On some sheaves of special groups

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1 Introduction and basic notions

We consider sheaves of special groups (mainly over Boolean spaces). These are connected to the sheaves of abstract Witt rings considered by Marshall in [9], used therein in particular to classify spaces of orderings with a finite number of accumulation points. Our approach allows us to show that the so-called "question 1" (see [11, 1]) has a positive answer for these spaces. We conclude these notes by computing the behaviour of the Boolean hull functor when applied to a sheaf of special groups. The author would like to thank the referee for some very helpful comments.

Our references are [3] for special groups and the Boolean hull functor, and [10] for spaces of orderings. The functorial link between (reduced) special groups and spaces of orderings –a categorical duality– can be found in [3, Chapter 3]. If G is a (reduced) special group we denote by (X_G, G) (or simply X_G) its associated space of orderings. Conversely, if Y is a space of orderings, we denote by G_Y its associated reduced special group.

Definition 1.1 *Let G be a special group and let $\mathrm{Sat}_f(G)$ be the set of saturated subgroups of finite index in G. We say that G is pure in the product of its finite quotients (ppfq, for short) if the canonical map*

$$\begin{array}{rcl} G & \rightarrow & \prod_{\Delta \in \mathrm{Sat}_f(G)} G/\Delta \\ a & \mapsto & (a/\Delta)_{\Delta \in \mathrm{Sat}_f(G)} \end{array}$$

is pure, that is, reflects positive existential formulas with parameters in G. We naturally refer to this property as the property ppfq, and say that G is a ppfq special group.

Then question 1 in [11] is simply if all reduced special groups are ppfq. Note that Gladki and Marshall have shown in [5] that the answer is in general negative, i.e. that some reduced special groups are not ppfq.

Definition 1.2 (see [4], Part I) *Let L be a first-order language containing a binary relation symbol for the equality and let X be a non-empty topological space. We consider the category of opens in X where the morphisms are the inclusions, and the category of L-structures where the morphisms are the morphisms of L-structures. A* **sheaf of L-structures** *over X is a contravariant functor Γ from the category of opens in X to the category of L-structures (we denote the image of an open C by $\Gamma(C)$ and the image of an inclusion $C \subseteq D$ of opens by $r_{D,C} : \Gamma(D) \to \Gamma(C)$) which satisfies the following properties:*

1. *$\Gamma(\emptyset)$ is the L-structure with only one element, in which every atomic formula is true;*
2. *(**Sheaf property**) For any $C = \cup_{i\in I} C_i$ where C, C_i are opens in X for i in some index set I and any $n \in \mathbb{N}$:*
   If $\bar{g}_i \in \Gamma(C_i)^n$, $i \in I$, are such that $r_{C_j, C_j\cap C_k}(\bar{g}_j) = r_{C_k, C_j\cap C_k}(\bar{g}_k)$ for every $j, k \in I$ then there is $\bar{g} \in \Gamma(C)^n$ such that $r_{C,C_i}(\bar{g}) = \bar{g}_i$ for every $i \in I$, and for every atomic L-formula $\theta(\bar{x})$ (the tuples $\bar{x}$ and $\bar{g}$ may be of different lengths):
   $\Gamma(C) \models \theta(\bar{g})$ if and only if for every $i \in I$ $\Gamma(C_i) \models \theta(\bar{g}_i)$.
   (Remark that this implies that $\bar{g}$ is unique since L contains the equality.)

If Γ is a sheaf over X, $D \subseteq C$ are two opens in X and $f \in \Gamma(C)$, we will often write f instead of $r_{C,D}(f)$ if the context is clear enough to prevent any confusions.

For $x \in X$, the opens C containing x form a direct system and the direct limits $\Gamma(x) := \lim_{\to x\in C} \Gamma(C)$ are called **the stalks** of the sheaf. If C is an open containing $x \in X$, we denote by $r_{C,x}$ the canonical L-morphism from $\Gamma(C)$ to $\Gamma(x)$, and for convenience we write $f(x)$ for $r_{C,x}(f)$ if $f \in \Gamma(C)$. The next lemma gathers a few well-known facts.

Lemma 1.3

1. *Let $x \in X$ and $a \in \Gamma(x)$. Then there exists an open C in X, $C \ni x$, and $f \in \Gamma(C)$ such that $f(x) = a$.*
2. *Let $C = \dot{\cup}_{i\in I} C_i$ where C, C_i are opens in X for i in some index set I. Then $\Gamma(C) = \prod_{i\in I} \Gamma(C_i)$ as L-structures (the map used for the identification being $g \mapsto (r_{C,C_i}(g))_{i\in I}$).*
3. *Let C be open in X. Then the map $\Gamma(C) \to \prod_{x\in C} \Gamma(x)$, $f \mapsto (f(x))_{x\in C}$ identifies $\Gamma(C)$ with an L-substructure of $\prod_{x\in C} \Gamma(x)$.*

PROOF: (1) is clear by the definition of $\Gamma(x)$ as a direct limit, and (2) is a direct consequence of the sheaf property.

(3) Let $\theta(\bar{g})$ be an atomic L-formula with parameters $\bar{g} \in \Gamma(C)$. Suppose that $\Gamma(C) \models \theta(\bar{g})$. Then by definition of $\Gamma(x)$ as direct limit we get $\Gamma(x) \models \theta(\bar{g}(x))$, for every $x \in C$, which implies $\prod_{x\in C} \Gamma(x) \models \theta(\bar{g})$.
Suppose now that $\prod_{x\in C} \Gamma(x) \models \theta(\bar{g})$, i.e. $\Gamma(x) \models \theta(\bar{g}(x))$, for every $x \in C$. Let $x \in C$. By definition of $\Gamma(x)$ there is some open D_x in C such that $\Gamma(D_x) \models \theta(\bar{g})$. We have $C = \cup_{x\in C} D_x$, and by the sheaf property $\Gamma(C) \models \theta(\bar{g})$ if and only if $\Gamma(D_x) \models \theta(\bar{g})$ for every $x \in C$, which is true. □

Remark: Using lemma 1.3, we see that an equivalent formulation of the sheaf property is the conjunction of the two conditions:

1. Let $C = \cup_{i\in I} C_i$ where C, C_i are opens in X for i in some index set I and let $g_i \in \Gamma(C_i)$, $i \in I$, be such that $r_{C_j, C_j\cap C_k}(g_j) = r_{C_k, C_j\cap C_k}(g_k)$ for every $j, k \in I$. Then there is $g \in \Gamma(C)$ such that $r_{C,C_i}(g) = g_i$ for every $i \in I$ ("gluing" property).

2. $\Gamma(C)$ is an L-substructure of $\prod_{x\in C} \Gamma(x)$, for every open C in X.

Finally, we recall that a positive-primitive formula (pp-formula for short) in a language L is of the form

$$\exists \bar{x} \bigwedge_{i=1}^{n} R_i(\bar{t}_i(\bar{x})),$$

where the R_i are relations symbols of L and the $\bar{t}_i(\bar{x})$ are tuples of L-terms. It is easy to check that if $f : M \to N$ is an L-morphism between two L-structures M and N, f is reflects positive existential formulas with parameters in M if and only if f reflect pp-formulas with parameters in M. In particular, it is enought to consider pp-formulas in the definition of property ppfq.

2 Concerning positive-primitive formulas and property ppfq

The following proposition is an easy extension of [12, Proposition 3.4]:

Proposition 2.1 *Let Γ be a sheaf of L-structures over a Boolean space X. Then the inclusion of $\Gamma(X)$ in $\prod_{x\in X} \Gamma(x)$ is a pure L-morphism.*

PROOF: Let $\bar{g} \in \Gamma(X)$ and let $\phi(\bar{g})$ be a positive-primitive formula of the form $\exists \bar{v}\ \theta(\bar{g}, \bar{v})$ where θ is a conjunction of atomic formulas with parameters $\bar{g}$, such that $\prod_{x\in X} \Gamma(x) \models \phi(\bar{g})$, i.e. $\Gamma(x) \models \phi(\bar{g}(x))$ for every $x \in X$

(since $\phi(\bar{g})$ is positive-primitive). For $x \in X$ let $\bar{v}_x \in \Gamma(x)$ be such that $\Gamma(x) \models \theta(\bar{g}(x), \bar{v}_x)$. Then by lemma 1.3 (1) there is a clopen neighbourhood N_x of x and $\bar{\lambda}_x \in \Gamma(N_x)$ such that $\bar{\lambda}_x(x) = \bar{v}_x$. By the definition of $\Gamma(x)$ as direct limit there is a clopen M_x containing x such that $M_x \subseteq N_x$ and $\Gamma(M_x) \models \theta(\bar{g}, \bar{\lambda}_x)$.
The sets M_x ($x \in X$) cover X which has dimension 0 since it is Boolean. So by [6] p. 54 B) there is a finite partition $\{M_1, \dots, M_n\}$ of X into clopen sets such that for every $1 \leq i \leq n$ there exists $x \in X$ with $M_i \subseteq M_x$. In particular there is some $\bar{\lambda}_i \in \Gamma(M_i)$ such that $\Gamma(M_i) \models \theta(\bar{g}, \bar{\lambda}_i)$.

Let $\bar{\lambda} := (\bar{\lambda}_1, \dots, \bar{\lambda}_n)$ in the decomposition $\Gamma(X) = \prod_{i=1}^n \Gamma(M_i)$ (see lemma 1.3 (2)). We have $\Gamma(X) \models \theta(\bar{g}, \bar{\lambda})$ (since $\Gamma(M_i) \models \theta(\bar{g}, \bar{\lambda}_i)$ for every i and θ is an atomic formula), so $\Gamma(X) \models \phi(\bar{g})$. $\square$

By a **sheaf of special groups**, respectively reduced special groups, we mean a sheaf such that the image of a non-empty open set is always a special group, respectively a reduced special group (and the image of the empty set is the only special group with one element).

If we consider a sheaf Γ of special groups (respectively reduced special groups) over X, the stalks will be special groups (respectively reduced special groups) since these two classes are closed under direct limits.

Conversely, if Γ is a sheaf of L_{SG}-structures over X Boolean and the stalks of Γ are all special groups (repectively reduced special groups), then $\Gamma(X)$ is a special group (respectively a reduced special group) since $\Gamma(X)$ is pure in $\prod_{x \in X} \Gamma(x)$ which is a special group (and is reduced if and only if all the stalks are reduced).

Suppose now that Γ is a sheaf of reduced special groups over X (which need not be Boolean) and consider a clopen C in X. Then by lemma 1.3 we have $\Gamma(X) = \Gamma(C) \times \Gamma(X \setminus C)$ as special groups, so $\Gamma(C)$ is a Pfister quotient of $\Gamma(X)$. If $x \in X$ has a system of neighborhoods consisting of clopen sets C_i, $i \in I$, then $\Gamma(x) = \varinjlim \Gamma(C_i)$ is also a Pfister quotient of $\Gamma(X)$. In terms of spaces of orderings: $X_{\Gamma(C)}$ and $X_{\Gamma(x)}$ are subspaces of $X_{\Gamma(X)}$.

Theorem 2.2 *Let Γ be a sheaf of reduced special groups over a Boolean space X and let C be a clopen in X.*

1. *If every stalk $\Gamma(x)$, for $x \in C$, is a ppfq reduced special group, then $\Gamma(C)$ is a ppfq reduced special group.*
2. *The following two conditions are equivalent:*
   - *(a) $\Gamma(x)$ is ppfq, for every $x \in C$.*
   - *(b) For every $x \in C$, there is a clopen D in C, $D \ni x$, such that $\Gamma(D)$ is ppfq.*

PROOF:

1. Since the class of special groups with the ppfq property is closed under products (it can be seen using, for instance, [1, Proposition 5]), $\prod_{x\in C}\Gamma(x)$ is a special group with the ppfq property. But $\Gamma(C)$ is pure in it by Proposition 2.1, so is ppfq by [1, Proposition 3].

2. "(a) $\Rightarrow$ (b)" is clear since the first point of the theorem tells us that for every clopen D of C, $\Gamma(D)$ is ppfq.
   "(b) $\Rightarrow$ (a)" Let $x \in C$ and let D be a clopen in C containing x such that $\Gamma(D)$ is ppfq. By the paragraph preceding the lemma, $\Gamma(x)$ is a Pfister quotient of $\Gamma(D)$, so is ppfq by [1, Proposition 6]. □

We now wish to use Marshall's results in [9, Chapter 8]. For this we check briefly that sheaves of reduced Witt rings correspond to sheaves of reduced special groups via the functor sending reduced Witt rings to reduced special groups (see [7] or [2]):
Let W be a sheaf of reduced Witt rings over a Boolean space X, and write $W(C)$ for the image of the clopen C under W and $r^W_{D,C}$ for the image under W of the inclusion of clopens $C \subseteq D$. Let G be the functor which sends a clopen C in X to $G(C)$ the reduced special group of $W(C)$, and an inclusion $C \subseteq D$ of clopens in X to $r^G_{D,C}$ the morphism of special groups associated to the morphism of Witt rings $r^W_{D,C}$. We only need to check the sheaf property for G (we use the reformulation presented after Lemma 1.3):
Let $C = \cup_{i\in I} C_i$, where C, C_i are clopens in X and $g_i \in \Gamma(C_i)$ (for all $i \in I$), such that $r^G_{C_j,C_j\cap C_k}(g_j) = r^G_{C_k,C_j\cap C_k}(g_k)$ for every $j,k \in I$. Since for $A \subseteq B$ clopens in X, $G(A) \subseteq W(A)$ and $r^G_{B,A} = r^W_{B,A} \upharpoonright G(B)$, we can apply the sheaf property for W. This gives an element $g \in W(C)$ such that $r^W_{C,C_i}(g) = g_i$ for every $i \in I$ and we have to check that $g \in G(C)$. For this, observe that $r^W_{C,C_i}(g) = g_i \in G(C_i) \subseteq W(C_i)$ for every $i \in I$, which implies that $g_i(x)$ belongs to the reduced special group of $W(x)$ for every $x \in C_i$ and every $i \in I$. So $g(x)$ belongs to the reduced special group of $W(x)$ for every $x \in C$. But morphisms of abstract Witt rings preserve the underlying special group, so we can consider sheaves of abstract Witt rings to be in the language L_{WR} of rings expanded by a predicate for the special group. Since $W(C)$ is an L_{WR}-substructure of $\prod_{x\in C} W(x)$ by Lemma 1.3, we get that g is in $G(C)$, the special group of $W(C)$.
The rest of the sheaf property comes easily from the fact that if G is a special group and $a, b \in G$ then $a \in D_G\langle 1, b\rangle$ if and only if $a(1 + b) = 1 + b$ in the Witt ring of G.

Following Marshall's terminology (see [9] p. 217), we say that a special group of the form $\Gamma(X)$, where Γ is a sheaf of special groups over a Boolean space X, is obtained by **sheaf formation** from the stalks of the sheaf.

Corollary 2.3 *Let G be a reduced special group whose space of orderings has only a finite number of accumulation points. Then G is ppfq.*

PROOF: By [9, Proposition 8.17 together with Corollary 6.25] and using the correspondence between abstract Witt rings and special groups, such a special group is built up from the special group $\mathbb{Z}_2$ by using a finite number of times the operations of extension and sheaf formation. Since both operations preserve the property ppfq (see [11, Proposition 2.3] for the extension; Theorem 2.2 proves it for sheaf formations) and the reduced special group $\mathbb{Z}_2$ is ppfq, the result follows. $\square$

3 Boolean hull and sheaves

We conclude this note by computing the behaviour of the Boolean hull functor (see [3, Chapter 4, Section 2]) on sheaves and is related to [3, Theorem 6.34], which computes the Boolean hull of Boolean filtered powers of reduced special groups. We use the following notation: If G is a special group with associated space of orderings X_G and $\bar{g} = (g_1, \ldots, g_n) \in G$, $X_G(\bar{g})$ denotes the following basic clopen of X_G:

$$\{\sigma \in X_G \mid \sigma(g_1) = \cdots = \sigma(g_n) = 1\}.$$

Proposition 3.1 *Let Γ be a sheaf of special groups over a Boolean space X. Let Γ_B be the map which sends a clopen C in X to the Boolean algebra $B_{\Gamma(C)}$, the empty set to the one-element Boolean algebra, and which associates to every $C \subseteq D$ clopens in X the morphism of Boolean algebras $B(r_{D,C}) : B_{\Gamma(D)} \to B_{\Gamma(C)}$. Then Γ_B is a sheaf of Boolean algebras.*

PROOF: For ease of notation we write X_U for $X_{\Gamma(U)}$. Since the Boolean hull operation is a functor, the only property we have to check to see that Γ_B is a sheaf is the sheaf property. We check its reformulation stated after lemma 1.3:
Let $C = \cup_{j \in J} C_j$ where C and C_j, $j \in J$, are clopens in X, and suppose we have $b_j \in B_{\Gamma(C_j)}$, $j \in J$, such that for every $j, k \in J$ $B(r_{C_j, C_j \cap C_k})(b_j) = B(r_{C_k, C_j \cap C_k})(b_k)$. Is there then some $b \in B_{\Gamma(C)}$ such that $B(r_{C,C_j})(b) = b_j$ for every $j \in J$?
We now reformulate the hypothesis using the definition of the Boolean hull functor: If $U \subseteq V$ are clopens in X, X_U is a subspace of X_V and $B_{\Gamma(U)}$, $B_{\Gamma(V)}$ are the Boolean algebras of clopens in X_U, X_V. Since the map $r_{V,U}$ is the Pfister quotient which corresponds to the inclusion of X_U in X_V as a

subspace, the map $B(r_{V,U})$ from $B_{\Gamma(V)}$ to $B_{\Gamma(U)}$ just sends a clopen in X_V to its intersection with X_U. So the hypothesis of the sheaf property can be reformulated as follows:
We have b_j clopens in X_{C_j}, $j \in J$, such that for every $j,k \in J$ $b_j \cap X_{C_j \cap C_k} = b_k \cap X_{C_j \cap C_k}$. And we are looking for some clopen b in X_C such that for every $j \in J$ $b \cap X_{C_j} = b_j$.

Using [9, Proposition 8.2], we know that $X_D = \dot{\cup}_{x \in D} X_x$ for every clopen D in X, (where X_x is the space of orderings of $\Gamma(x)$). In particular $X_C = \cup_{j \in J} X_{C_j}$ and we define $b = \cup_{j \in J} b_j$. We first check that b is clopen in X_C, and for this we use the following:
Fact 1: Let $j \in J$ and let A be open in X_{C_j}. Since $X_{C_j} \subseteq X_C$ we have $A \subseteq X_C$. Then A is open in X_C.
Proof of the fact: Let $\sigma \in A$. Since A is open in X_{C_j} there is $\bar{g}_j \in \Gamma(C_j)$ such that $\sigma \in X_{C_j}(\bar{g}_j) \subseteq A$. Define $\bar{g} = (\bar{g}_j, \overline{-1}) \in \Gamma(C) = \Gamma(C_j) \times \Gamma(C \setminus C_j)$. Then $X_C(\bar{g}) = X_{C_j}(\bar{g}_j)$ and $\sigma \in X_C(\bar{g}) \subseteq A$. End of the proof of Fact 1.

Applying Fact 1, each b_j is open in X_C and then $b = \cup_{j \in J} b_j$ is open in X_C. We now need a second fact:
Fact 2: $X_C \setminus b = \cup_{j \in J}(X_{C_j} \setminus b_j)$.
Proof of the fact: "$\subseteq$" Let $\sigma \in X_C \setminus b$. Then $\sigma \in X_{C_j}$ for some $j \in J$ and $\sigma \notin b_j$ (otherwise $\sigma \in b$).
"$\supseteq$" Let $\sigma \in X_{C_j} \setminus b_j$ for some $j \in J$. Since $X_{C_j} \subseteq X_C$ we have $\sigma \in X_C$. Suppose $\sigma \in b$. Then $\sigma \in b_k \subseteq X_{C_k}$ for some $k \neq j$. Now $\sigma \in X_{C_j} \cap X_{C_k} = X_{C_j \cap C_k}$ (because for D clopen $X_D = \cup_{x \in D} X_x$ by [9, Proposition 8.2]). So $\sigma \in b_k \cap X_{C_j \cap C_k}$ which is equal to $b_j \cap X_{C_j \cap C_k}$ by hypothesis and we get $\sigma \in b_j$, a contradiction. End of the proof of Fact 2.

Since the b_j are clopens in X_{C_j}, the two facts then describe $X_C \setminus b$ as a union of opens of X_C, so $X_C \setminus b$ is open in X_C. We now check that $b \cap X_{C_j} = b_j$ for every $j \in J$: $b \cap X_{C_j} = (\cup_{k \in J} b_k) \cap X_{C_j} = \cup_{k \in J}(b_k \cap X_{C_j})$, which clearly contains $b_j \cap X_{C_j} = b_j$. For the other inclusion we have
$b_k \cap X_{C_j} = b_k \cap X_{C_j} \cap X_{C_k} = b_k \cap X_{C_j \cap C_k} = b_j \cap X_{C_j \cap C_k} \subseteq b_j$.

The Boolean hull operation is a functor, so the map $r_{C,x} : \Gamma(C) \to \Gamma(x) = \lim_{\to D \text{ clopen} \ni x} \Gamma(D)$ is sent to the map from $B_{\Gamma(C)}$ to $B_{\Gamma(x)} = \Gamma_B(x)$ given by the definition of $\Gamma_B(x)$ as direct limit, i.e. this last map is $B(r_{C,x})$. So to check the second condition of the reformulation of the sheaf property, we have to check that $f : B_{\Gamma(C)} \to \prod_{x \in C} B_{\Gamma(x)}$, $b \mapsto (B(r_{C,x})(b))_{x \in C}$ is a monomorphism of Boolean algebras. Since $r_{C,x}$ is the map corresponding to the inclusion of X_x in X_C as a subspace, the map $B(r_{C,x})$ just sends a clopen of X_C to its intersection with X_x. So the map f is in fact:
$f : \{\text{clopens of } X_C\} \to \prod_{x \in C}\{\text{clopens of } X_x\}$, $b \mapsto (b \cap X_x)_{x \in C}$. This is clearly a monomorphism of Boolean algebras since $X_C = \dot{\cup}_{x \in C} X_x$ by [9,

Proposition 8.2]. □

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