steepness ε is defined as

$$\varepsilon = \tfrac{1}{2}[\eta(0,0) - \eta(\pi,0)].$$

Schwartz and Whitney (1981) developed a semi-analytic algorithm for calculating the coefficients in a formal power series expansion of a standing wave to any order, provided that a resonance does not occur in executing the N^2 step in the algorithm, for any integer N. They obtained the 25th order solution for very large standing waves by a time-dependent conformal mapping method. Using Domb and Sykes's method and Padé approximants, they estimated the maximum wave steepness to be between 0.641 and 0.669. Vanden-Broeck and Schwartz (1981) computed standing waves on water of finite depth. Their numerical procedure involves series truncation combined with collocation.

Mercer and Roberts (1992) obtained extremely steep standing waves by a method based on the semi-Lagrangian approach. Their results showed that the standing wave has a maximum wave steepness of 0.6202. Tsai and Jeng (1994) calculated the highest standing waves of finite depth with a truncated double Fourier series up to 16th order. They estimated the maximum wave steepness to be 0.641 in deep water. Okamura (2003) used a perturbation series on only the velocity potential up to 30th order in the wave amplitude to obtain the nearly limiting standing wave. Okamura's method is an extension of Tsai and Jeng's method. However, Okamura used a Galerkin method while Tsai and Jeng used a collocation method. Okamura's conclusions are that the enclosed crest angle of the limiting wave is 90° and that the maximum wave steepness is 0.6272 when the limiting wave profile is reached. The discrepancy among the various estimates of the maximum wave steepness still is an open problem.

Vanden-Broeck (1984) provided numerical results for capillary-gravity standing waves by using the same numerical method as in Vanden-Broeck and Schwartz (1981). Bryant and Stiassnie (1994) computed standing waves with several dominant modes by using Fourier series expansions.

4.2. Three-dimensional standing waves

The only papers in this direction are those of Bryant and Stiassnie (1995) and of Zhu et al. (2003). The analysis of Zhu et al. employed the transition matrix approach and a high-order spectral element method.

The method was used to study the stability of three-dimensional perturbations of a two-dimensional standing wave.

5. Statistical description of water waves

The statistical description of water waves is an important topic, because of the variety of its practical applications, including the theory of wind-driven sea waves and the theory of freak waves (cf. Janssen, 2004). The full three-dimensional water-wave equations are a subject of major interest, but the simpler two-dimensional equations are interesting too. It is well-known from observations that the spectrum of wind-driven sea waves near the leading frequency is narrow in angle. Thus, the two-dimensional approach makes sense. The statistical description goes back to the pioneering work of Hasselmann (1962, 1963a, 1963b) who derived the kinetic equation for surface waves. Details on the nonlinear dynamics of water waves can be found in the monograph by Zakharov et al. (1992) and the reviews by Yuen and Lake (1982) and Zakharov et al. (2004).

Several researchers have used Zakharov's equation for a statistical description of water waves. For the description of Zakharov's equation, we restrict the analysis to gravity waves in deep water. Fourier transforms are introduced

$$\Phi_k(t) = \frac{1}{2\pi}\int \Phi(\mathbf{x},t)\,\mathrm{e}^{-i\mathbf{k}\cdot\mathbf{x}}\,\mathrm{d}\mathbf{x}, \qquad \eta_k(t) = \frac{1}{2\pi}\int \eta(\mathbf{x},t)\mathrm{e}^{-i\mathbf{k}\cdot\mathbf{x}}\,\mathrm{d}\mathbf{x}.$$

One can introduce the normal variables

$$\eta_k = \frac{1}{\sqrt{2}}\left(\frac{|\mathbf{k}|}{g}\right)^{1/4}(a_k + a^*_{-k}), \qquad \Phi_k = \frac{\mathrm{i}}{\sqrt{2}}\left(\frac{g}{|\mathbf{k}|}\right)^{1/4}(a_k - a^*_{-k}), \tag{34}$$

where $(*)$ stands for complex conjugation. Using the normal variables (34), the Hamiltonian evolution system (20) written in Fourier space becomes

$$\frac{\partial a_k}{\partial t} + \mathrm{i}\frac{\delta H}{\delta a^*_k} = 0. \tag{35}$$

The expression for the Hamiltonian is

$$H = H_0 + H_1 + H_2 + \cdots, \tag{36}$$

where

$$H_0 = \frac{1}{2}\int (|\mathbf{k}||\Phi_k|^2 + g|\eta_k|^2)\,\mathrm{d}\mathbf{k},$$

$$H_1 = -\frac{1}{4\pi}\int (\mathbf{k}_1\mathbf{k}_2 + |\mathbf{k}_1||\mathbf{k}_2|)\Phi_1\Phi_2\eta_k\delta(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k})\,\mathrm{d}\mathbf{k}_1\,\mathrm{d}\mathbf{k}_2\,\mathrm{d}\mathbf{k},$$

$$H_2 = -\frac{1}{(4\pi)^2}\int \left[|\mathbf{k}_1| + |\mathbf{k}_2| - \frac{1}{2}(|\mathbf{k}_1+\mathbf{k}_3| + |\mathbf{k}_2+\mathbf{k}_3| + |\mathbf{k}_1+\mathbf{k}| + |\mathbf{k}_2+\mathbf{k}|)\right]$$
$$\times\, |\mathbf{k}_1||\mathbf{k}_2|\Phi_1\Phi_2\eta_3\eta_k\delta(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3+\mathbf{k})\,\mathrm{d}\mathbf{k}_1\,\mathrm{d}\mathbf{k}_2\,\mathrm{d}\mathbf{k}_3\,\mathrm{d}\mathbf{k}.$$

One can perform a canonical transformation that eliminates the cubic terms in the Hamiltonian. This is a cumbersome transformation, which is described in detail in the article by Krasitskii (1990). In the new variable, say b_k, the equation still has the canonical form

$$\frac{\partial b_k}{\partial t} + \mathrm{i}\frac{\delta H}{\delta b_k^*} = 0, \tag{37}$$

where

$$H = H_0 + H_{\mathrm{int}}, \tag{38}$$

with

$$H_0 = \int \omega_k |b_k|^2 \, \mathrm{d}\mathbf{k}, \quad \omega_k = \sqrt{g|\mathbf{k}|},$$

$$H_{\mathrm{int}} = \frac{1}{4} \int T_{123k}^{WW} b_1 b_2 b_3^* b_k^* \delta(\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k}_3 - \mathbf{k}) \, \mathrm{d}\mathbf{k}_1 \, \mathrm{d}\mathbf{k}_2 \, \mathrm{d}\mathbf{k}_3 \, \mathrm{d}\mathbf{k}.$$

In the expression for H_{int} and consequently in Eq. (38), only the terms corresponding to four-wave interactions have been kept. Higher-order wave interactions have been neglected. The coupling coefficient T_{123k}^{WW} is a complicated homogeneous function of $\mathbf{k}_1$, $\mathbf{k}_2$, $\mathbf{k}_3$ and $\mathbf{k}$:

$$T^{WW}(\xi\mathbf{k}_1, \xi\mathbf{k}_2, \xi\mathbf{k}_3, \xi\mathbf{k}) = \xi^3 T^{WW}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3, \mathbf{k}), \quad \xi > 0. \tag{39}$$

The full expression for the interaction coefficient T_{123k}^{WW} can be found for example in Krasitskii (1990) (see also the earlier work by Zakharov, 1968).

It seems natural to solve numerically Eq. (37). However, the complexity of the coupling coefficient T_{123k}^{WW} does not allow the use of the most efficient spectral codes. It is more economical to solve directly Eq. (20). The statistical description of 3D water waves is a tremendous task. The first attempt due to Pushkarev and Zakharov (1996) (see also Pushkarev and Zakharov, 2000, and Dyachenko et al., 2003a) was devoted to pure capillary waves. Ensembles of capillary waves are less difficult to compute because the dominant wave interactions are the three-wave interactions. Recently, several groups have tried to solve Eq. (20) on the computer: Onorato et al. (2002), Dyachenko et al. (2003b, 2004), and also Janssen (2003) for computations on the Zakharov equations. Onorato et al. (2002) and Dyachenko et al. (2003b) used a wave field contained in a square domain with a 256×256 resolution, while Dyachenko et al. (2004) used a 512×512 resolution. They achieved some success but found that the whole business is tricky, and the computations easily become unstable. All these numerical experiments are consistent with weak-turbulence theory.

The verification of Hasselman's energy transfer among surface gravity waves was performed by direct numerical simulation by Tanaka (2001). Tanaka and Yokoyama (2004) studied the effects of discretization of the spectrum on the evolution of weak turbulence of surface gravity waves, again by direct numerical simulation. In both papers, the high-order spectral method developed independently by West et al. (1987) and Dommermuth and Yue (1987) was used. The technique uses a slope expansion of the velocity potential at the free surface. More precisely, the formal expansion about the free surface of Bernoulli's equation and the kinetic boundary condition are retained to arbitrary order. The numerical integration of these latter equations is done by taking products of field quantities in configuration space (the surface displacement, velocity potential, and gradients of these terms), fast Fourier transforming the configuration equations and time incrementing the transformed equations to obtain the components of the appropriate field variables, then transforming back to configuration space to again evaluate the nonlinear products and start the process again.

6. Numerical wave tanks

The idea behind numerical wave tanks is to reproduce (or even replace) laboratory experiments with computer simulations. A typical 3D numerical wave tank has several boundaries: a wave-maker at one end, an absorbing beach at the other end, an impervious boundary at the bottom and on the sides, and a free surface at the top. However, the terminology 'wave tank' is used below abusively even for methods that can only include fixed vertical boundaries.

6.1. Two-dimensional numerical wave tanks

Boundary integral methods described in Section 2.2 have been used successfully for two-dimensional waves by Longuet-Higgins and Cokelet (1976), Vinje and Brevig (1981) and New et al. (1985). Most codes have been based on their formulations. Calculations in 2D have been extensive. Dommermuth et al. (1988) were able to reproduce overturning waves generated by frequency focusing all the way to breaking. As shown in Section 2, Laplace's equation is solved with a high-order BIE method, based either on Green's identity or on Cauchy's integral theorem formulations. Time integration of the free-surface boundary conditions expressed in a mixed Eulerian–Lagrangian formulation is performed using a time marching Runge–Kutta scheme, or a predictor–corrector scheme, or both (Longuet-Higgins and Cokelet), or a Taylor series expansion method (Dold and Peregrine, 1986, and Dold, 1992). Early computations following this approach were restricted to space-periodic waves over constant depth. Later computations were able to accommodate both arbitrary incident waves and complex bottom topographies (Ohyama and Nadaoka, 1991). State-of-the-art models directly work in a physical space region, in which incident waves are generated at one extremity and reflected, absorbed, or radiated at the other extremity (see for example Grilli and Subramanya, 1996; Grilli and Horrillo, 1997). The code first developed by Dold and Peregrine (1986) has been used for various computations by Banner and Tian (1998), Henderson et al. (1999) and Song and Banner (2002), and extended by Tanaka et al. (1987). Grilli and Subramanya (1996), who used a high-order boundary element method, provide a comparison between three types of boundary elements: iso-parametric, quasi-spline and MII/MCI (where MII stands for Mid Interval Interpolation and MCI stands for MII for the potential and spline for the geometry).

Efficient time-dependent numerical codes based on the formulation presented in Section 2.5 have been developed by Chalikov and Sheinin (1998) and Zakharov et al. (2002). Precursors to this work include Whitney (1971) and Fornberg (1980). An anonymous referee reports that Fornberg (1980) eventually gave up the direction of numerical conformal mapping because of the inefficiency due to the propensity of the method to disperse points away from sharp and overturning crests.

As said in Section 5, higher-order spectral methods have been developed by West et al. (1987) and Dommermuth and Yue (1987). Skandrani et al. (1996) applied the numerical method developed by Dommermuth and Yue (1987) to nonlinear gravity waves in the presence of weak viscous effects and surface tension and observed the occurrence of downshift.

6.2. Three-dimensional numerical wave tanks

The boundary element method (BEM) is the predominant method of choice for the spatial discretization of 3D numerical wave tanks. Wave tanks based on BEM have been designed by Romate (1990a, 1990b), Romate and Zandbergen (1989), Broeze and Romate (1992), Broeze et al. (1993), Grilli et al. (2001). There is currently much activity in this area, with much potential for new developments. For example, three-dimensional wave tanks have been used by Guyenne and Grilli (2006) to study overturning waves in shallow water, by Grilli et al. (2002) to study tsunami generation, by Fructus et al. (2005) to compute three-dimensional instabilities of Stokes waves.

6.2.1. Time integration

Once the boundary-value problem at each time step is solved and the velocity on the free surface obtained from Φ and $\Phi_{(n)}$, the free-surface boundary conditions (12) and (13) can be integrated in time in a straightforward manner. Effectively, the problem has been reduced to a very large system of ordinary differential equations.

Xue et al. (2001) used a fourth-order Adams–Bashforth–Moulton integrator coupled with a fourth-order Runge–Kutta scheme. Grilli et al. (2001) used a second-order explicit Taylor series expansion. Zeroth-order coefficients are given by the geometry and the solution of the BIE (11) at time t. First-order coefficients are then directly obtained from the boundary conditions (12) and (13). Second-order coefficients are obtained from the material derivative of Eqs. (12) and (13), which requires solving a BIE similar to (11) for $\partial\phi/\partial t$. The BIEs for ϕ and $\partial\phi/\partial t$ are solved at time t and thus correspond to the same boundary geometry and have the same discretized form (see Section 6.2.3). The use of second-order terms leads to a better accuracy of the time scheme. The time step Δt is adapted at each time as a function of the minimum distance between two neighbouring nodes on the free surface and the magnitude of the maximum (nodal) velocity on the free surface. An optimal Courant number (typically less than 1) is used.

In mixed Eulerian–Lagrangian simulations, in the absence of numerical damping, saw-tooth instabilities eventually develop on the free surface as nonlinearity increases, as observed first by Longuet-Higgins and Cokelet (1976). The presence of saw-tooth instabilities can be expected in the theory for nonlinear systems without dissipation, wherein energy flows from low to high wavenumbers and accumulates at the highest wavenumber associated with the discretization. It is said sometimes that the appearance of saw-tooth instabilities in the simulation of overturning waves depends on the BIE formulation. With the Green-theorem formulation, severe saw-tooth instabilities usually appear near the crests of steep waves. On the other hand, no apparent saw-tooth instability effect is observed with the Cauchy-integral formulation, as emphasized for example by Dold (1992). A possible reason for this is that with the Green-theorem/Cauchy-integral formulation, one encounters the first/second-kind Fredholm integral equation. Usually, it is easier to obtain an accurate solution to the second-kind equation than to the first-kind equation. Another possible reason is that use of explicit integrators such as fourth-order Runge–Kutta results in high-wavenumber instabilities unless unnatural restrictions are put on the time step (see the next subsection for discussion of this point). To remove instabilities, several smoothing techniques have been proposed.

The lack of saw-tooth instabilities observed by Dold (1992) or Grilli et al. (2001) may also be related to the use of Taylor series expansions for time stepping, or the high-order accuracy of spatial derivatives, or their combination. Dold and Peregrine (1986) in particular used an 11 point formula for derivatives.

6.2.2. Towards symplectic integrators for water waves

A long overlooked problem in the numerical simulation of water waves is neglect of the Hamiltonian structure in the design of numerical method. In celestial mechanics and molecular dynamics the use of symplectic integrators is pervasive because they give better performance over long times and respect the energy budget better (see for example the recent monograph by Leimkuhler and Reich, 2005).

There are two principal reasons why symplectic integrators might improve the performance of numerical wave tanks. First, they show excellent energy conservation properties over extraordinarily large time intervals. Second they perform much better than non-symplectic methods for systems of oscillators. In other words, they do far less damage to the Fourier spectrum. This latter point can be seen explicitly on the *linear* problem as follows.

After semi-discretization, the linear water-wave problem is reduced to a system of linear ordinary differential equations of the form $\mathbf{u}_t = \mathbf{A}\mathbf{u}$, where $\mathbf{u} \in \mathbb{R}^N$ for some large N. For purposes of discussion take $\mathbf{A}$ to be a constant matrix. As is easily verified, the classical fourth-order explicit Runge–Kutta method reduces to the following one step method (when $\mathbf{A}$ is constant),

$$\mathbf{u}^{n+1} = \left[\mathbf{I} + h\mathbf{A} + \frac{1}{2}h^2\mathbf{A}^2 + \frac{1}{3!}h^3\mathbf{A}^3 + \frac{1}{4!}h^4\mathbf{A}^4\right]\mathbf{u}^n \tag{40}$$

for one time step h. Apply this to the linear 2D water wave problem in finite depth, linearized about the still water state. In this case, the water wave problem is an infinite coupled system of harmonic oscillators of the form

$$\dot{q}_k = -\omega_k p_k, \quad \dot{p}_k = \omega_k q_k, \quad k = 1, 2, \ldots, \quad \text{where } \omega_k^2 = gk\,\tanh(kh).$$

The numerical issue is: what happens to each harmonic oscillator under discretization by (40)? In this case $\mathbf{A}$ decomposes into 2×2 blocks, each of which has a simple form,

$$\mathbf{A} = \mathbf{A}_1 \oplus \cdots \oplus \mathbf{A}_k \oplus \cdots, \quad \text{with } \mathbf{A}_k = \omega_k \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

The orbit of each mode is circular with the radius determined by initial data, since

$$\frac{\mathrm{d}}{\mathrm{d}t} I_k = 0 \quad \text{where } I_k = q_k^2 + p_k^2.$$

The *continuous* system preserves each I_k for all time. In other words, in each two-dimensional subplane (q_k, p_k) lie on a circle. Does the numerical algorithm reproduce this? Using the discretization (40) applied to a 2×2 sub-block shows that

$$I_k^{n+1} = \left(1 - \frac{(h\omega_k)^6}{72} + \frac{(h\omega_k)^8}{576}\right) I_k^n. \tag{41}$$

The ratio I_k^{n+1}/I_k^n is plotted as a function of $h\omega_k$ in Fig. 1.

To summarize, the linear water wave problem (in a bounded domain, linearized about the flat state) is essentially an infinite system of harmonic oscillators, and the classical fourth-order Runge–Kutta method amplifies the amplitude of each component oscillator by the formula (41).

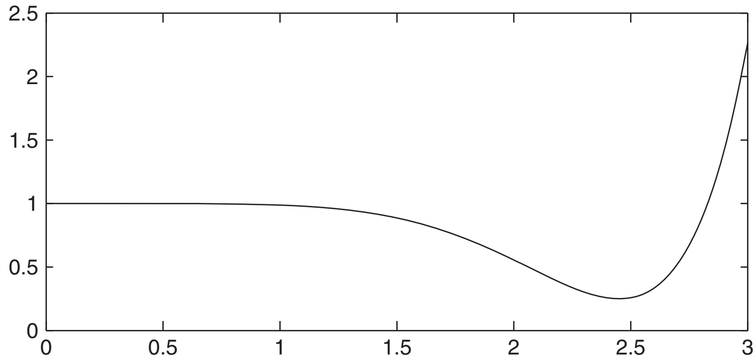

Fig. 1. Plot of the ratio I_k^{n+1}/I_k^n as a function of $h\omega_k$, showing the amplification of oscillatory modes given by (41).

Now, note that

$$\frac{I_k^{n+1}}{I_k^n} \leqslant 1 \quad \text{requires } \omega_k h < \sqrt{8},$$

so if $0 < \omega_k h < \sqrt{8}$ the amplitude of the oscillator is damped, and if $\omega_k h > \sqrt{8}$ the amplitude is growing. Since $I_k^{n+1} = I_k^n$ for at most one choice of k, only one Fourier mode can be preserved exactly, the rest being damped, or unstable. It is clear from this example that severe damage is being done to the Fourier spectrum.

Contrast this fourth-order Runge–Kutta method with for example the implicit midpoint method:

$$\mathbf{u}^{n+1} = \left[\mathbf{I} - \frac{h}{2}\mathbf{A}\right]^{-1} \left[\mathbf{I} + \frac{h}{2}\mathbf{A}\right]\mathbf{u}^n,$$

which will satisfy $I_k^{n+1} = I_k^n$ for all n and all k! The implicit midpoint rule is an example of a symplectic integrator. It is evident that further work is needed to clarify the potentialities of symplectic integrators for water wave simulation.

6.2.3. *Spatial discretization*

In three dimensions, integral equations are usually solved by a BEM (see for example Forbes, 1989; Părău and Vanden-Broeck, 2002; Părău et al., 2005, for other methods). Existing BIE solvers commonly employ piecewise-constant approximations for the unknowns, piecewise-linear approximation of the boundary, and collocation at panel centroids. This so-called 'constant-panel' method has a number of shortcomings. To circumvent these shortcomings, higher-order panel or BIE methods must be sought. Xue et al. (2001) used an iso-parametric quadratic BEM, based on piecewise bi-quadratic representation of both the boundary and the potential and its normal derivative on the boundary. The boundary panels are curvilinear quadrilaterals. The boundary is discretized into N collocation nodes and M high-order elements are used to interpolate in between m of these nodes. Within each element, the boundary geometry and the field variables are discretized using polynomial shape functions. Various elements can provide a high-order approximation within their area of definition but only offer C_0 continuity in between elements.

The integrals on the boundary are converted into a sum on the elements, each one being calculated on the reference element. Depending on the boundary type on which the current node lies, the potential, or its normal derivative, is either specified by a boundary condition or is an unknown of the problem.

The matrices are built with the numerical computation of the integrals on the reference element. When the collocation node does not belong to the integrated element, a standard Gauss–Legendre quadrature method is used.[7] When the collocation node belongs to the element, special methods are needed to take account of the singularities in the evaluation of the Green's functions. Some of the regular integrals can be nearly singular because of the accumulation of nodes. An adaptive technique that is based on recursive subdivisions is needed.

The linear algebraic systems resulting from the two BIEs (one for ϕ and one for $\partial\phi/\partial t$) are full and non-symmetric. Assembling the matrix as well as performing the integrations accurately are time consuming tasks. The assembling is done only once at each time step, since the same matrix is used for both systems.

Solving the linear system is another time consuming task. A classical algorithm using a direct solver for the linear system turns out to be very expensive with an $O(N^3)$ complexity. This is why the generalized minimal residual (GMRES) algorithm with preconditioning is often used to solve the linear systems iteratively (see Saad and Schultz, 1986). The computational complexity is reduced to $O(N^2)$, which is the same as the complexity of the assembling phase. Note that since there are only small changes in domain shape and variables at each time step, iterative methods for solving the full linear system can be effective. However, there is a minor drawback of the iterative method when a time stepping scheme based on Taylor series expansion is used. As said above (see Section 6.2.1), two systems of equations must be solved at each time step—one for ϕ and one for $\partial\phi/\partial t$. While the solution of the second system takes only a few per cent of the time needed to solve the first system with a direct method, two full systems must be solved with the iterative method. Nevertheless, iterative methods are faster for large systems.

Fochesato and Dias (2006) recently incorporated the fast multipole algorithm in the numerical wave tank of Grilli et al. (2001), reducing the complexity to near $O(N)$. The fast multipole algorithm was first used for 3D problems governed by a Laplacian by Greengard and Rokhlin (1997) (see also Cheng et al., 1999). For 2D free-surface flows, the fast multipole algorithm was used by Graziani and Landrini (1999), but they did not use boundary elements to discretize the problem. They used the Euler–McLaurin quadrature formula.

There are several results available on the convergence of boundary integral methods (see for example Beale, 2000; Beale et al., 1996; Hou and Zhang, 2001, 2002).

6.3. Three-dimensional wave tank based on Fourier expansions

The methodology described here has its roots in the unidirectional wave model proposed by Fenton and Rienecker (1982), which itself builds on the time-marching procedures originally proposed by Longuet-Higgins and Cokelet (1976). It was further developed by Bateman et al. (2001) and Johannessen and Swan (2003) in order to include the effects of directional spreading.

[7] Note that approximation of integrals over non-periodic domains by Gauss–Legendre quadrature is equivalent to the trapezoidal rule for integrals over a periodic domain (see Chapter 12 of Trefethen, 2000)

The water depth h is assumed to be constant. If the wave field is periodic in space (x, y) with large fundamental wavelengths $\lambda_x = 2\pi/k$ and $\lambda_y = 2\pi/l$, where k and l are the fundamental wavenumbers in the x- and y-directions, a solution for ϕ that satisfies both Laplace equation and the bottom boundary condition at $z = -h$ is given by

$$\phi = \sum_{m=0}^{M} \cos(mly) \left[A_{0m} + \sum_{n=1}^{N-1} \left((A_{nm} \cos(nkx) + B_{nm} \sin(nkx)) \frac{\cosh(k_{nm}(z+h))}{\sinh(k_{nm}h)} \right) + \left(A_{Nm} \cos(Nkx) \frac{\cosh(k_{Nm}(z+h))}{\sinh(k_{Nm}h)} \right) \right], \tag{42}$$

where $k_{nm} = (n^2k^2 + m^2l^2)^{1/2}$ and the unknown coefficients A_{nm} and B_{nm} are functions of time only. It is further assumed that there are no contributions to ϕ from components beyond the truncation wavenumbers Nk and Ml. The corresponding free-surface elevation is of the form

$$\eta = \sum_{m=0}^{M} \cos(mly) \left(a_{0m} + \sum_{n=1}^{N-1} (a_{nm} \cos(nkx) + b_{nm} \sin(nkx)) + a_{Nm} \cos(Nkx) \right), \tag{43}$$

where a_{nm} and b_{nm} are again functions of time only. Eqs. (42) and (43) are based on the assumption that the wave field is symmetric about $y = 0$. This imposed symmetry is not essential. It only provides a reduction of the computational effort.

The nonlinear free-surface boundary conditions define ϕ_t and η_t in terms of η and the spatial derivatives of η and ϕ evaluated at the water surface. A spatial representation of η and ϕ at some initial time $t = t_0$ allows ϕ_t and η_t to be defined and the solution successfully time-marched to $t = t_0 + \Delta t$. Repeated application allows the evolution of the wave field to be predicted over large times. The free-surface boundary conditions are applied at $2N(M + 1)$ spatial locations within a grid defined by

$$[x, y]_{ij} = \left[\frac{(i-1)\lambda_x}{2N}, \frac{(j-1)\lambda_y}{2M} \right], \quad i = 1, 2, \ldots, 2N, \quad j = 1, 2, \ldots, M + 1.$$

If M and N are integer powers of 2, the unknown time-derivatives within the description of η, $(a_{nm})_t$ and $(b_{nm})_t$, can be solved rapidly using standard fast Fourier transform techniques. In contrast, a similar approach cannot be applied to determine the unknown time-derivatives within the description of ϕ, $(A_{nm})_t$ and $(B_{nm})_t$, since the expression for ϕ incorporates the vertical structure of the flow. As a result, ϕ becomes a function of η when calculated at the water surface. Accordingly, $(A_{nm})_t$ and $(B_{nm})_t$ are evaluated using lower–upper factorization and back substitution. Therefore, the scheme is not comparable in computational efficiency with the best available unidirectional time-stepping procedures, notably those due to Dold (1992) and Craig and Sulem (1993). The coefficients of ϕ and η are updated using the Adams–Bashford–Moulton predictor-corrector method similar to that employed by Longuet-Higgins and Cokelet.

Clamond and Grue (2001) developed an efficient numerical wave tank, based on the vortex sheet method. Initially development was for two-dimensional deep water waves, with recent extension to 3D. The 2D version is based on Eqs. (18) and (19). Typically an iterative solution requires $O(N^2)$ operations but Clamond and Grue were able to reduce it to $O(N \log N)$ operations by reformulating the boundary integrals. It was extended to finite depth by Grue (2002). The 3D version was given in Section 7 of Clamond and Grue (2001) and implemented in Fructus et al. (2005).[8]

Bingham and Agnon (2005) developed a Fourier–Boussinesq method for nonlinear water waves. The equations are essentially of Boussinesq type but they include a generalized Hilbert transform in order to remove any limitation with respect to relative water depth. The Hilbert transform is evaluated via the fast Fourier transform.

7. Concluding remarks

The greatest progress in the numerical computation of water waves witnessed in the last five years has been the development of efficient three-dimensional wave tanks. A distinction must be made between methods that cannot represent wave overturning, such as those that use spatial Fourier representations, or assuming single values of surface elevation, and those that can describe wave overturning. The former may have problems giving fully convincing results. But the most common drawback of most existing fully nonlinear methods is that the computational schemes are generally explicit and so require very short time steps. It is still unclear whether one will be able to circumvent this difficulty, by using implicit methods for example, or if this difficulty is inherent to the water-wave problem. Recent developments such as the use of multipole expansion (Fochesato and Dias, 2006) may however speed up the internal time for each time step. Within a given method, it has been noticed too that tiny changes in the formulation can make big changes in the accuracy or the stability of the method. The computations by Fuhrman et al. (2004) based on a fully nonlinear and highly dispersive Boussinesq formulation combined with the efficient solution strategies developed in Fuhrman and Bingham (2004) are also promising. Although most computations have been based on the Euler equations, some computations that use the Navier–Stokes equations are emerging (see for example the fully nonlinear three-dimensional breaking waves recently computed by Lubin et al. (2003) by solving the Navier–Stokes equations in air and water). Finally, there is an urgent need for performing unbiased comparisons between the various 3D numerical wave tanks that are now available.

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[8] In the appendix of West et al. (1987), a generalization of the vortex sheet method to three dimensions is given. West et al. provide a brief comparison between their high-order spectral method and the (not yet numerically implemented) 3D vortex sheet method.

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