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Structural degree-degree dependencies in large networks

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Joint work with
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Ribno, 22-09-2016

Heavy-tailed degree distributions

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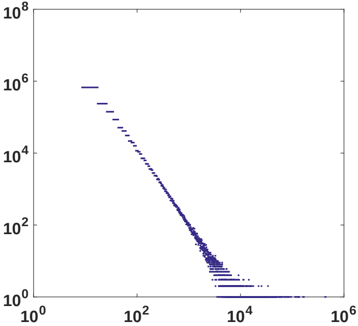

Loglog plot distribution in-degrees of English Wikipedia (data from U.Milan)

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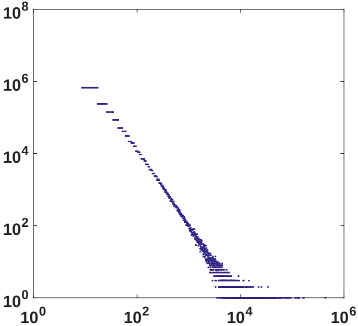

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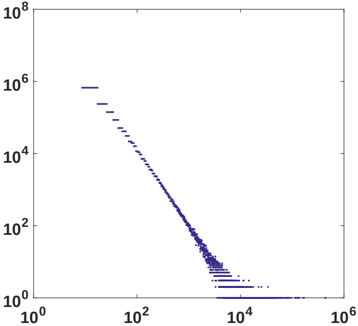

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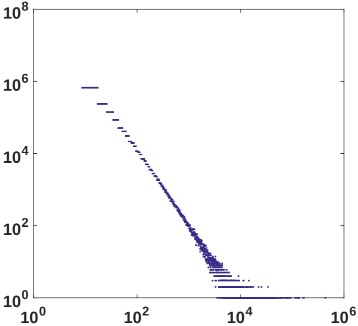

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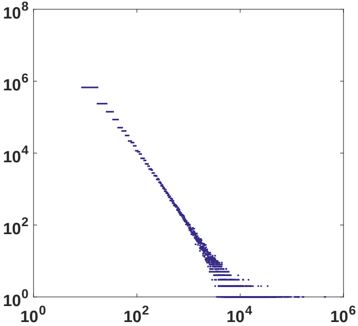

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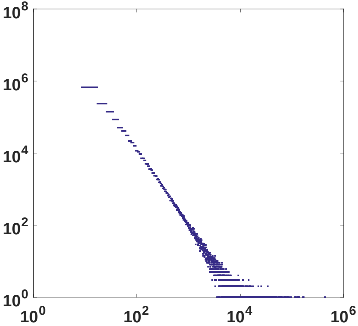

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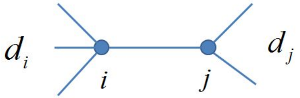
Assortativity coefficient



- $G = (V, E)$ undirected graph of n nodes, E'– directed edges
- D_i degree of node $i = 1, 2, \ldots, n$

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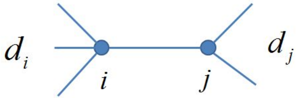

- $G = (V, E)$ undirected graph of n nodes, E'– directed edges
- D_i degree of node $i = 1, 2, \dots, n$
- Newman (2002): assortativity measure $\rho(G)$

$$\rho(G) = \frac{\frac{1}{|E'|}\sum_{(i,j)\in E'} D_i D_j - \left(\frac{1}{|E'|}\sum_{(i,j)\in E'} \frac{1}{2}(D_i + D_j)\right)^2}{\frac{1}{|E'|}\sum_{(i,j)\in E'} \frac{1}{2}(D_i^2 + D_j^2) - \left(\frac{1}{|E'|}\sum_{(i,j)\in E'} \frac{1}{2}(D_i + D_j)\right)^2}$$

- Statistical estimation of the Pearson's correlation coefficient between degrees on two ends of a random edge

Motivation

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- Information flow neural networks.
- Stability of P2P networks under attack.
- Epidemics on networks.
- Network Observability.
- Opinion dynamics based on social influence.
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Assortative and disassortative graphs

- Newman(2003)

	network	type	size n	assortativity r	error σ_r	ref.
social	physics coauthorship	undirected	52 909	0.363	0.002	a
	biology coauthorship	undirected	1 520 251	0.127	0.0004	a
	mathematics coauthorship	undirected	253 339	0.120	0.002	b
	film actor collaborations	undirected	449 913	0.208	0.0002	c
	company directors	undirected	7 673	0.276	0.004	d
	student relationships	undirected	573	−0.029	0.037	e
	email address books	directed	16 881	0.092	0.004	f
technological	power grid	undirected	4 941	−0.003	0.013	g
	Internet	undirected	10 697	−0.189	0.002	h
	World-Wide Web	directed	269 504	−0.067	0.0002	i
	software dependencies	directed	3 162	−0.016	0.020	j
biological	protein interactions	undirected	2 115	−0.156	0.010	k
	metabolic network	undirected	765	−0.240	0.007	l
	neural network	directed	307	−0.226	0.016	m
	marine food web	directed	134	−0.263	0.037	n
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- Technological and biological networks are disassortative, $\rho(G) < 0$
- Social networks are assortative, $\rho(G) > 0$
- Note: large networks are never strongly disassortative... Dorogovtsev et al. (2010), Raschke et al. (2010)

$\rho(G)$ via moments of the degrees

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- Write

$$\sum_{(i,j)\in E'} \tfrac{1}{2}(D_i + D_j) = \sum_{i=1}^{n} D_i^2, \quad \sum_{(i,j)\in E'} \tfrac{1}{2}(D_i^2 + D_j^2) = \sum_{i=1}^{n} D_i^3$$

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- Then

$$\rho(G) = \frac{\sum_{(i,j)\in E} D_i D_j - \frac{1}{|E|}\Big(\sum_{i=1}^{n} D_i^2\Big)^2}{\sum_{i=1}^{n} D_i^3 - \frac{1}{|E|}\Big(\sum_{i=1}^{n} D_i^2\Big)^2}.$$

Scaling of the terms in $\rho(G)$

$$\rho(G) = \frac{\textit{crossproducts} - \textit{expectation}^2}{\textit{variance}} \geqslant - \frac{\textit{expectation}^2}{\textit{variance}} = \rho^-(G)$$

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- We have $\sum_{i=1}^n D_i^3 \geqslant cn^{3/\gamma}$
- But also

$$\frac{1}{|E'|}\left(\sum_{i=1}^n D_i^2\right)^2 \leqslant (C^2/c)n^{\max\{4/\gamma-1,1\}}.$$

- $\rho^-(G) \to 0$ as $n \to \infty$ in ANY power law graph with $\gamma \in (1,3)$

Web and social networks

Dataset	Description	# nodes	max d	$\rho(G_n)$	$\rho(G_n)^{\mathrm{rank}}$	$\rho^{-}(G_n)$
stanford-cs	web domain	9,914	340	-0.1656	-0.1627	-0.4648
eu-2005	.eu web crawl	862,664	68,963	-0.0562	-0.2525	-0.0670
uk@100,000	.uk web crawl	100,000	55,252	-0.6536	-0.5676	-1.117
uk@1,000,000	.uk web crawl	1,000,000	403,441	-0.0831	-0.5620	-0.0854
enron	e-mailing	69,244	1,634	-0.1599	-0.6827	-0.1932
dblp-2010	co-authorship	326,186	238	0.3018	0.2604	-0.7736
dblp-2011	co-authorship	986,324	979	0.0842	0.1351	-0.2963
hollywood	co-starring	1,139,905	11,468	0.3446	0.4689	-0.6737

All graphs are made undirected

Convergence of $\rho(G)$ to a non-negative value

Theorem (L & vdHofstad 2013)

Let $(G_n)_{n\geqslant 1}$ be a sequence of graphs of size n satisfying that there exist $\gamma \in (1,3)$ and $0 < c < C < \infty$ such that

$$cn \leqslant |E| \leqslant Cn,$$
$$cn^{1/\gamma} \leqslant \max_{i=1,\dots,n} D_i \leqslant Cn^{1/\gamma},$$
$$cn^{(2/\gamma)\vee 1} \leqslant \sum_{i=1}^{n} D_i^2 \leqslant Cn^{(2/\gamma)\vee 1}.$$

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- Large scale-free graphs are never disassortative!
- Alternative: **rank correlations**

Degree-degree correlations in directed networks

- Generalize to directed networks
- Use rank correlations
- Null-model: Directed Configuration Model (DCM)
- Rank correlations on DCM: asymptotics and finite-size effects

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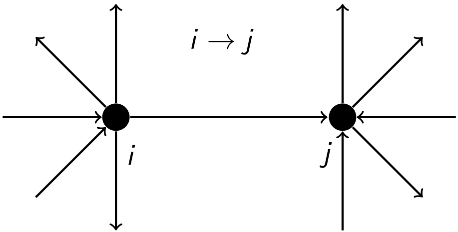

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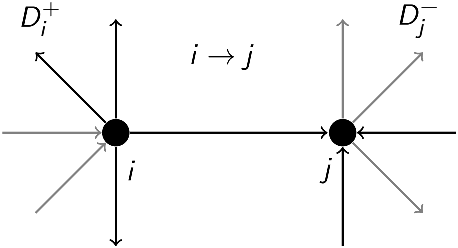

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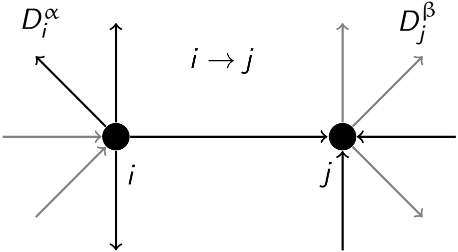

Index degree type by $\alpha, \beta \in \{+, -\}$.

Four types of degree-degree correlation

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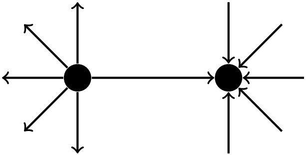
Out-In



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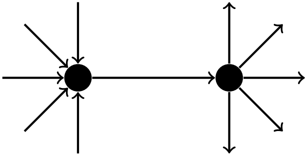

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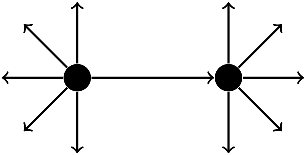

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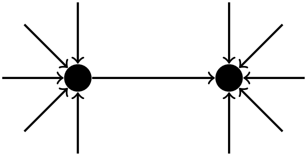

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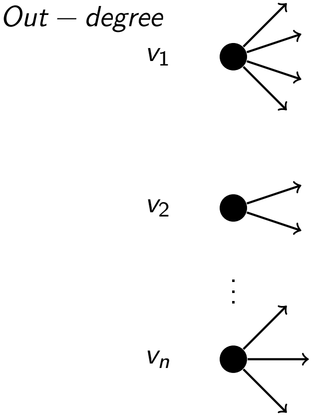

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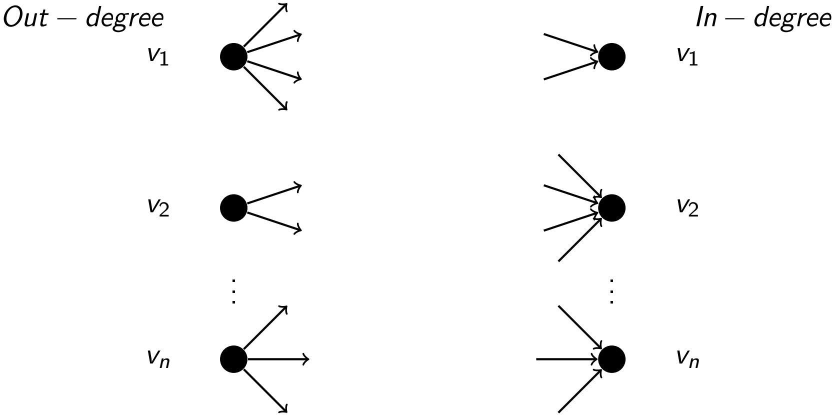

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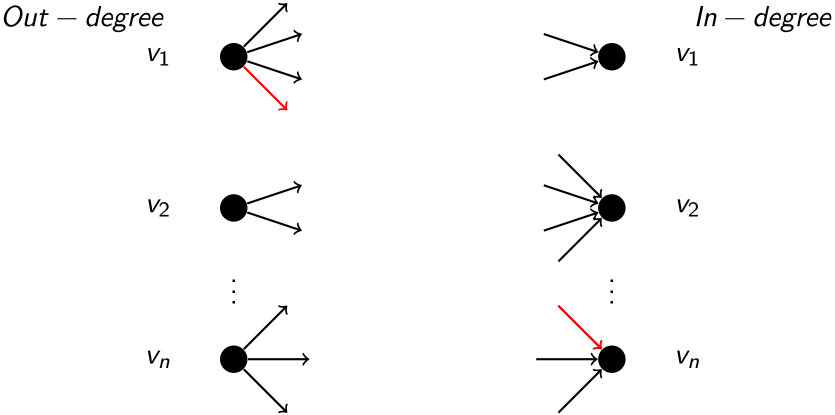

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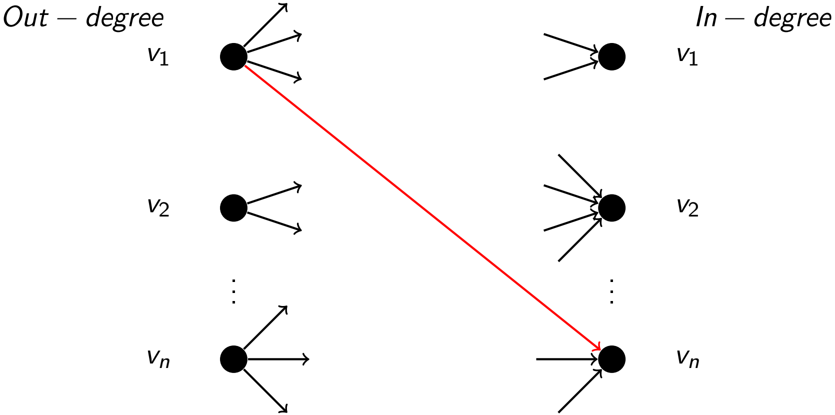

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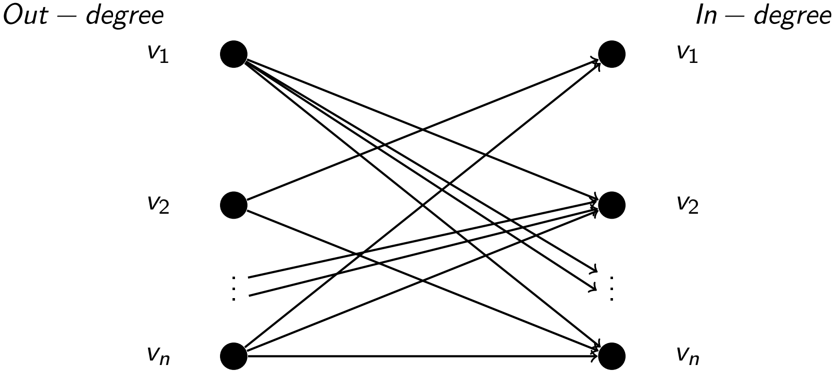

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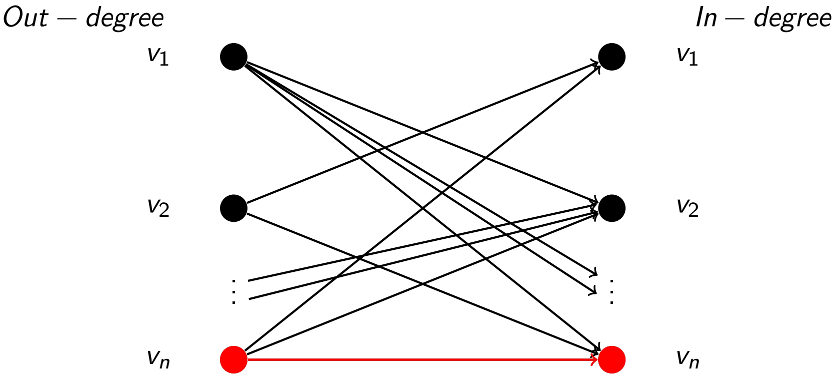

Remove self-loops and double edges. The result is a simple graph

Rank correlations: Spearman's rho

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$$\rho_\alpha^\beta(G_n) := r(R^\alpha, R^\beta)$$

Statistical consistency Spearman's rho

Theorem (vdHoorn and L 2014)

Let $\{G_n\}_{n\in\mathbb{N}}$ be a sequence of random graphs, $\alpha, \beta \in \{+, -\}$ and suppose there exist integer valued random variables $\mathcal{D}^\alpha$ and $\mathcal{D}^\beta$ such that

$$p_\alpha^\beta(k,\ell) \xrightarrow{\mathbb{P}} \mathbb{P}\left(\mathcal{D}^\alpha = k, \mathcal{D}^\beta = \ell\right) \quad \text{as } n \to \infty.$$

Then, as $n \to \infty$,

$$\rho_\alpha^\beta(G_n) \xrightarrow{\mathbb{P}} \rho\left(\mathcal{D}^\alpha, \mathcal{D}^\beta\right)$$

Structural correlations

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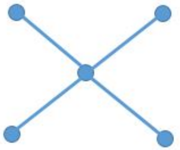

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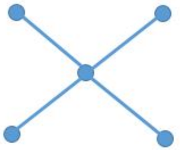

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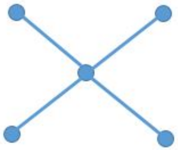

- There is only one way to make it a simple graph, and it is disassortative
- This phenomenon is called 'structural correlations'
- How large are structural correlations in the erased Directed Configuration Model?

Spearman's rho in the Erased Configuration Model

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Theorem (vdHoorn and L 2014)

Let $\{G_n\}_{n\in\mathbb{N}}$ be a sequence of graphs of size n, generated by either the Repeated or Erased Configuration Model and $\alpha, \beta \in \{+, -\}$. Then, as $n \to \infty$,

$$\rho_\alpha^\beta(G_n) \xrightarrow{\mathbb{P}} 0.$$

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- Number of erased edges of a node converges in distribution to zero. Chen and Olvera-Cravioto, 2013

Structural correlations in the Erased model

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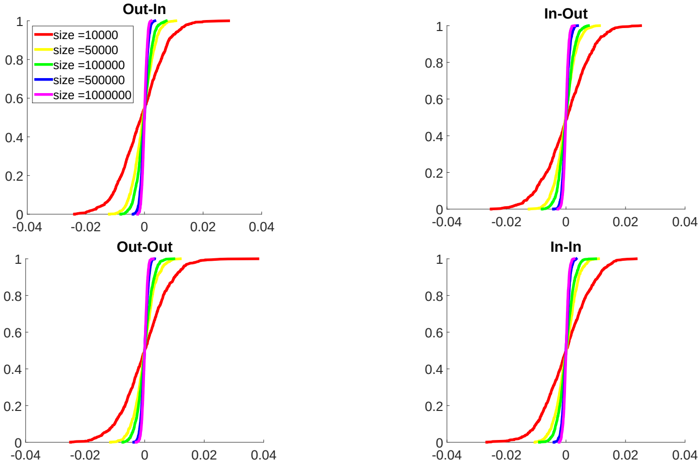

Figure : Empirical cdf of $\rho_\alpha^\beta(G_n)$ for ECM graphs with $\gamma_\pm = 2.1$

Structural correlations in the Erased model

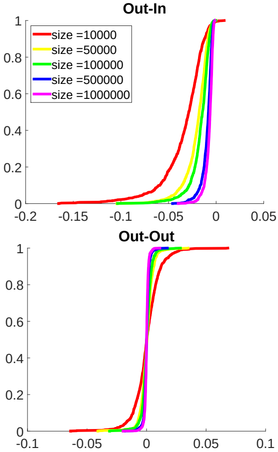

Figure : Empirical cdf of $\rho_{\alpha}^{\beta}(G_n)$ for ECM graphs with $\gamma_{\pm} = 1.5$

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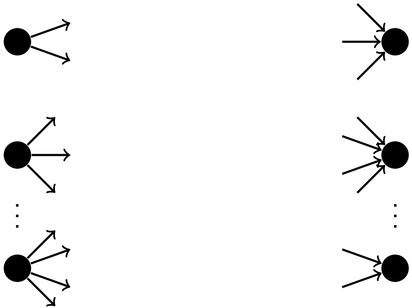

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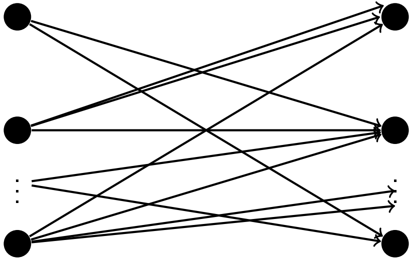

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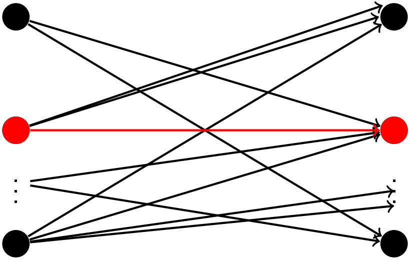

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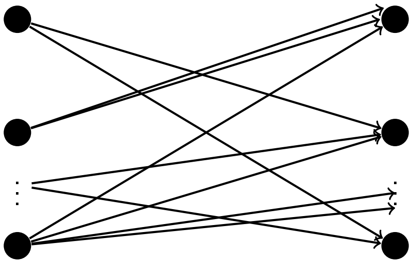

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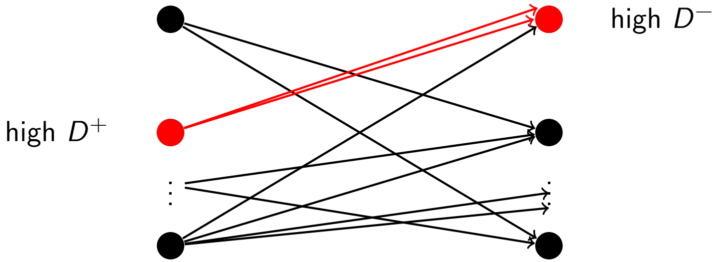

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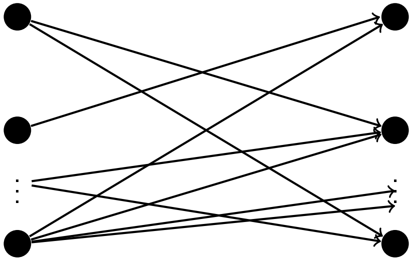

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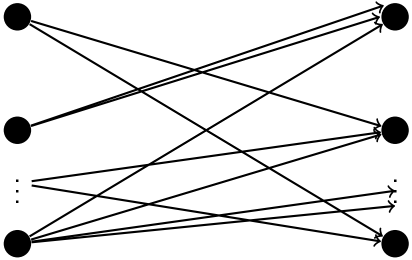

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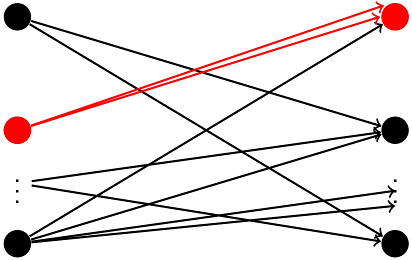

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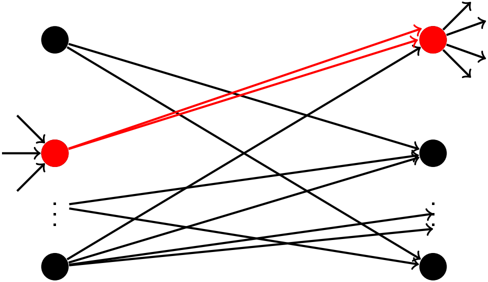

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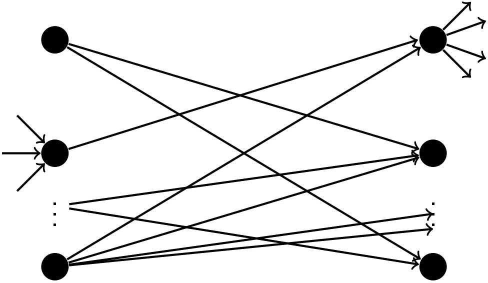

How large are the structural correlations?

- We want a result of the type:

$$\frac{\rho_+^-(G_n) - \mathbb{E}\left[\rho_+^-(G_n)\right]}{n^{f(\gamma_+,\gamma_-)}} \xrightarrow{d} W,$$

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- However, none of $f(\gamma_+,\gamma_-)$ scalings works
- Different $f(\gamma_+,\gamma_-)$ in different areas of (γ_+,γ_-)

Upper bounds

$$\frac{1}{E}\sum_{i,j=1}^{n}\mathbb{E}_n\left[E_{ij}^c\right] \leqslant \sum_{i,j=1}^{n}\frac{(D_i^+)^2(D_j^-)^2}{E^3} + \sum_{i=1}^{n}\frac{D_i^+ D_i^-}{E^2}$$
$$= O\left(n^{\frac{2}{\gamma_+}+\frac{2}{\gamma_-}-3}\right) + O\left(n^{-1}\right)$$

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CLT for Spearman's ρ

$$\rho_+^-(G_n) = O\left(\rho_+^-(G_n^*)\right) = O\left(n^{-1/2}\right)$$

Phase transition in the scaling of $\rho_{+}^{-}(G_n)$

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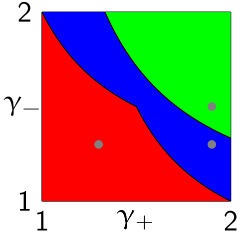

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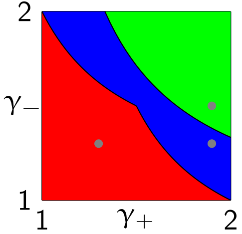

$$\frac{\rho_+^-(G_n) - \mathbb{E}\left[\rho_+^-(G_n)\right]}{n^{f(\gamma_+,\gamma_-)}}$$

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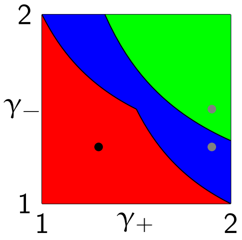

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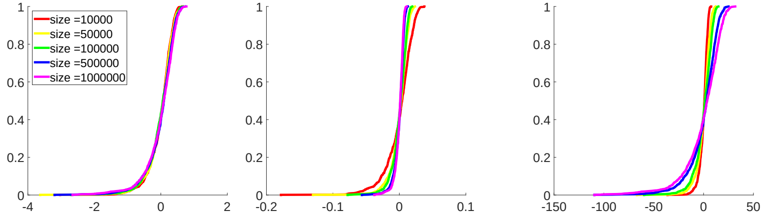

(a) $n^{-1+1/(\gamma_+\wedge\gamma_-)}$ (b) $n^{(2/\gamma_+)+(2/\gamma_-)-3}$ (c) $n^{-1/2}$

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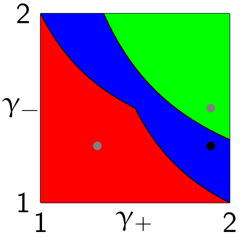

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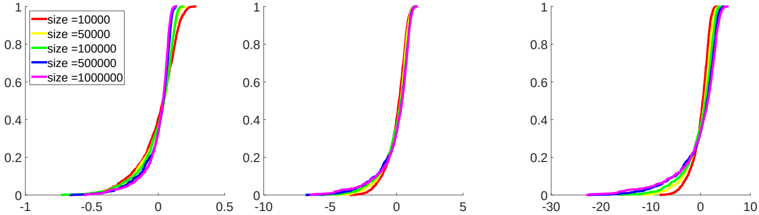

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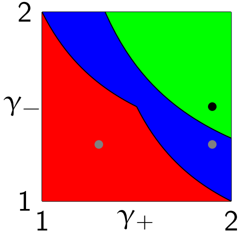

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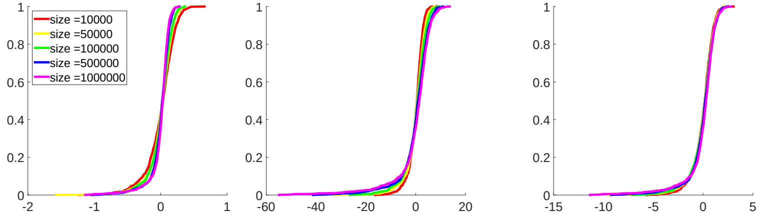

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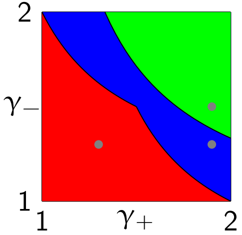

$$\frac{\rho_{-}^{+}(G_n) - \mathbb{E}\left[\rho_{-}^{+}(G_n)\right]}{n^{f(\gamma_+,\gamma_-)}}$$

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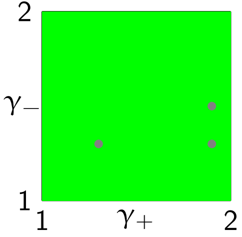

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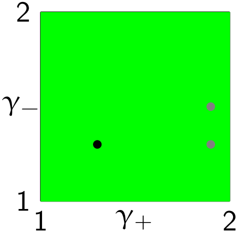

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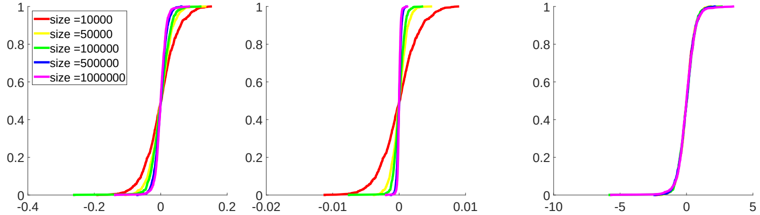

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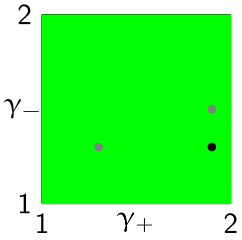

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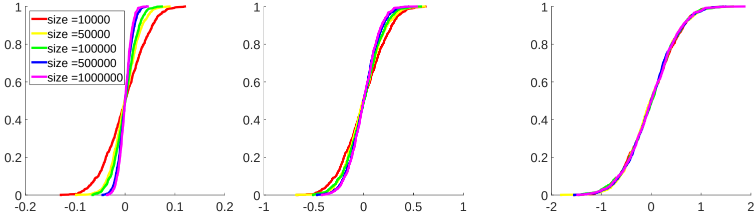

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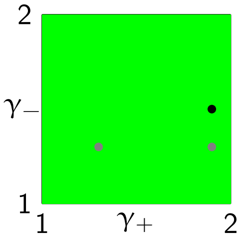

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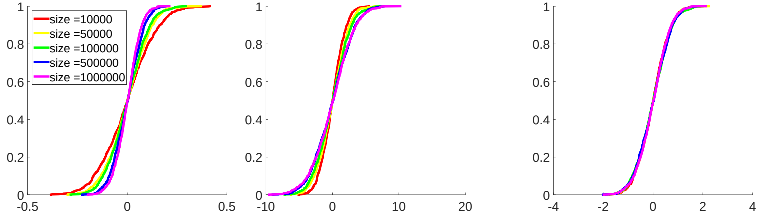

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Looks like an empirical form of

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where $P_\xi(k)\sim k^{-(\gamma_{\min}+1)}$, $\gamma_{\min}=\min\{\gamma_+,\gamma_-\}$.

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Tauberian Theorem Bingham and Doney 1974

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Theorem (vdHoorn, vdHofstad, Stegehuis, L 2016)

$$\frac{1}{E}\sum_{i,j=1}^{n}\mathbb{E}_n\left[E_{ij}^c\right] = O(n^{-\gamma_{\min}}).$$

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- How much erased edges affect the neutral mixing in a graph?
- Work in progress.

Thank you!