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Elementary equivalence of some rings of definable functions

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Abstract We characterize elementary equivalences and inclusions between von Neumann regular real closed rings in terms of their boolean algebras of idempotents, and prove that their theories are always decidable. We then show that, under some hypotheses, the map sending an L-structure R to the L-structure of definable functions from R^n to R preserves elementary inclusions and equivalences and gives a structure with a decidable theory whenever R is decidable. We briefly consider structures of definable functions satisfying an extra condition such as continuity.

Keywords von Neumann regular ring · elementary equivalence · ring of definable functions

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1 Introduction and notation

Let L be a first-order language. If R is an L-structure and $n \in \mathbb{N}$, we consider $\mathrm{def}_L(R^n, R)$ the set of L-definable maps from R^n to R, which can be seen as an L-structure via its inclusion in R^{R^n} (the functions, constants and relations are defined coordinate by coordinate). We show that, under appropriate hypotheses, the map which sends R to $\mathrm{def}_L(R^n, R)$ preserves elementary equivalences and inclusions, and gives a decidable L-structure if R is decidable (corollary 4, proposition 4 and corollary 5).

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The first of these results is proved when L contains the language of rings, R is a real closed field and only the ring structure is considered on $\mathrm{def}_L(R^n, R)$. For this we use the representation of von Neumann regular commutative rings as rings of global continuous sections of a sheaf to characterize elementary inclusions between von Neumann regular real closed rings and to show that the first-order theory of such a ring (in the language of rings) is always decidable (theorem 1). By real closed rings we mean real closed rings in the sense of Schwartz (see [9]).

We then consider the L-structures of L-definable functions from R^n to R satisfying some additional property (the motivation here is given by the ring of continuous definable functions, when R is a ring equipped with some topology). We get a simple result concerning the preservation of elementary inclusions when both structures are ω-saturated (proposition 5) and briefly mention the limits of the method used in the proof.

We use the following conventions and notation throughout this paper: We will not make any notational distinction between an L-structure and its underlying set, $L_R := \{0, 1, +, -, \cdot\}$ is the language of rings, $L_{BA} := \{\bot, \top, \wedge, \neg\}$ is the language of boolean algebras, and "definable" will always mean definable with parameters. If M is an L-structure, $Th_L(M)$ is the set of L-formulas without parameters which are true in M. Similarly, if $\{M_i\}_{i \in I}$ is a collection of L-structures, $Th_L(\{M_i \mid i \in I\})$ is the set of L-formulas without parameters which are true in every M_i, for $i \in I$.
If M is an L-structure and C is a set consisting of elements of M, of functions defined on some finite cartesian power of M with values in M and of subsets of some finite cartesian power of M, then $L(C)$ denotes the language L expanded by new constant symbols, function symbols and relation symbols for each element of C, which are then interpreted in M by the element of C they represent.

If we want to make clear that an elementary inclusion or elementary equivalence between M and N is meant in the language L, we write $M \prec_L N$ or $M \equiv_L N$. Finally, if $\bar{x}$ is a tuple, $l(\bar{x})$ is the length of $\bar{x}$.

2 Boolean algebras of definable subsets

Let L be a first-order language, $\mathscr{A}$ an L-structure and $n \in \mathbb{N}$. We consider $\mathrm{def}_L(\mathscr{A}^n)$ the boolean algebra of definable subsets of $\mathscr{A}$ (which we identify with the boolean algebra of $L(\mathscr{A})$-formulas modulo equivalence in $\mathscr{A}$). If $\mathscr{B}$ is another L-structure, the well-knowed elementary invariants of boolean algebras (see [5]) give us that $\mathrm{def}_L(\mathscr{A}^n)$ and $\mathrm{def}_L(\mathscr{B}^n)$ are elementarily equivalent.
The goal of this section is to point out that the obvious generalisation to elementary inclusions still holds: If $\mathscr{B}$ is an elementary extension of $\mathscr{A}$, then the map sending the subset of $\mathscr{A}^n$ defined by the formula $\phi(x_1, \ldots, x_n)$ to the subset of $\mathscr{B}^n$ defined by $\phi(x_1, \ldots, x_n)$ is an elementary inclusion of boolean algebras from $\mathrm{def}_L(\mathscr{A}^n)$ to $\mathrm{def}_L(\mathscr{B}^n)$.

We only give a sketch of the proof since it is a direct consequence of a simple extension of [5, Proposition 18.10]. If B is a boolean algebra then $\mathrm{inv}(B)$ denotes its elementary invariant, and for $b \in B$, $B \upharpoonright b$ denotes the boolean algebra $\{x \in B \mid x \subseteq b\}$.

Proposition 1 *Let A, B and C be boolean algebras with C subalgebra of A, and assume there is an L_{BA}-monomorphism $f : C \to B$ such that for every $c \in C$* $\mathrm{inv}(A \upharpoonright$

$c) = \mathrm{inv}(B \upharpoonright f(c))$. Then $A \equiv B$ in the language $L_{BA}(C)$ (where an element $c \in C$ is interpreted in B by $f(c)$).

Proof We simply follow the proof of [5, Proposition 18.10]. We fix a finite tuple $\bar{c} \in C$ and replace it by the finite subalgebra it generates. We also replace A and B by ω-saturated elementary extensions, and then use the same back-and-forth argument as in [5, Proposition 18.10]. The only modification is that we take for P the set of L_{BA}-isomorphisms $p : A_0 \to B_0$ where A_0 is a finite substructure of A and B_0 is a finite substructure of B, such that $\bar{c}$ is a substructure of A_0, p extends $f \upharpoonright \bar{c}$ and $\mathrm{inv}(A \upharpoonright a) = \mathrm{inv}(B \upharpoonright p(a))$ for every $a \in A_0$. □

Corollary 1 *If $\mathscr{A}$ and $\mathscr{B}$ are two L-structures such that $\mathscr{A} \prec_L \mathscr{B}$ then $\mathrm{def}_L(\mathscr{A}^n) \prec \mathrm{def}_L(\mathscr{B}^n)$ as boolean algebras, for every $n \in \mathbb{N}$.*

3 Von Neumann regular real closed rings

We use the notions and notation introduced in [2] concerning sheaves of L-structures (see also [7]). However, we will also use some results of [1] on sheaves of rings, where the definition of sheaves comes from [8], and is slightly weaker than Comer and Macintyre's (when their definition is written for the language of rings). We present this weaker definition in the case of L-structures, and remark that the main result in [2] still holds with this modified definition. Except for the different definition, everything until and including corollary 3 can be found in [2] and [7].

Definition 1 A sheaf of L-structures is a triple (X, S, π) where

1. X and S are topological spaces (we will always assume that X is non-empty);
2. π is a local homeomorphism from S onto X;
3. $S_x := \pi^{-1}(x)$ is an L-structure for each $x \in X$;
4. For each constant symbol $c \in L$ the map from X to S that sends $x \in X$ to the interpretation of c in S_x is continuous;
5. For each n-ary function symbol $f \in L$ the map from $\bigcup_{x\in X}(S_x^n)$ to S sending $(s_1, \ldots, s_n) \in S_x^n$ to the interpretation of $f(s_1, \ldots, s_n)$ in S_x is continuous (where $\bigcup_{x\in X}(S_x^n)$ is equipped with the topology induced by its inclusion in $(\bigcup_{x\in X} S_x)^n = S^n$);
6. For each n-ary relation symbol $R \in L$, $\{(s_1, \ldots, s_n) \in \bigcup_{x\in X}(S_x^n) \mid R(s_1, \ldots, s_n)\}$ is open in $\bigcup_{x\in X}(S_x^n)$.

Remark 1 This definition coincides with the one given in [8] when L is the language of rings (point 6 is easily checked for the equality), but is weaker than Comer and Macintyre's which requires the subset defined in condition 6 to be clopen.

If N is open in X, a continuous section from N to S is a continuous map $f : N \to S$ such that $f(x) \in S_x$ for every $x \in N$. We denote by $\Gamma(N, S)$ the set of continuous sections from N to S. Remark that if $x_0 \in X$ and $a \in S_{x_0}$ then there is an open N of X containing x_0 and $f \in \Gamma(N, S)$ such that $f(x_0) = a$: By definition there is an open V of S such that $a \in V$, $\pi(V)$ is open in X and $\pi \upharpoonright V : V \to \pi(V)$ is an homeomorphism. In particular we can define a map $f : \pi(V) \to S$ sending $x \in \pi(V)$

to the only $s \in V$ such that $\pi(s) = x$. Taking $N := \pi(V)$, we have $f \in \Gamma(N,S)$, $x_0 \in N$ and $f(x_0) = a$.

If $\theta(\bar{u})$ is an L-formula, N is open in X and $\bar{f} \in \Gamma(N,S)^{\bar{u}}$ we define

$$\|\theta(\bar{f})\|_N := \{x \in N \mid S_x \models \theta(\bar{f}(x))\}.$$

We say that an L-theory T is positively model-complete if every L-formula (so possibly with free variables) is equivalent modulo T to a positive existential formula.

Lemma 1 *Let (X,S,π) be a sheaf of L-structures and let $\theta(\bar{u})$ be an L-formula. Let N be open in X and let $\bar{f} \in \Gamma(N,S)^{\bar{u}}$.*

1. *If $\theta(\bar{u})$ is positive existential then $\|\theta(\bar{f})\|_N$ is open in N;*
2. *If $Th_L(\{S_x \mid x \in N\})$ is positively model-complete then $\|\theta(\bar{f})\|_N$ is clopen in N.*

Proof 1. We first assume that $\theta(\bar{u})$ is atomic, so $\theta(\bar{f})$ is $R(t_1(\bar{f}),\ldots,t_n(\bar{f}))$, where $R \in L$ is a relation symbol and $t_1,\ldots,t_n$ are L-terms. The result is then a direct consequence of items 4, 5 and 6 in definition 1, and extends naturally to the case of $\theta(\bar{u})$ quantifier-free positive.
Assume now that $\theta(\bar{u})$ is $\exists \bar{v}\, \tau(\bar{u},\bar{v})$ where $\tau(\bar{u},\bar{v})$ is quantifier-free positive. Let $x_0 \in \|\exists \bar{v}\, \tau(\bar{f},\bar{v})\|_N$. Then there exists $\bar{a} \in (S_{x_0})^{\bar{v}}$ such that $S_{x_0} \models \tau(\bar{f}(x_0),\bar{a})$. As observed before the lemma, there is then an open U in N and $\bar{g} \in \Gamma(U,S)^{\bar{v}}$ such that $\bar{g}(x_0) = \bar{a}$. Since $\tau(\bar{u},\bar{v})$ is quantifier-free positive, $\|\tau(\bar{f},\bar{g})\|_U$ is open. But it also contains x_0 and is included in $\|\exists \bar{v}\, \tau(\bar{f},\bar{v})\|_N$, which completes the proof.
2. Since $Th_L(\{S_x \mid x \in N\})$ is positively model-complete we can assume that $\theta(\bar{u})$ is positive existential, so $\|\theta(\bar{f})\|_N$ is open in N.
Let $\tau(\bar{u})$ be a positive existential formula such that $Th_L(\{S_x \mid x \in N\}) \vdash \forall \bar{u}\, (\neg\theta(\bar{u})) \leftrightarrow \tau(\bar{u})$. Then $N \setminus \|\theta(\bar{f})\|_N = \|\neg\theta(\bar{f})\|_N = \|\tau(\bar{f})\|_N$, which is open by the first point of this lemma. □

Let L_B be any fixed language expanding the language L_{BA} of boolean algebras and assume that clopens(X), the set of clopen subsets of X, is an L_B-structure. We consider on $\Gamma(X,S)$ the predicates induced by the generalized product $\prod_{x\in X} S_x$, as introduced in [3] (see the second page of [2]):

Definition 2 1. An acceptable sequence is a sequence $\langle \Phi(X_1,\ldots,X_m), \theta_1(\bar{z}),\ldots,\theta_m(\bar{z})\rangle$ where m is in $\mathbb{N}$, Φ is an L_B-formula with m free variables and $\theta_1(\bar{z}),\ldots,\theta_m(\bar{z})$ are L-formulas with the same free variables $\bar{z}$.
2. The language L_{gp} (for "generalized product") consists of all $l(\bar{z})$-ary relations R_ξ for all acceptable sequences $\xi := \langle \Phi(X_1,\ldots,X_m), \theta_1(\bar{z}),\ldots,\theta_m(\bar{z})\rangle$. R_ξ is interpreted in $\Gamma(X,S)$ by

$$\{\bar{f} \in \Gamma(X,S)^{\bar{z}} \mid \text{clopens}(X) \models \Phi(\|\theta_1(\bar{f})\|_X,\ldots,\|\theta_m(\bar{f})\|_X)\}.$$

We recall a special case of [2, Theorem 1.1] and its consequences (see [2, Theorem 1.3] and [2, Theorem 1.5]; we also use property (C) defined on the second page of [2]):

Proposition 2 *There is an effective procedure which associates to any L_{gp}-formula $\sigma(\bar{z})$ an acceptable sequence $\langle \Phi(X_1,\dots,X_m), \theta_1(\bar{z}),\dots,\theta_m(\bar{z})\rangle$ such that the following holds: For every sheaf of L-structures (X,S,π) with X boolean topological space and* $\mathrm{Th}_L(\{S_x\}_{x\in X})$ *positively model-complete, every L_B-structure on* clopens(X) *(recall that L_B has been previously fixed) and every $\bar{f}\in\Gamma(X,S)^{\bar{z}}$,*

$$\Gamma(X,S)\models\sigma(\bar{f})$$
if and only if
$$\mathrm{clopens}(X)\models\Phi(\|\theta_1(\bar{f})\|_X,\dots,\|\theta_m(\bar{f})\|_X).$$

Proof Comer's proof in [2] works perfectly, even with our weaker definition, because every set of the form $\|\theta(\bar{g})\|_X$ is clopen in X, for every L-formula $\theta(\bar{z})$ and every $\bar{g}\in\Gamma(X,S)^{\bar{z}}$ (by lemma 1). □

Corollary 2 *Let $\mathscr{S}=(X,S,\pi)$, $\mathscr{S}'=(X',S',\pi')$ be sheaves of L-structures such that for every $x\in X$ and $y\in X'$ $S_x\equiv_L S'_y$ and their common theory is positively model-complete. Suppose X and X′ are boolean spaces and* $\mathrm{clopens}(X)\equiv\mathrm{clopens}(X')$ *in L_B. Then $\Gamma(X,S)\equiv\Gamma(X',S')$ in L_{gp}.*

Corollary 3 *Let (X,S,π) be a sheaf of L-structures with* $\mathrm{Th}_L(\{S_x\}_{x\in X})$ *positively model-complete and X boolean, and such that the theories* $\mathrm{Th}_L(\{S_x\}_{x\in X})$ *and* $Th_{L_B}(\mathrm{clopens}(X))$ *are both decidable. Then $Th_{L_{gp}}(\Gamma(X,S))$ is decidable.*

While corollary 2 gives assumptions under which we get elementary equivalence, we need to be a bit more precise to deal with the case of elementary inclusions: Let $\mathscr{S}_1=(X_1,S_1,\pi_1)$, $\mathscr{S}_2=(X_2,S_2,\pi_2)$ be sheaves of L-structures and assume we have a continuous map $p: X_2\to X_1$ such that $S_{1p(x)}$ is a substructure of S_{2x} for every $x\in X_2$. It induces a map

$$\begin{aligned} p^*:\Gamma(X_1,S_1)&\to\Gamma(X_2,S_2)\\ f&\mapsto f\circ p. \end{aligned}$$

Using that $\Gamma(X_i,S_i)$ is an L-substructure of $\prod_{x\in X_i}S_{ix}$ we easily check that p^* is an L-morphism and that p^* is an L-monomorphism whenever p is surjective. Note that if p is surjective it induces an embedding of boolean algebras:

$$\begin{aligned} p^{BA}:\mathrm{clopens}(X_1)&\to\mathrm{clopens}(X_2)\\ U&\mapsto p^{-1}(U). \end{aligned}$$

Proposition 3 *Let $\mathscr{S}_1=(X_1,S_1,\pi_1)$, $\mathscr{S}_2=(X_2,S_2,\pi_2)$ be sheaves of L-structures such that* $\mathrm{Th}(\{S_{1x}\}_{x\in X_1})=\mathrm{Th}(\{S_{2x}\}_{x\in X_2})$ *is positively model-complete and X_1, X_2 are boolean topological spaces. Assume that we have a surjective continuous map $p: X_2\to X_1$ such that $S_{1p(x)}\prec_L S_{2x}$ for every $x\in X_2$ and p^{BA} is an elementary embedding in L_B.*
Then $p^*:\Gamma(X_1,S_1)\to\Gamma(X_2,S_2)$ is elementary in the language L_{gp}.

Proof Let $\sigma(\bar{z})$ be an L_{gp}-formula with free variables $\bar{z}$ and let $\langle\Phi(X_1,\dots,X_m),\theta_1(\bar{z}),\dots,\theta_m(\bar{z})\rangle$ be the acceptable sequence associated to it by proposition 2. Let $\bar{f}\in\Gamma(X_1,S_1)^{\bar{z}}$.

Claim: For every L-formula $\theta(\bar{z})$ we have $p^{BA}(\|\theta(\bar{f})\|_{X_1}) = \|\theta(p^*(\bar{f}))\|_{X_2}$.
Let $x \in X_2$. We have the following sequence of equivalences:

$$\begin{aligned} x \in p^{BA}(\|\theta(\bar{f})\|_{X_1}) &\Leftrightarrow p(x) \in \|\theta(\bar{f})\|_{X_1} \\ &\Leftrightarrow S_{1p(x)} \models \theta(\bar{f}(p(x))) \\ &\Leftrightarrow S_{2x} \models \theta(\bar{f}(p(x))), \end{aligned}$$

which proves the claim.

Using the claim we get

$$\text{clopens}(X_1) \models \Phi(\|\theta_1(\bar{f})\|_{X_1}, \dots, \|\theta_m(\bar{f})\|_{X_1})$$
if and only if
$$\text{clopens}(X_2) \models \Phi(p^{BA}(\|\theta_1(\bar{f})\|_{X_1}), \dots, p^{BA}(\|\theta_m(\bar{f})\|_{X_1}))$$
if and only if
$$\text{clopens}(X_2) \models \Phi(\|\theta_1(p^*(\bar{f}))\|_{X_2}, \dots, \|\theta_m(p^*(\bar{f}))\|_{X_2}).$$

And this proves the result, by proposition 2. □

We now apply these results to von Neumann regular real closed rings. For this we recall a few notions and a lemma, taken from [1], section 2:

We only consider commutative rings with 1. If R is a ring, we denote by B_R the boolean algebra of its idempotents (endowed with the following operations: $a \wedge b := ab$, $a \vee b := a + b - ab$ and the complement of a is $1 - a$).
If $e \in B_R$ then $R = Re \times R(1-e)$, $Re \cong R/(1-e)$, and Re and $R(1-e)$ are rings with respective units e and $1-e$.

To fix the notation, we denote as follows the Stone duality for a boolean algebra B: X_B is the boolean topological space of maximal ideals of B, or equivalently of morphisms from B onto $\{\perp, \top\}$, and an element b of B is identified with the continuous map defined on X_B by $b(x) := x(b)$.

Let R be a ring, $B := B_R$ and let X_B be the Stone space of B. For any $x \in X_B$ the stalk R_x of R at x is defined to be the direct limit, over elements e of B such that $e \notin x$, of the rings Re. They form an inductive system of (non-injective) morphisms of rings since for $e_1, e_2 \in B$ we have a morphism $\lambda_{e_1 e_2} : Re_1 \to Re_1 e_2$, $ae_1 \mapsto ae_1 e_2$ (and $e_1 e_2 \notin x$, using that x is maximal).
We get in this way a sheaf of rings (X_B, S, π), where $S_x = R_x$ for every $x \in X_B$, and R is the ring of continuous global sections $\Gamma(X_B, S)$ (all this can be found in [1], section 2 on central idempotents).

Recall that a commutative ring R is von Neumann regular if for every $a \in R$ there exists $a^* \in R$ such that $a^2 a^* = a$.

Lemma 2 *Assume that R is von Neumann regular and let $x \in X_B$. We denote by (x) the ideal of R generated by the elements of x. Then*

1. *$(x) = \bigcup_{f \in x} Rf$ and (x) is a maximal ideal of R;*
2. *$R_x \cong R/(x)$. In particular R_x is a field.*

Proof 1. The inclusion from right to left is clear. Let $a \in (x)$, $a = a_1 f_1 + \dots + a_n f_n$ with $a_1, \dots, a_n \in R$ and $f_1, \dots, f_n \in x$. Let $f = f_1 \vee \dots \vee f_n$. $f \in x$ since x is an

ideal and $a \in Rf$, indeed $a = (a_1 f_1 + \cdots + a_n f_n) f$ because $f_i f = f_i \wedge (f_1 \vee \cdots \vee f_n) = f_i$.
We check that (x) is maximal. Let $a \notin x$. Then $aa^* \notin x$ (otherwise $a = a(aa^*) \in x$, a contradiction). But aa^* is an idempotent (recall that R is commutative), so $1 - aa^* \in x$ since x is a maximal ideal of B_R (and thus for every $e \in B_R$, either $e \in x$ or $\neg e = 1 - e \in x$). This gives $1 = aa^* + (1 - aa^*)$ is in the ideal generated by a and (x), so (x) is a maximal ideal.

2. Let $e \notin x$. Since $Re \cong R/(1-e)$ and $1 - e \in x$, we have a canonical morphism $\mu_e : Re \to R/(x)$ satisfying, for $e_1, e_2 \notin x$, $\mu_{e_1} = \mu_{e_2} \circ \lambda_{e_1 e_2}$, and we have $\varinjlim_{e \notin x} Re = \varinjlim_{e \notin x} R/(1-e) = \varinjlim_{f \in x} R/(f)$. The universal property of the direct limit produces then a morphism of rings from $\varinjlim_{e \notin x} Re$ to $R/(x)$, which is explicitly given by

$$\begin{array}{c} \varinjlim_{f \in x} R/(f) \to R/(x) \\ \text{the class of } a \text{ in some } R/(f) \mapsto a/(x). \end{array}$$

This map is clearly surjective. We check that it is injective: Suppose $a/(x) = 0$ where a in $R/(f)$ for some $f \in x$. Then $a \in (f_1, \ldots, f_n)$ for some $n \in \mathbb{N}$ and $f_1, \ldots, f_n \in x$, which gives $a \in (f_1 \vee \cdots \vee f_n)$, and therefore $a = 0$ in $\varinjlim_{f \in x} R/(f)$ since $f_1 \vee \cdots \vee f_n \in x$. □

The following theorem characterizes elementary equivalences and inclusions of von Neumann regular real closed rings in terms of their boolean algebras of idempotents.

Theorem 1 *Let A_1, A_2 be two von Neumann regular real closed rings and let B_1, B_2 be their respective boolean algebras of idempotents. Then*

1. *$Th_{L_R}(A_1)$ is decidable;*
2. *$A_1 \equiv A_2$ as rings if and only if $B_1 \equiv B_2$ as boolean algebras;*
3. *$A_1 \prec A_2$ as rings if and only if $A_1 \subseteq A_2$ as rings and $B_1 \prec B_2$ as boolean algebras.*

Proof First note that the two left to right implications in (2) and (3) are clear since B_i is definable without parameters in A_i for $i = 1, 2$. Let $\mathscr{S}_1 = (X_{B_1}, S_1, \pi_1)$, $\mathscr{S}_2 = (X_{B_2}, S_2, \pi_2)$ be sheaves such that $A_i = \Gamma(X_{B_i}, S_i)$ for $i = 1, 2$.

1. We apply corollary 3: $S_{1x} = A_1/(x)$ for every $x \in X_{B_1}$ by lemma 2, and is a real closed field as quotient of a real closed ring by a maximal ideal (see point (d) of the definition of real closed rings in the introduction of [9]). So $\mathrm{Th}_{L_R}(\{S_{1x} \mid x \in X_{B_1}\})$ is the theory of real closed fields, which is decidable and positively model-complete. $Th_{L_{BA}}(X_{B_1})$ is determined by the elementary invariant ξ of X_{B_1}, i.e. $Th_{L_{BA}}(X_{B_1})$ is the theory $T(\xi)$ of boolean algebras with elementary invariant ξ (see [5, Proposition 18.10]). Since $T(\xi)$ has a recursive set of axioms by [5, 18.8], it is decidable and the decidability of $Th_{L_R}(A_1)$ follows by corollary 3.
2. By lemma 2, the stalks of $\mathscr{S}_1$, $\mathscr{S}_2$ are quotients of A_1, A_2 by maximal ideals, and are thus real closed fields since A_1 and A_2 are real closed rings. It proves that the theories of the stalks are the theory of real closed fields, which is positively model-complete. We can then apply corollary 2.

3. Assume now that $A_1 \subseteq A_2$ as rings and $B_1 \prec B_2$ as boolean algebras. Let $X_i := X_{B_i}$ for $i = 1,2$. We need to make more precise the terminology introduced in lemma 2: If C is a subset of A_1, $(C)_{A_1}$ is the ideal generated by C in A_1 and $(C)_{A_2}$ is the ideal generated by C in A_2.
Let $p : X_2 \to X_1$, $p(x) = x \cap B_1$. p is continuous and surjective. For p to induce an embedding p^* from $\Gamma(X_1,S_1)$ into $\Gamma(X_2,S_2)$, we first have to find an embedding of $A_{1p(x)} = A_1/(x\cap B_1)_{A_1}$ into $A_{2x} = A_2/(x)_{A_2}$, for every $x \in X_2$.
Claim: $(x\cap B_1)_{A_1} = (x)_{A_2} \cap A_1$.
We clearly have $(x\cap B_1)_{A_1} \subseteq (x)_{A_2} \cap A_1$. But $x \cap B_1$ is a maximal ideal of B_1. Applying lemma 2, we get that $(x\cap B_1)_{A_1}$ is a maximal ideal of A_1, which implies $(x\cap B_1)_{A_1} = (x)_{A_2} \cap A_1$.

The claim gives a canonical embedding of the stalks

$$\begin{aligned} A_1/(x\cap B_1)_{A_1} = A_1/((x)_{A_2} \cap A_1) &\to A_2/(x)_{A_2} \\ a/((x)_{A_2} \cap A_1) &\mapsto a/(x)_{A_2}, \end{aligned} \tag{1}$$

for every $x \in X_2$.
In particular, p then induces an embedding p^* from $\Gamma(X_1,S_1)$ into $\Gamma(X_2,S_2)$ which coincides with the inclusion $A_1 \subseteq A_2$ (the verification is straightforward using (1)).
Moreover p^{BA} : clopens$(X_1) \to$ clopens(X_2) is exactly the inclusion $B_1 \subseteq B_2$ so is elementary by assumption. Proposition 3 will give the desired result if we only check that the inclusion in (1) is elementary, but it is the case since both $A_{1p(x)}$ and A_{2x} are real closed fields (as quotients of real closed rings by maximal ideals). □

Remark 2 1. The same result holds (using the same arguments) for von Neumann regular rings in which every monic polynomial has a root, because the stalks will be algebraically closed fields, and the theory of algebraically closed fields is positively model-complete. In particular the theory of such rings without minimal idempotents is model-complete since the theory of atomless boolean algebras is model-complete (and is decidable since the theory of algebraically closed fields is decidable). We recover in this way parts of Theorems 3 and 4 of [6].
2. Since a real closed ring is an f-ring and its order relation is definable in the language of rings we also recover the fact that the theory of von Neumann regular real closed f-rings without minimal idempotents is model-complete and decidable (see [7], Section 6, and [12], Theorem 4.17), once again because the theory of atomless boolean algebras is model-complete.

4 *L*-structures of definable functions

4.1 The case of real closed rings

Definition 3 Let L be a first-order language containing the language of rings, R an L-structure and $n \in \mathbb{N}$. We denote by $\mathrm{def}_L(R^n,R)$ the structure of definable L-functions from R^n to R.

(Remark that if $R \prec_L S$ we have a natural inclusion of rings from $\mathrm{def}_L(R^n,R)$ to $\mathrm{def}_L(S^n,S)$.)

Corollary 4 *1. Let $L_1 \subseteq L_2$ be expansions of the language of rings and let R_i be a real closed field which is an L_i-structure for $i=1,2$. Suppose $R_1 \prec_{L_1} R_2$. Then for every $n \in \mathbb{N}$, $\mathrm{def}_{L_1}(R_1^n,R_1) \prec \mathrm{def}_{L_2}(R_2^n,R_2)$ as rings.*
Moreover $Th_{L_R}(\mathrm{def}_{L_1}(R_1^n,R_1))$ is decidable.

2. In particular:

(a) If $R \subseteq S$ are real closed fields and $n \in \mathbb{N}$ then $\mathrm{def}_{L_R}(R^n,R) \prec \mathrm{def}_{L_R}(S^n,S)$ as rings and $Th_{L_R}(\mathrm{def}_{L_R}(R^n,R))$ is decidable.

(b) If R is a real closed field with an exponential in the language $L(\exp)$ where L is an expansion of the language of rings, then $\mathrm{def}_L(R^n,R) \prec \mathrm{def}_{L(\exp)}(R^n,R)$ as rings and by (a) the theory $Th_{L_R}(\mathrm{def}_{L(\exp)}(R^n,R))$ is decidable.

Proof We only prove the first part, since the second follows from it. The rings $\mathrm{def}_{L_1}(R_1^n,R_1)$ and $\mathrm{def}_{L_2}(R_2^n,R_2)$ are real closed (see [11], or [10], example 10.18 and section 12) and are von Neumann regular. We can then apply theorem 1. Let $B_1 := \mathrm{def}_{L_1}(R_1^n)$ be the boolean algebra of idempotents of $\mathrm{def}_{L_1}(R_1^n,R_1)$, and $B_2 := \mathrm{def}_{L_2}(R_2^n)$ be the boolean algebra of idempotents of $\mathrm{def}_{L_2}(R_2^n,R_2)$. We only have to check that the natural inclusion $B_1 \to B_2$ is elementary in L_{BA}. By proposition 1, it is enough to check that for every $c \in B_1$, $\mathrm{inv}(B_1 \restriction c) = \mathrm{inv}(B_2 \restriction c)$. From the definition of the elementary invariant we easily see that $\mathrm{inv}(B_1 \restriction c) = (0,0,\alpha_1)$ and $\mathrm{inv}(B_2 \restriction c) = (0,0,\alpha_2)$, where α_i is the minimum of ω and the number of atoms of $B_i \restriction c$, for $i=1,2$. We have $\alpha_1 = \alpha_2$ since $R_1 \prec_{L_1} R_2$.

The decidability is a direct consequence of theorem 1 (1). □

Remark 3 Let $\chi(x_1,\dots,x_n)$ be an L-formula with parameters in R and let $\mathrm{def}_L(\chi(R^n),R)$ be the ring of L-definable functions from $\chi(R^n)$ to R. It is possible to apply the previous results to $\mathrm{def}_L(\chi(R^n),R)$, since it is definable in $\mathrm{def}_L(R^n,R)$, using the function $1_{\chi(R^n)}$.

4.2 General L-structures

Let L be a first-order language and let R be an L-structure. Then $\mathrm{def}_L(R^n,R)$ is naturally an L-structure (where the interpretations of the function, constant and relation symbols are defined coordinate by coordinate, i.e. $\mathrm{def}_L(R^n,R)$ is an L-substructure of R^{R^n}). To deal with this new language we need an additional assumption and use a straightforward adaptation of the proof of [3, Theorem 3.1]:

Proposition 4 *Let T be an L-theory with definable Skolem functions, and let R be a model of T. There is a procedure which associates to any L-formula $\sigma(\bar{z})$ an L_{BA}-formula $\Phi(X_1,\dots,X_m)$ and L-formulas $\theta_1(\bar{z}),\dots,\theta_m(\bar{z})$ such that for every L-structure R and every $\bar{f} \in \mathrm{def}_L(R^n,R)^{\bar{z}}$:*

$$\mathrm{def}_L(R^n,R) \models \sigma(\bar{f})$$
if and only if
$$\mathrm{def}_L(R^n) \models \Phi(\|\theta_1(\bar{f})\|_R,\dots,\|\theta_m(\bar{f})\|_R),$$

where $\|\theta_i(\bar{f})\|_R = \{\bar{x} \in R^n \mid R \models \theta_i(\bar{f}(\bar{x}))\}$.
This procedure is effective if there is an effective way to associate a definable Skolem function to any given existential formula. In this case, if $Th_L(R)$ *is decidable then* $Th_L(\mathrm{def}_L(R^n, R))$ *is decidable.*

Proof The first part is obtained with the same argument as [3, Theorem 3.1], which we reproduce here for clarity. We proceed by induction on σ:

- $\sigma(\bar{z})$ is atomic, of the form $R(t_1(\bar{z}), \dots, t_k(\bar{z}))$, where R is a relation symbol in L and the t_i are L-terms. We take $X_1 = \top$ for $\Phi(X_1)$ and $R(t_1(\bar{z}), \dots, t_k(\bar{z}))$ for θ_1.
- $\sigma(\bar{z})$ is $\neg\sigma'(\bar{z})$. By induction we get an L_{BA}-formula $\Phi'(X_1, \dots, X_m)$ and L-formulas $\theta'_1(\bar{z}), \dots, \theta'_m(\bar{z})$ associated to σ', which satisfy the conclusion of the proposition. We then take $\Phi(X_1, \dots, X_m) = \neg\Phi'(X_1, \dots, X_m)$ and $\theta_i(\bar{z}) = \theta'_i(\bar{z})$ for $i = 1, \dots, m$.
- $\sigma(\bar{z})$ is $\sigma_1(\bar{z}) \wedge \sigma_2(\bar{z})$. We obtain, by induction and for $i = 1, 2$, an L_{BA}-formula $\Phi_i(X_1, \dots, X_{m_i})$ and L-formulas $\theta_{i1}(\bar{z}), \dots, \theta_{im_i}(\bar{z})$ associated to σ_i, which satisfy the conclusion of the proposition. We then define

$$\Phi(X_1, \dots, X_{m_1+m_2}) = \Phi_1(X_1, \dots, X_{m_1}) \wedge \Phi_2(X_{m_1+1}, \dots, X_{m_1+m_2}), \text{ and}$$

$$\theta_1(\bar{z}) = \theta_{11}(\bar{z}), \dots, \theta_{m_1}(\bar{z}) = \theta_{1m_1}(\bar{z}),$$

$$\theta_{m_1+1}(\bar{z}) = \theta_{21}(\bar{z}), \dots, \theta_{m_1+m_2}(\bar{z}) = \theta_{2m_2}(\bar{z}).$$

- $\sigma(\bar{z})$ is $\exists x\ \sigma'(x, \bar{z})$. By induction there is an L_{BA}-formula $\Phi'(X_1, \dots, X_m)$ and there are L-formulas $\theta'_1(x, \bar{z}), \dots, \theta'_m(x, \bar{z})$ associated to σ', which satisfy the conclusion of the proposition.
  Fact: We can assume that the formulas θ'_i form a partition, i.e.

$$\forall 1 \le i \ne j \le m \quad \vdash \forall x, \bar{z}\ \neg(\theta'_i(x, \bar{z}) \wedge \theta'_j(x, \bar{z})),$$

$$\vdash \forall x, \bar{z} \bigvee_{i=1}^{m} \theta'_i(x, \bar{z}).$$

  Proof of the fact: For $\bar{\varepsilon} \in \{\emptyset, \neg\}^m$, let $\theta'_{\bar{\varepsilon}}(x, \bar{z})$ be $\bigwedge_{i=1}^m \varepsilon_i \theta'_i(x, \bar{z})$, where $\emptyset\theta'_i$ is θ'_i and $\neg\theta'_i$ is $\neg\theta'_i$. The formulas $\theta'_{\bar{\varepsilon}}$ are the atoms of the finite boolean algebra of formulas generated by $\theta'_1, \dots, \theta'_m$, they form a partition, and each formula θ'_i can be expressed (as a conjunction) in terms of the formulas $\theta'_{\bar{\varepsilon}}$. Upon replacing the θ'_i by the $\theta'_{\bar{\varepsilon}}$ and modifying Φ' accordingly, we can assume that the θ'_i form a partition, which proves the fact.

  We take for $\Phi(X_1, \dots, X_m)$ the following formula

$$\exists Y_1, \dots, Y_m \bigwedge_{i=1}^{m} Y_i \subseteq X_i \wedge (\bigcup_{i=1}^{m} Y_i = \top) \wedge \bigwedge_{1 \le i \ne j \le m} (Y_i \cap Y_j = \bot) \wedge \Phi'(Y_1, \dots, Y_m),$$

  and for $\theta_i(\bar{z})$ the formula $\exists x\ \theta'_i(x, \bar{z})$, for $i = 1, \dots, m$. We check that these formulas have the required properties:
  "$\Rightarrow$" Assume $\mathrm{def}_L(R^n, R) \models \exists x\ \sigma'(x, \bar{f})$, i.e. there is $g \in \mathrm{def}_L(R^n, R)$ such

that $\mathrm{def}_L(R^n,R) \models \sigma'(g,\bar{f})$. By induction hypothesis we have $\mathrm{def}_L(R^n) \models \Phi'(\|\theta_1'(g,\bar{f})\|_R,\dots,\|\theta'(g,\bar{f})\|_R)$. Using that the θ_i' form a partition we get

$$\bigcup_{i=1}^{m} \|\theta_i'(g,\bar{f})\|_R = R^n \text{ and } \bigwedge_{1\le i\neq j\le m} (\|\theta_i'(g,\bar{f})\|_R \cap \|\theta_j'(g,\bar{f})\|_R = \emptyset).$$

We then take $Y_i = \|\theta_i'(g,\bar{f})\|_R$, which is possible since $\|\theta_i'(g,\bar{f})\|_R \in \mathrm{def}_L(R^n)$.
"$\Leftarrow$" Assume

$$\mathrm{def}_L(R^n) \models \exists Y_1,\dots,Y_m \bigwedge_{i=1}^{m} Y_i \subseteq \|\exists x\, \theta_i'(x,\bar{f})\|_R \wedge (\bigcup_{i=1}^{m} Y_i = \top) \wedge \bigwedge_{1\le i\neq j\le m} (Y_i \cap Y_j = \bot) \wedge \Phi'(Y_1,\dots,Y_m),$$

and let $Y_1,\dots,Y_n$ be the definable subsets whose existence is asserted by this formula.
For $1 \le i \le m$ let $g_i(\bar{z})$ be a definable Skolem function for the formula $\theta_i'(x,\bar{z})$. We define a function g from R^n to R by $g(\bar{a}) = g_i(\bar{a})$ where i is the unique element of $\{1,\dots,m\}$ such that $\bar{a} \in Y_i$. The function g is in $\mathrm{def}_L(R^n,R)$ and, using that the $\|\theta_i'(g,\bar{f})\|_R$ as well as the Y_i form a partition of R^n, we obtain $Y_i = \|\theta_i'(g,\bar{f})\|_R$ for every $i = 1,\dots,m$, from which follows $\mathrm{def}_L(R^n) \models \Phi(\|\theta_1'(g,\bar{f})\|_R,\dots,\|\theta_m'(g,\bar{f})\|_R)$, i.e. $\mathrm{def}_L(R^n,R) \models \sigma(g,\bar{f})$.

For the second part of the proposition, in case there is an effective way to associate a definable Skolem function to any given existential formula: we know that $Th_{L_{BA}}(\mathrm{def}_L(R^n))$ is always decidable. Indeed, if $\xi := \mathrm{inv}(\mathrm{def}_L(R^n))$ then by [5, Proposition 18.10] $Th_{L_{BA}}(\mathrm{def}_L(R^n))$ is $T(\xi)$ the theory of boolean algebras with elementary invariant ξ, which has a recursive set of axioms (see [5, 18.8]), so is decidable. Now if σ is an L-formula without free variables, the formulas θ_i associated to it by the first part of the proposition are without free variables, so $\|\theta_i\|_R \in \{\bot,\top\}$, and $(\|\theta_1\|_R,\dots,\|\theta_m\|_R) = \bar{\varepsilon} \in \{\bot,\top\}^m$. If $Th_L(R)$ is decidable then $\|\theta_i\|_R = \top$ is decidable (it is equivalent to $Th_L(R) \vdash \theta_i$), so there is an algorithm to decide $(\|\theta_1\|_R,\dots,\|\theta_m\|_R) = \bar{\varepsilon}$, and then $Th_L(\mathrm{def}(R^n,R)) \vdash \sigma$ is equivalent to $T(\xi) \vdash \Phi(\bar{\varepsilon})$. □

Corollary 5 *Let R and S be models of an L-theory with definable Skolem functions and assume $R \prec_L S$, respectively $R \equiv_L S$. Then* $\mathrm{def}_L(R^n,R) \prec_L \mathrm{def}_L(S^n,S)$*, respectively* $\mathrm{def}_L(R^n,R) \equiv_L \mathrm{def}_L(S^n,S)$*, for every $n \in \mathbb{N}$.*

Proof We only prove the first part of the corollary. Since $R \prec_L S$, the natural inclusion $\lambda : \mathrm{def}_L(R^n) \to \mathrm{def}_L(S^n)$ is elementary by corollary 1. Let $\sigma(\bar{z})$ be an L-formula, and let $\bar{f} \in \mathrm{def}_L(R^n,R)^{\bar{z}}$. Applying proposition 4 and the fact that λ is elementary with $\lambda(\|\theta(\bar{f})\|_R) = \|\theta(\bar{f})\|_S$ for every L-formula $\theta(\bar{x})$, we successively get

$$\begin{aligned}\mathrm{def}_L(R^n,R) \models \sigma(\bar{f}) &\Leftrightarrow \mathrm{def}_L(R^n) \models \Phi(\|\theta_1(\bar{f})\|_R,\dots,\|\theta_m(\bar{f})\|_R)\\ &\Leftrightarrow \mathrm{def}_L(S^n) \models \Phi(\lambda(\|\theta_1(\bar{f})\|_R),\dots,\lambda(\|\theta_m(\bar{f})\|_R))\\ &\Leftrightarrow \mathrm{def}_L(S^n) \models \Phi(\|\theta_1(\bar{f})\|_S,\dots,\|\theta_m(\bar{f})\|_S)\\ &\Leftrightarrow \mathrm{def}_L(S^n,S) \models \sigma(\bar{f}).\end{aligned}$$

□

5 Continuous semi-algebraic functions

We can also try to get results similar to those obtained in corollaries 4 and 5, but for structures consisting of definable functions satisfying some additional property P_0 (the most natural one, which motivates this section, is probably continuity when R is equipped with some topology). We present here a very partial result, using infinitary languages, when both structures are ω-saturated.

Let L be a first-order language. To keep our arguments simple, we assume that L only contains relation symbols (replacing, if necessary, every k-ary function symbol in L by a $k+1$-ary relation symbol corresponding to the graph of the function).

Let $n \in \mathbb{N}$, let F be a new n-ary function symbol and let $P_0(F)$ be a first-order $L(F)$-formula. If S is an L-structure, we define

$$P_0(\mathrm{def}_L(S^n, S)) := \{f \in \mathrm{def}_L(S^n, S) \mid S \models P_0(f)\}.$$

We then define the language $L(P) := L \cup \{P\}$, where P is a new unary relation symbol, and turn $\mathrm{def}(S^n, S)$ into an L_P-structure by interpreting P by $P_0(\mathrm{def}_L(S^n, S))$.

We use the following convention: Suppose $\bar{G}$ is a tuple of new n-ary function symbols and ψ is an $L(\bar{G})$-formula. We write $\psi(\bar{G})$ instead of ψ if we want to make explicit the use of the symbols $\bar{G}$ in ψ. In this case, if S is an L-structure and $\bar{g}$ is a tuple of functions from S^n to S of the same length as $\bar{G}$, then $\psi(\bar{g})$ means that ψ is interpreted in S using the functions $\bar{g}$ as the interpretation of the symbols $\bar{G}$.

Let $n \in \mathbb{N}$ and let $\{\phi_i(\bar{x}, y, \bar{z}_i)\}_{i \in I}$ be the list of all L-formulas with at least $\bar{x}$ and y as free variables, with $l(\bar{x}) = n$.

Lemma 3 *Let S_1 be an L-structure. Let $\bar{f} \in \mathrm{def}_L({S_1}^n, S_1)$ and let $\bar{a} \in S_1$ be the elements of S_1 used to define $\bar{f}$. Let $\psi(\bar{u}, \bar{v})$ be an $L(P)$-formula with $\bar{v}$ of length $l \in \mathbb{N} \cup \{0\}$. Then there is an $L(\bar{G})_{\infty\omega}$-formula $\psi^*(\bar{a}, \bar{G})$, where $\bar{G}$ is a tuple of length l of new n-ary function symbols, such that for every L-structure S_2 such that $S_1 \prec_L S_2$ and every $\bar{g} \in \mathrm{def}_L({S_2}^n, S_2)$:*

$$\mathrm{def}_L({S_2}^n, S_2) \models \psi(\bar{f}, \bar{g}) \Leftrightarrow S_2 \models \psi^*(\bar{a}, \bar{g}),$$

where in the right hand side, the tuple $\bar{g}$ is used as a tuple of n-ary functions interpreting $\bar{G}$ in S_2.

Proof We proceed by induction on the formula ψ. We use $\bar{f}$ as a tuple of n-ary function symbols in S_2, which we can do since the elements of $\bar{f}$ are definable in S with parameters $\bar{a}$.

- $\psi(\bar{u}, \bar{v})$ is atomic of the form $R(\bar{u}, \bar{v})$, where R is a relation symbol in L (recall that L only contains relation symbols). Then $\psi^*(\bar{a}, \bar{G})$ is

$$\forall \bar{x}, \bar{y}, \bar{z}\ (\bigwedge_{i=1}^{l(\bar{f})} f_i(\bar{x}) = y_i \wedge \bigwedge_{i=1}^{l} G_i(\bar{x}) = z_i) \to R(\bar{y}, \bar{z}).$$

- $\psi(\bar{u}, \bar{v})$ is $P(u_i)$. Then $\psi^*(\bar{a}, \bar{G})$ is $P_0(f_i)$.
- $\psi(\bar{u}, \bar{v})$ is $P(v_i)$. Then $\psi^*(\bar{a}, \bar{G})$ is $P_0(G_i)$.

- The negation and conjunction cases are clear by induction.
- $\psi(\bar{u},\bar{v})$ is $\exists w\ \psi_1(\bar{u},\bar{v},w)$. By induction we have a formula $\psi_1^*(\bar{a},\bar{G},H)$ associated to ψ_1 (H is another new n-ary function symbol). We use the following notation: If $\phi_i(\bar{x},y,\bar{s})$ is an L-formula with parameters $\bar{s}\in S_2$ which defines the graph of a function from ${S_2}^n$ to S_2, we denote this function by $f_{\phi_i(\bar{x},y,\bar{s})}$. A straightforward verification now shows that the following formula $\psi^*(\bar{a},\bar{G})$ has the required property:

$$\bigvee_{i\in I}\left\{\exists\bar{z}_i \text{ "}\phi_i(\bar{x},y,\bar{z}_i) \text{ defines a total } n\text{-ary function"} \wedge \psi_1^*(\bar{a},\bar{G},f_{\phi_i(\bar{x},y,\bar{z}_i)})\right\}.$$

□

Remark 4 The same result also holds if $P_0(F)$ or ψ are $L_{\infty\omega}$-formulas.

We deduce

Proposition 5 *Let $n\in\mathbb{N}$ and let $S_1\prec_L S_2$ be two ω-saturated L-structures. Then* $\mathrm{def}_L({S_1}^n,S_1)\prec_{L(P)}\mathrm{def}_L({S_2}^n,S_2)$. *In particular, if S_1 and S_2 are ordered structures and the order is L-definable in the same way in S_1 and S_2 then* $\mathrm{cdef}({S_1}^n,S_1)\prec_L \mathrm{cdef}({S_2}^n,S_2)$, *where* cdef *denotes the set of continuous definable functions.*

Proof From the ω-saturation we get $S_1\equiv_{\infty\omega}S_2$ in $L\cup\{\bar{a}\}$ for every finite tuple $\bar{a}\in S_1$ (see [4, exercise 8 p. 488]). We then apply lemma 3 for every $L(P)$-formula $\psi(\bar{f})$ with parameters $\bar{f}\in\mathrm{def}_L(S_1^n,S_1)$.
For the second part of the statement, we just have to take for $P_0(F)$ the L-formula "F is continuous". Then $\mathrm{cdef}(S_i^n,S_i)$ becomes $L(P)$-definable in $\mathrm{def}(S_i^n,S_i)$. □

Unfortunately, the method used to get proposition 5 cannot be obviously extended to the general case of (non-necessarily ω-saturated) L-structures, not even when these L-structures are real closed fields:

First note that lemma 3 can clearly be extended to the case of the language $L\cup\{P_1,\dots,P_m\}$, where $P_1,\dots,P_m$ are obtained in the same way as P. In particular, if we take $P_1(F)$ to be "F is continuous" and $P_2(F)$ to be "F is constant", we get the following extension of proposition 5:

Proposition 6 *Let $n\in\mathbb{N}$ and let $S_1\prec_L S_2$ be two ω-saturated ordered structures, such that the order is L-definable in the same way in S_1 and S_2. Then* $\mathrm{cdef}({S_1}^n,S_1)\prec_{L(C)}\mathrm{cdef}({S_2}^n,S_2)$, *where C is a new unary relation symbol interpreted in both structures by the set of constant functions.*

But a result of Tressl (as yet unpublished) shows that if S is a real closed field, then the set

$$N(S):=\{f\in\mathrm{cdef}(S,S)\mid f \text{ is constant and its value is in } \mathbb{N}\}$$

is definable in $\mathrm{cdef}(S,S)$ in the language $L(C)$.
It follows that if S_1 is an archimedean real closed field and S_2 is a non-archimedean real closed field such that $S_1\prec S_2$, then $\mathrm{cdef}(S_1,S_1)\not\prec\mathrm{cdef}(S_2,S_2)$ in $L(C)$ ($N(S_1)$ is cofinal in C^{S_1}, but $N(S_2)$ is not cofinal in C^{S_2}).

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