NONASSOCIATIVE QUATERNION ALGEBRAS OVER RINGS

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Abstract. Non-split nonassociative quaternion algebras over fields were first discovered over the real numbers independently by Dickson and Albert. They were later classified over arbitrary fields by Waterhouse. These algebras naturally appeared as the most interesting case in the classification of the four-dimensional nonassociative algebras which contain a separable field extension of the base field in their nucleus. We investigate algebras of constant rank 4 over an arbitrary ring R which contain a quadratic étale subalgebra S over R in their nucleus. A generalized Cayley-Dickson doubling process is introduced to construct a special class of these algebras.

Introduction

Let k be a field. A *non-split nonassociative quaternion algebra* over k is a four-dimensional unital k-algebra A whose nucleus is a separable quadratic field extension of k. Non-split nonassociative quaternion algebras were early examples of nonassociative division algebras which are neither power-associative nor quadratic and were first considered by Dickson [D] in 1935, and by Albert [A] in 1942, both times over the reals. In 1987, Waterhouse [W] completely classified these algebras as well as the corresponding *split nonassociative quaternion algebras* (defined to be four-dimensional unital simple k-algebras whose nucleus is isomorphic to the split quadratic étale algebra $k \oplus k$) over arbitrary base fields, and computed their automorphisms and derivations. Lee and Waterhouse [L-W], [L] later investigated maximal R-orders in a nonassociative quaternion algebra A over $k = \text{Quot}(R)$, R a Dedekind domain, and classified certain isomorphism classes of these orders.

Let l be a quadratic étale algebra over k. A unital nonassociative k-algebra A is called *l-associative* if l is contained in its nucleus. Given a separable quadratic field extension l of the base field k, non-split nonassociative

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quaternion algebras over k naturally appeared as the only interesting (i.e., formerly unknown) case in the classification of the l-associative algebras of dimension 4 over k in [W]. This was already observed by Althoen-Hansen-Kugler [A-H-K] in the special case that $k = \mathbb{R}$. Althoen-Hansen-Kugler classified the $\mathbb{C}$-associative $\mathbb{R}$-algebras of dimension 4, and hence in particular also the non-split nonassociative quaternion algebras over $\mathbb{R}$.

Read in the right way, Waterhouse's classification reveals that every - split or non-split - nonassociative quaternion algebra over a field k is a classical Cayley-Dickson doubling of its nucleus S, where the scalar chosen for this doubling process is an invertible element in S, not contained in the base field. Hence this construction canonically extends the one known for classical quaternion algebras over k.

Let R be a commutative associative unital ring. Let S be a quadratic étale algebra over R. The term R-algebra refers to unital nonassociative algebras over R which are finitely generated projective of constant rank as R-modules. An R-algebra A is called *S-associative* if S is contained in its nucleus. We obtain a general construction method for certain S-associative algebras which contain S as a direct summand. Furthermore, we define *nonassociative quaternion algebras* more generally as R-algebras which can be generated by a generalized Cayley-Dickson process: over a ring R, every "classical" (i.e., associative) quaternion algebra containing a quadratic étale subalgebra S over R, can be realized by a Cayley-Dickson doubling process $\mathrm{Cay}(S, P, h)$ with P a finitely generated right S-module of rank 1, and $h : P \times P \to S$ a nondegenerate 1-hermitian form (cf. Petersson [P]). This doubling process can be adapted and then yields a special class of nonassociative S-associative algebras over R: The idea is to take a nondegenerate ε-hermitian form $b : P \times P \to S$ instead of the nondegenerate 1-hermitian form $h : P \times P \to S$. In particular, take an invertible scalar $\mu \in S$ not contained in R then the form $\mu h : P \times P \to S$ is such an ε-hermitian form with $\varepsilon = \bar{\mu}\mu^{-1}$. Thus nonassociative quaternion algebras are closely related to the classical quaternion algebras.

Detailed examples are obtained considering special classes of rings.

1. Preliminaries

Let R be a unital commutative associative ring. The term "R-algebra" refers to unital nonassociative algebras over R which are finitely generated projective as R-modules of constant (local) rank. Let S be a quadratic étale algebra over R (i.e., a separable quadratic R-algebra in the sense of [Knu, p. 4]) with canonical involution $\sigma\colon S\to S$, also written as $\sigma={}^-$, and with nondegenerate norm $n_S\colon S\to S$, $n_S(s)=s\overline{s}=\overline{s}s$. S is a unital commutative associative algebra over R, finitely generated projective of constant rank 2 as R-module. $R\times R$ (with the diagonal action of R) is a quadratic étale algebra. Its canonical involution is given by $(x,y)\mapsto(y,x)$. A quadratic étale algebra S which is isomorphic to the algebra $R\times R$ is called *split*, otherwise it is called *non-split*.

For an R-algebra A, associativity in A is measured by the *associator* $[x,y,z]\colon=(xy)z-x(yz)$. The *nucleus* of A is defined as $N(A)\colon=\{x\in A\,|\,[x,A,A]=[A,x,A]=[A,A,x]=0\}$. The nucleus is an associative subalgebra of A (it may be zero), and $x(yz)=(xy)z$ whenever one of the elements x,y,z is in $N(A)$.

An R-algebra A is *S-associative* if S is contained in the nucleus $N(A)$. $\mathbb{C}$-associative algebras over $\mathbb{R}$ of dimension 4 were classified in [A-H-K], l-associative algebras over an arbitrary base field k, l a separable quadratic field extension of k in [W]. The only nonassociative algebras which appear in these classification are the non-split nonassociative quaternion algebras over $\mathbb{R}$ (or, respectively, over k). These algebras become split nonassociative quaternion algebras under suitable base field extensions in the sense defined before (e.g. under the field extension $\mathbb{C}$ if $k=\mathbb{R}$ as in [A-H-K]). We rephrase this classification [W, Theorem 4, Theorem 5] as follows:

Lemma 1. *Let k be a field and S be a quadratic étale algebra over k with canonical involution $\sigma={}^-$. For every $\mu\in S^\times\setminus k$, the vector space*

$$\mathrm{Cay}(S,\mu)=S\oplus S$$

becomes a (split or non-split) nonassociative quaternion algebra over k with $S=N(A)$ under the multiplication

$$(u,v)(u',v')=(uu'+\mu\overline{v}'v,\,v'u+v\overline{u}')$$

for $u,u',v,v'\in S$.
Given any (split or non-split) nonassociative quaternion algebra A over k with nucleus $N(A)=S$, there exists an element $\mu\in S^\times\setminus k$ such that $A\cong\mathrm{Cay}(S,\mu)$.

Proof. For $J\colon=(0,1)\in S\oplus S$ we obtain $A=\mathrm{Cay}(S,\mu)=S\oplus SJ$ and the multiplication of A becomes

$$(u+vJ)(u'+v'J)=(uu'+\mu\overline{v}'v)+(v'u+v\overline{u}')J.$$

In particular, $J^2 = \mu$, $(1,0)$ is the identity of A and $Jx = \overline{x}J$ for all $x \in S$.

If S is non-split, the assertion follows from [W, Theorem 1].

If S is split, identify S with $k \times k$. An element $\mu = (\mu_1, \mu_2) \in S$ lies in k if and only if $\mu_1 = \mu_2$. It is invertible if and only if $\mu_1\mu_2 \neq 0$. The idempotents $e_1 = (1,0)$ and $e_2 = (0,1)$ in $k \times k$ are a basis for $k \times k$. Following [W, Proof of Theorem 5] define $e_3 = e_1 J$, $e_4 = e_2 J$ to obtain a basis for A. Then all products involving e_1 or e_2 are known, and $e_3^2 = 0$, $e_4^2 = 0$, $e_3 e_4 = e_1 J^2 \overline{e}_2 = \mu e_1 \overline{e}_2 = \mu(1,0)\overline{(0,1)} = \mu(1,0)(1,0) = \mu e_1 = (\mu_1, 0)$ for $\mu = (\mu_1, \mu_2) \in k \times k$, $e_4 e_3 = \mu e_2 \overline{e_1} = \mu(0,1)(0,1) = \mu e_2 = (0, \mu_2)$. Moreover, $e_3 e_4 + e_4 e_3 = (e_3 + e_4)^2 = J^2 = \mu \in S$. By replacing e_3 by the scalar multiple $\frac{1}{\mu_1} e_3$ we get $e_3 e_4 = e_1$, and $e_4 e_3 = \frac{\mu_2}{\mu_1} e_2$. Put $\lambda\colon = \frac{\mu_2}{\mu_1}$, then the algebra A is of the type described in the multiplication table in [W, Theorem 4], and thus a split nonassociative quaternion algebra. The rest of the assertion follows from [W, Theorem 4]. □

This obviously extends the construction of "classical" quaternion algebras over fields starting from a quadratic étale subalgebra: the case $\mu = (\mu_1, \mu_2) \in S = k \times k$ such that $\mu_1 = \mu_2 \neq 0$ yields the split (associative) quaternion algebra $\mathrm{Mat}_2(k)$ over k ($\mu_1 = \mu_2$ is equivalent to $\mu = (\mu_1, \mu_2) \in k$ here). The case $\mu \in k$ for a separable quadratic field extension S of k yields either the associative quaternion algebra $\mathrm{Cay}(S, \mu)$ over k (if $\mu \neq 0$) or the "degenerate associative quaternion algebra" $\mathrm{Cay}(S, 0)$. All non-split nonassociative quaternion algebras over a field are division algebras which are neither power-associative nor quadratic [W, p. 369].

2. S-associative algebras

We will mostly study S-associative algebras of rank 4.

For each prime ideal $p \in \mathrm{Spec}\, R$, the residue class field is defined as $K(p) = R_p / pR_p$. Let A be an R-algebra. Its localization is given by $A_p = A \otimes R_p$ and its residue class algebra is given by $A(p) = A \otimes R_p / pR_p$ for each $p \in \mathrm{Spec}\, R$.

Lemma 2. *Let S be a quadratic étale R-algebra. If A is an S-associative R-algebra, then the localization A_p is an S_p-associative R_p-algebra for all $p \in \mathrm{Spec}\, R$, and the residue class field algebra $A(p)$ is an $S(p)$-associative $K(p)$-algebra for all $p \in \mathrm{Spec}\, R$.*

Proof. We have $N(A) \otimes_R R' \subset N(A \otimes_R R')$ for any ring extension R' of R, and any unital R-algebra A. Hence $N(A)_p \subset N(A_p)$ and $N(A) \otimes_R$

$K(p) \subset N(A(p))$ for all $p \in \operatorname{Spec} R$. In particular, this implies that the algebra $A_p = A \otimes_R R_p$ over the local ring R_p has the quadratic étale algebra S_p in its nucleus for each $p \in \operatorname{Spec} R$, and that the residue class algebra $A(p) = A \otimes_R K(p)$ over $K(p)$ has the quadratic étale algebra $S(p)$ in its nucleus for each $p \in \operatorname{Spec} R$. □

Let S be a quadratic étale R-algebra with canonical involution $\sigma = ^{-}$. Let A be an R-algebra which contains S in its nucleus. Then A is an S-bimodule via $S \times A \to A$, $(s, a) \to sa$ and $A \times S \to A$, $(a, s) \to as$. This implies in particular that A must have even constant rank as an R-module.

The S-bimodule structure on A can be viewed as a left (or right) module structure for the ring $S \otimes S^{\mathrm{op}}$. This ring is a quadratic étale algebra over S and we have $S \otimes S^{\mathrm{op}} \cong S \times S$ via $x \otimes y \to (xy, x\bar{y})$ [Knu, p. 127].

The finitely generated projective left modules of rank one over the split quadratic étale algebra $S \times S$ are known. Any such module is isomorphic to the direct sum of two invertible modules L and M in $\operatorname{Pic}(S)$ (see the proof of [P, 2.7]). We conclude:

Lemma 3. *Let A be an S-associative algebra of constant rank 4 over a ring R. Suppose that A is finitely generated projective of rank one as an $S \times S$-module. Then*

$$A \cong L \oplus M$$

for two invertible S-modules L and M.

To achieve a classification of the S-associative algebras of rank 4 over arbitrary rings seems to be a complex problem, even if we require them to be finitely generated projective of rank one as $S \times S$-modules. Contrary to the case studied in [W] (where only three types of bimodule structures were possible provided that S was a separable quadratic field extension and R a field) it is probably also not true anymore that the quadratic étale R-algebra S is always a direct summand of A. To achieve at least a partial classification of S-associative algebras one promising line of attack seems to be to focus on certain classes of S-associative algebras over R. Another one might be to restrict our attention to certain rings R and algebras S where for instance the Picard group $\operatorname{Pic}(S)$ is small or even trivial.

One important class of S-associative R-algebras where S indeed is a direct summand of A can be constructed as follows:

Theorem 4. *Let P be a finitely generated projective right S-module of constant rank m carrying a sesquilinear form $h\colon P\times P\to S$. Then the R-module*

$$A = S\oplus P$$

becomes an S-associative R-algebra of constant even rank $2m+2$ via the multiplication

$$(u,w)(u',w') = (uu' + h(w',w), w'\cdot u + w\cdot \overline{u'})$$

for $u,u'\in S$, $w,w'\in P$. We write $A=(S,P,h)$ for this algebra.

Proof. It is easy to check that $S\subseteq N(A)$, hence $A=(S,P,h)$ is an S-associative algebra over R. □

Here are some easy examples of S-associative R-algebras of rank 4:

Example 1. *(i) Let A be the direct product of S and any associative commutative R-algebra A_0 of rank 2. Then $A=S\oplus A_0$ is a commutative associative S-associative R-algebra of rank 4. This "trivial" type of an S-associative algebra appeared in Waterhouse's classification in [W, Proposition 1] as type 1).*

Indeed, if A is an S-associative R-algebra of rank 4 and if its S-bimodule structure is given by $A=S\oplus A_0$ where $xa=ax=0$ for all $x\in S$ and $a\in A_0$, Waterhouse's argument yields analogously that A must be the direct product of S and A_0.
(ii) Let $2\in R^\times$. Consider the rank 2 S-algebra (S,P,h) with $P\in\operatorname{Pic}S$, and $h: P\times P\to S$ a (perhaps degenerate) 1-hermitian form. Then A is a commutative associative S-associative R-algebra of rank 4 (see [W, Proposition 2], with bimodule structure of type 2)).

(iii) Suppose that S is a non-split quadratic étale algebra over R such that the residue class algebras $S(p)=S_p\otimes K(p)$ are non-split for all $p\in\operatorname{Spec}R$. Let A be an S-associative R-algebra of rank 4. We know that A is an S-bimodule. If A also is an S-algebra (of constant rank 2) with identity $1\in S$ (as it can happen when R is a field and S a separable quadratic field extension of R, see [W, Proposition 2], type 2) in his classification), then A must be commutative and associative, since these properties hold for all residue class algebras $S(p)$ [W, Proposition 2]. If, additionally, A is even finitely generated projective and faithful as an S-module, then $S1_A$ is a direct summand of A [Knu p. 4].

A simple calculation verifies the following result:

Lemma 5. *Let $A = (S, P, h)$. If $N(A) \cap P = \{0\}$ then $N(A) = S$.*

Lemma 6. *Let $A = (S, P, h)$. If $N(A)$ is a quadratic étale algebra over R, then $S = N(A)$.*

Proof. By assumption, $N(A)$ is a quadratic étale R-algebra. Since S is a quadratic étale subalgebra of $N(A)$, both must be equal: this is obvious after passing to the stalks and then to the residue class algebras over $K(p)$. By Nakayama's Lemma, the fact that these algebras are isomorphic over $K(p)$ carries over to local rings. □

If $h : P \times P \to S$ is an (ε-)hermitian form and P a finitely generated projective right S-module of rank one, the R-algebra $A = (S, P, \mu h)$ with $\mu \in S$ is also denoted by $\mathrm{Cay}(S, P, \mu h)$. The algebra $A = \mathrm{Cay}(S, P, \mu h)$ is called a *(generalized) Cayley-Dickson doubling of S.* For $\varepsilon = 1$, $\mu \in R$ and h nondegenerate, this construction method is due to Petersson [P].

The algebra S itself is canonically a (free) right S-module of rank one which carries a nondegenerate 1-hermitian form. Any nondegenerate 1-hermitian form $h\colon S \times S \to S$ is similar (with some similarity factor $r \in R^\times$) to the canonical 1-hermitian form given by the involution, i.e., to $h_0 : (w, w') \mapsto \overline{w}w'$. In this special case the *"classical"* Cayley-Dickson doubling process $\mathrm{Cay}(S, \mu) = \mathrm{Cay}(S, S, \mu h_0)$ with $\mu \in S$ is obtained, which for $\mu \in R^\times$ is due to Albert [A].

Remark 7. *By [P, 2.5], every "classical" (i.e., associative) quaternion algebra C over R containing a quadratic étale algebra S can be constructed with the help of such a Cayley-Dickson doubling process, i.e., there exists a nondegenerate 1-hermitian form $h\colon P \times P \to S$ on $P = S^\perp$, the orthogonal complement of S in C relative to its norm n_C, such that $C \cong \mathrm{Cay}(S, P, h)$.*

We later often restrict our attention to the special case that P is a finitely generated projective right S-module of rank one carrying a nondegenerate 1-hermitian form $h\colon P \times P \to S$. On the one hand, for each invertible $\mu \in S$, the form $\mu h\colon P \times P \to S$ is a nondegenerate ε-hermitian form, with $\varepsilon = \overline{\mu}/\mu$. (In particular, $\varepsilon \neq 1$ if and only if $\mu \notin R$, and $\varepsilon = -1$ if and only if $\mu \in \mathrm{Skew}(S, ^-)$.)

On the other hand, the study of ε-hermitian forms over S can be reduced to the study of 1-hermitian ones by scaling in several important cases: let

$\mu \in S^\times$ such that $\frac{\overline{\mu}}{\mu} = \varepsilon$. Then μh is a hermitian form, for any ε-hermitian form h. Such a μ exists if $H^1(\mathbb{Z}/2\mathbb{Z}, S^\times) = 0$ (Hilbert's Theorem 90), e.g. if $\operatorname{Pic} R = 0$ [Knu, p. 300].

Lemma 8. *(i) For $\mu \in R$ and h a nondegenerate 1-hermitian form, $A = \operatorname{Cay}(S, P, \mu h)$ is a quadratic associative R-algebra with norm $n_A((u, w)) := n_S(u) - \mu h(w, w)$. It is a quaternion algebra if and only if $\mu \in R^\times$.*
(ii) Let $A = \operatorname{Cay}(S, \mu)$ be a classical Cayley-Dickson doubling with scalar $\mu \in S$. Then A is nonassociative if and only if $\mu \in S \setminus R$.

Proof. (i) Let $\mu \in R$. Then $n_A : A \to R$, $n_A((u, w)) = n_S(u) - \mu h(w, w)$ is a quadratic form satisfying $n_A((1, 0)) = 1$, and the equation $a^2 - n_A(1_A, a)a + n_A(a)1_A = 0$ holds for each $a \in A$ (here, $n_A(x, y) = n_A(x+y) - n_A(x) - n_A(y)$ is the symmetric bilinear form induced by n_A). It is easy to check that, for $\mu \in R$, A is associative. A is a quaternion algebra if and only if n_A is nondegenerate [Mc, 4.6] which is equivalent to $\mu \in R^\times$.
(ii) If $\mu \in R$ then A is associative. Suppose now that A is associative. It follows that for every $w_1, w_2, w_3 \in S$,
$(0, w_1)[(0, w_2)(0, w_3)] = [(0, w_1)(0, w_2)](0, w_3)$, i.e.,
$(0, w_1)(\mu\bar{w}_3 w_2, 0) = (\mu\bar{w}_2 w_1, 0)(0, w_3)$, i.e., $(0, w_1\bar{\mu}\bar{w}_2 w_3) = (0, w_3\mu\bar{w}_2 w_1)$.
With $w_i = 1$ we get $\bar{\mu} = \mu$, i.e., $\mu \in R$. □

Lemma 8 (ii) also holds for generalized Cayley-Dickson doublings:

Corollary 9. *Let $A = \operatorname{Cay}(S, P, \mu h)$ be a Cayley-Dickson doubling with scalar $\mu \in S$ and a nondegenerate 1-hermitian form h. Then A is associative if and only if $\mu \in R$.*

Proof. $A = \operatorname{Cay}(S, P, \mu h)$ is an associative R-algebra if and only if its localizations $A_p = \operatorname{Cay}(S_p, \mu_p h_p)$ are associative R_p-algebras for each $p \in \operatorname{Spec} R$. The 1-hermitian form h_p is nondegenerate for all p. Thus this is equivalent to $\mu_p \in R_p$ for each $p \in \operatorname{Spec} R$ by Lemma 8(ii). This proves the assertion. □

3. Nonassociative quaternion algebras

Remark 10. *Consider the S-associative algebra $A = (S, P, h)$ where $h : P \times P \to S$ is a sesquilinear form. The residue class algebras are given by $A(p) \cong (S(p), S(p), h(p))$, where $h(p) : S(p) \times S(p) \to S(p)$ is the sesquilinear form on $S(p)$ induced by h. This form is of the kind $h(p)(x, y) = \bar{x}ay$ for some $a \in S(p)$ (a depending on the chosen p of course). Hence we have*

$h(p) = ah_0$ with $h_0(x, y) = \bar{x}y$ the canonical nondegenerate 1-hermitian form on $S(p)$. In other words,

$$A(p) \cong \mathrm{Cay}(S, a).$$

If $a \in S^\times$ then $h(p)$ is ε-hermitian with $\varepsilon = \bar{a}/a$ and $A(p)$ is a split or non-split nonassociative quaternion algebra as in Waterhouse's classification, or a split or non-split quaternion algebra (if $a \in R^\times$). If $a = 0$ then $A(p)$ is an S-associative algebra of type 1) in [W]. If $a \neq 0$ and a is not invertible in $S(p)$, then $S(p)$ must be a split quadratic étale algebra over $K(p)$ and $A(p)$ is an $S(p)$-associative algebra which does not appear in Waterhouse classification of the split nonassociative quaternion algebras, because he restricted his classification to simple algebras, as soon as the nucleus was split. (By [W], the split nonassociative quaternion algebras over $K(p)$ are exactly the Cayley-Dickson doublings of the split algebra $S(p)$ with an invertible scalar in $S(p)$.)

Now let $h : P \times P \to S$ be an ε-hermitian form. Then each $h(p) : S(p) \times S(p) \to S(p)$ is an $\varepsilon(p)$-hermitian form, with $\varepsilon(p) = \varepsilon_p \otimes 1 \in S(p)$. There exists an element $\mu \in S(p)^\times$ (depending on p) such that $\mu h(p)$ is a 1-hermitian form, either $\mu h(p) = h_0$ or $\mu h(p) = 0$. This implies that either $A(p) \cong \mathrm{Cay}(S(p), \mu^{-1})$ or $A(p) \cong \mathrm{Cay}(S(p), 0)$, i.e., that $A(p)$ is a nonassociative quaternion algebra as in Waterhouse's classification, a quaternion algebra, or an S-associative algebra of type 1) in [W].

In particular, if $h : P \times P \to S$ is a nondegenerate 1-hermitian form and $\mu \in S^\times$, then μh is an ε-hermitian form with $\varepsilon = \bar{\mu}/\mu$. However, given any ε-hermitian form it is not always possible to find an element μ to write this form in the above way, unless we make additional requirements on our ring R (cf. Example 4, where the non-trivial element in $\mathrm{Pic}\, S$ corresponds with an $\varepsilon \in S^\times$ satisfying $\varepsilon\bar{\varepsilon} = 1$, however, there is no element $\mu \in S^\times$ such that $\varepsilon = \bar{\mu}\mu^{-1}$ by [K, III.(2.8.1)]).

This observation makes it clear that Waterhouse restriction to simple S-associative algebras as soon as S was split is mirrowed in our more general setup in the choice of the form h. If we limit ourselves to considering only ε-hermitian forms $h : P \times P \to S$ we will obtain only simple residue class algebras $A(p)$ as soon as $S(p)$ is split (or the algebra $A(p) \cong \mathrm{Cay}(S(p), 0)$). If we allow the most general case of h being a sesquilinear form we also obtain S-associative algebras $A(p)$ which were not mentioned in [W], when S is split.

Definition 11. *Let S be a quadratic étale R-algebra. Let A be an S-associative R-algebra. A is a* nonassociative quaternion algebra *over R if $A \cong \operatorname{Cay}(S,P,h)$ with $h : P \times P \to S$ a nondegenerate ε-hermitian form. A nonassociative quaternion algebra over R is called split, if $S \cong R \times R$ is split, otherwise non-split.*

This implies: if A is a nonassociative quaternion algebra over R, the residue class algebras $A(p)$ are (nonassociative or associative) split or non-split quaternion algebras over the residue class field $K(p)$, or the "degenerate" algebra $A(p) = \operatorname{Cay}(S(p), 0)$, for each $p \in \operatorname{Spec} R$. Obviously, A has constant rank 4 as R-module. Furthermore, the algebra $A(p)$ is either simple, or $A(p) = \operatorname{Cay}(S(p), 0)$. Every non-split (associative or nonassociative) quaternion algebra over a field is indeed even a *central simple division* algebra.

For $R = k$ a field, our definition of a nonassociative quaternion algebra deviates slightly from the one given in [W]: it includes both the case of a split and a non-split nonassociative quaternion algebras, while Waterhouse distinguished the non-split nonassociative quaternion algebras from the split nonassociative quaternion algebras. Otherwise, it coincides with his definition, since we only allow nondegenerate 1-hermitian forms.

Lemma 12. *If A is a nonassociative quaternion algebra over R then A_p is a nonassociative quaternion algebra over R_p for each $p \in \operatorname{Spec} R$.*

Lemma 13. *Let A be a nonassociative quaternion algebra over a local ring R. Then $A \cong \operatorname{Cay}(S, a)$ is a classical Cayley-Dickson doubling of S with scalar $a \in S^\times$.*

The proofs are obvious.

Proposition 14. *Let $A = \operatorname{Cay}(S, P, \mu h)$ for some nondegenerate 1-hermitian form $h : P \times P \to S$, and a scalar $\mu \in S \setminus R$. If S is a domain (or if P is torsion-free and $\bar{\mu} - \mu$ not a zero divisor in S), then $N(A) = S$.*

Proof. Obviously, $S \subset N(A)$ is always true. Now let $(e, e') \in N(A)$ with $e \in S$ and $e' \in P$. We have to show that this implies $e' = 0$. The equation

$$(e, e')((u, w)(u', w')) = ((e, e')(u, w))(u', w')$$

implies that

$$e'\bar{\mu}h(w, w') = w'\mu h(w, e')$$

for all $w, w' \in P$ and thus (put $w' = e'$)

$$e'(\bar{\mu} - \mu)h(w, e') = 0$$

for all $w \in P$. If $h(w, e') = 0$ for all $w \in P$ then $e' = 0$ since h is nondegenerate and thus $(e, e') = (e, 0) \in S$. Otherwise, there is one $w \in P$ such that $h(w, e') \neq 0$. Then if S has no zero divisors, i.e., is a domain, P is torsion free and we have $e' = 0$ since $\bar{\mu} - \mu \neq 0$ here. The same holds if P is torsion free and $\bar{\mu} - \mu$ not a zero divisor in S. □

In case $2 \in R^\times$ is an invertible element, the ring R is a subalgebra of S, and $S \cong \mathrm{Cay}(R, L, \alpha)$ for a suitable selfdual element $L \in \mathrm{Pic}\, R$ and some nondegenerate R-quadratic form α on L (see for instance [P]). In particular, $S \cong R \oplus L$ as R-module. The choice of the scalar $\mu \in S$ is reflected in the residue class algebras of $A = \mathrm{Cay}(S, P, \mu h)$:

Proposition 15. *Let $2 \in R^\times$ be invertible. Assume that $S \cong \mathrm{Cay}(R, \eta)$ for some $\eta \in R^\times$. Let $\mu \in S^\times \setminus R$.*
(i) If $\mu_{(p)} \notin K(p)$ for each $p \in \mathrm{Spec}\, R$ then

$$\mu = (\mu_1, \mu_2) \in R \oplus R = S \quad \textit{with} \quad \mu_2 \in R^\times.$$

(ii) In case $S(p)$ is a non-split quadratic étale algebra over $K(p)$ for all $p \in \mathrm{Spec}\, R$,

$$\mu_{(p)} \notin K(p) \quad \textit{for each} \quad p \in \mathrm{Spec}\, R$$

if and only if

$$\mu = (\mu_1, \mu_2) \in R \oplus R = S \quad \textit{with} \quad \mu_2 \in R^\times.$$

Proof. (i) Let $\mu \in S^\times$. Write $\mu = (\mu_1, \mu_2) \in R \oplus R = S$. Then $\mu_{(p)} \notin K(p)$ for every $p \in \mathrm{Spec}\, R$ if and only if $\mu_{2,p} \notin pR_p$, for all $p \in \mathrm{Spec}\, R$ where $S(p)$ is non-split, and $\mu_{1,p} \neq \mu_{2,p}$, $\mu_{1,p} \notin pR_p$, $\mu_{2,p} \notin pR_p$ for all $p \in \mathrm{Spec}\, R$ where $S(p)$ is split. This in turn is equivalent to $\mu_{2,p} \in R_p^\times$ for all $p \in \mathrm{Spec}\, R$ where $S(p)$ is non-split and to $\mu_{1,p} \neq \mu_{2,p}$, $\mu_{1,p} \in R_p^\times$, $\mu_{2,p} \in R_p^\times$ for all $p \in \mathrm{Spec}\, R$ where $S(p)$ is split. It follows that $\mu_2 \in R^\times$.

(ii) Let S be such that $S(p)$ is a non-split quadratic étale algebra over $K(p)$ for all $p \in \mathrm{Spec}\, R$. Then $\mu_{(p)} \notin K(p)$ for every $p \in \mathrm{Spec}\, R$ if and only if $\mu_{2,p} \notin pR_p$, for all $p \in \mathrm{Spec}\, R$, which is equivalent to $\mu_{2,p} \in R_p^\times$ for all $p \in \mathrm{Spec}\, R$. This is equivalent to $\mu_2 \in R^\times$. □

Since $\mu \in S^\times$ is equivalent to $n_S(\mu) \in R^\times$, in the above setting of (ii) we have for $\mu = (\mu_1, \mu_2) \in R \oplus R = S$ that $\mu = (\mu_1, \mu_2) \in S^\times$ if and only if $\mu_1^2 - \eta\mu_2^2 \in R^\times$.

Localization does not affect the scalars:

Lemma 16. *Let R be a domain such that $2 \in R^\times$. Take $\mu \in S^\times$. Then $\mu \in S^\times \setminus R$ if and only if $\mu_p \in S_p^\times \setminus R_p$, for all $p \in \operatorname{Spec} R$.*

Proof. Let $\mu \in S^\times \setminus R$. Since $S = \operatorname{Cay}(R, L, \alpha)$ write $\mu = (\mu_1, \mu_2)$ with $\mu_1 \in R$, $0 \neq \mu_2 \in L$. (If S is non-split, the condition that $\mu \in S^\times \setminus R$ implies $0 \neq \mu_2 \in L$. If S is split, $\mu \in S^\times \setminus R$ automatically translates to $\mu_1 \neq \mu_2$ and $\mu_1\mu_2 \in R^\times$ which implies $0 \neq \mu_2 \in L$.) Consider the morphism Ψ given by the multiplication by μ_2, i.e., the sequence

$$\begin{aligned} \ker \Psi \longrightarrow R &\longrightarrow L, \\ r &\longmapsto \mu_2 r. \end{aligned}$$

Since L is projective, the sequence splits. Since R is a domain, $R \cong \ker \Psi \oplus \Psi(R)$ implies $\ker \Psi = 0$. Thus Ψ is injective, and hence so is Ψ_p. Suppose that S_p is non-split. The injectivity of Ψ_p implies that $\mu_p r_p = ((\mu_1)_p r_p, (\mu_2)_p r_p) \notin R_p$ for every $r_p \in R_p$ and every $p \in \operatorname{Spec} R$ where S_p is non-split. It follows that $\mu_p \notin R_p$ for every $p \in \operatorname{Spec} R$ where S_p is non-split. If, on the other hand, S_p is split, we get analogously that $\mu_p r_p = ((\mu_1) r_p, (\mu_2)_p r_p) \notin R_p$ for every $r \in R_p$ since $(\mu_1)_p r_p \neq (\mu_2)_p r_p$. It follows that $\mu_p \notin R_p$ for every $p \in \operatorname{Spec} R$ where S_p is split, too.

The converse is obvious. □

We take a closer look at the case where S is split.

Proposition 17. *Let S be a split quadratic étale algebra over R. Let $A \cong \operatorname{Cay}(S, P, \mu h)$ with a scalar $\mu \in S$ and a nondegenerate 1-hermitian form h.*
(i) If $\mu \in R^\times$ then there exists an invertible module $L \in \operatorname{Pic} R$ and an isomorphism such that

$$A \cong \operatorname{End}_R(R \oplus L) = \begin{pmatrix} R & L \\ L^\vee & R \end{pmatrix}$$

is an associative split quaternion algebra over R where the algebra structure of the right-hand-side is given by the ordinary matrix multiplication.
(ii) For $\mu = (\mu_1, \mu_2) \notin R$ and $\mu_i \in R^\times$ for $i = 1, 2$ (hence in particular $\mu \in S^\times$ here), there exists an invertible module $L \in \operatorname{Pic} R$ and an isomorphism

$$A \cong (\operatorname{End}_R(R \oplus L), \circ\mu) = \left(\begin{pmatrix} R & L \\ L^\vee & R \end{pmatrix}, \circ\mu\right)$$

sending $R \times R$ to the diagonal of $\mathrm{End}_R(R \oplus L)$*, where the multiplication on the module* $\mathrm{End}_R(R \oplus L)$ *is defined by*

$$\begin{pmatrix} a & s \\ \check{s} & b \end{pmatrix} \circ \mu \begin{pmatrix} c & t \\ \check{t} & d \end{pmatrix} = \begin{pmatrix} ac + \frac{\mu_1}{\mu_2}\check{t}(s) & ta + sd \\ \check{s}c + \check{t}b & \check{s}(t) + bd \end{pmatrix}$$

Proof. Let S be a split quadratic étale algebra over R. Let $A \cong \mathrm{Cay}(S, P, \mu h)$ be a Cayley-Dickson doubling where $\mu \in S$. Identify S with $R \times R$. By [Knu, V (6.2.2), p. 302], any nondegenerate 1-hermitian space (P, h) of rank one over S is hyperbolic, i.e., $(P, h) \cong (L \oplus L^\vee, \varepsilon\mathbb{H})$ for suitable $L \in \mathrm{Pic}\, R$, $\varepsilon \in R^\times$, where $\mathbb{H}$ denotes the hyperbolic hermitian form on $L \oplus L^\vee$ given by $(s, \check{s})(t, \check{t}) = \varepsilon(\langle s, \check{t}\rangle, \langle t, \check{s}\rangle)$ for $s, t \in L$, $\check{s}, \check{t} \in L^\vee$. The map $\langle , \rangle$ is the canonical pairing $L \times L^\vee \to R$ as in [P, 2.7]. (The quadratic form associated with $\mathbb{H}$ is the hyperbolic quadratic form on the R-module $L \oplus L^\vee$.) Hence $A \cong \mathrm{Cay}(S, P, \mu\varepsilon\mathbb{H}) \cong \mathrm{Cay}(S, P, \mu h)$. Put $\mu = (\mu_1, \mu_2) \in S = R \times R$.
(i) If $\mu \in R$ (which is equivalent to $\mu_1 = \mu_2$) then A is an associative algebra over R. If in addition μ is invertible, then this algebra is isomorphic to a split (associative) quaternion algebra over R, i.e., to the algebra $\mathrm{End}_R(R \oplus L)$ where the algebra structure is given by the ordinary matrix multiplication [P, 2.7].
(ii) If $\mu \notin R$ (which is equivalent to $\mu_1 \neq \mu_2$) then A is a nonassociative algebra over R. If in addition the μ_i, $i = 1, 2$ are invertible (which implies $\mu \in S^\times$), then this algebra is isomorphic to the following algebra over R: let $\mu = (\mu_1, \mu_2) \in S = R \times R$, $\mu_i \in R^\times$ for $i = 1, 2$. Consider the module $\mathrm{End}_R(L \oplus R) = \begin{pmatrix} R & L \\ L^\vee & R \end{pmatrix}$ equipped with the multiplication given by

$$\begin{pmatrix} a & s \\ \check{s} & b \end{pmatrix} \circ \begin{pmatrix} c & t \\ \check{t} & d \end{pmatrix} = \begin{pmatrix} ac + \frac{\mu_1}{\mu_2}\check{t}(s) & ta + sd \\ \check{s}c + \check{t}b & \check{s}(t) + bd \end{pmatrix}.$$

(This becomes again the ordinary matrix multiplication for $\mu_1 = \mu_2$, i.e., for $\mu \in R^\times$.) Denote this algebra by $(\mathrm{End}_R(L \oplus R), \circ\mu)$. Then

$$\varphi\colon \mathrm{Cay}(S, P, \mu n_0) \longrightarrow \left(\begin{pmatrix} R & L \\ L^\vee & R \end{pmatrix}, \circ\mu\right)$$

defined by

$$\varphi((a, b), (s, \check{s})) = \begin{pmatrix} a & \mu_1 s \\ \frac{\mu_2}{\mu_1}\check{s} & b \end{pmatrix}$$

for $a, b \in R$, $s \in L$, $\check{s} \in L^\vee$, is an isomorphism of R-algebras. □

The classical Cayley-Dickson doubling of a split algebra S is contained here as a special case: the algebra $\mathrm{Cay}(S, \mu)$ for $\mu = (\mu_1, \mu_2) \in S \setminus R$ and

$\mu_i \in R^\times$ for $i = 1, 2$ satisfies the multiplication table of [W, Theorem 4] with $\lambda = \bar{\mu}/\mu$ for the case that R is a field.

Remark 18. *Let $A \cong \mathrm{Cay}(S, P, \mu h)$ with S a non-split quadratic étale algebra over R, a nondegenerate 1-hermitian form h, and with $\mu \in S$. Let R' be a ring extension of R containing S. Then $S \otimes_R R'$ is the split quadratic étale algebra over R' and we have the canonical isomorphism $S \otimes_R R' \cong R' \times R'$ given by $b \otimes 1 \mapsto (b, \bar{b})$. If $\mu \in S$ is invertible, then so is $\mu' = (\mu, \bar{\mu})$ in $R' \times R'$ and in particular, we have $\mu, \bar{\mu} \in R'^\times$. Hence* $\mathrm{Cay}(S, P, \mu h) \otimes R' \cong \mathrm{Cay}(R' \times R', P \otimes R', \mu'(h \otimes R')) \cong (\mathrm{End}_R(R \oplus L), \circ\mu')$ *with $P \otimes R' \cong L \oplus L^\vee$ as $R' \times R'$-module, for a suitable $L \in \mathrm{Pic}\, R'$.*

For any R-algebra A which contains a quadratic étale subalgebra S, define $\widetilde{S} := \{x \in A \,|\, xs = \overline{s}x \text{ for all } s \in S\}$. Take for instance the classical quaternion algebra of Hamilton, the $\mathbb{R}$-division algebra $\mathbb{H} = \mathrm{Cay}(\mathbb{C}, -1)$. Here, $\widetilde{S} = \mathbb{C}j$ for $S = \mathbb{C}$, where $1, i, j, ij$ denotes the standard basis of $\mathbb{H}$.

$\tilde{S}$ becomes a right S-module by the action

$$\tilde{S} \times S \longrightarrow \tilde{S}, \quad (w, s) \longmapsto sw,$$

where the right-hand side is the multiplication in A. Obviously, $P \subset \widetilde{S}$ for $A = (S, P, h)$. The following lemma is needed to show that we even have equality here when $2 \in R^\times$ and R is a domain:

Lemma 19. *Let S be a quadratic étale algebra over R. Let A be an R-algebra containing S.*
(i) If S is split, then $S \cap \widetilde{S} = 0$.
(ii) Let $2 \in R^\times$ and let R be a domain, then $S \cap \widetilde{S} = 0$ also in case S is non-split.

Proof. (i) Assume first that S is a split quadratic étale algebra, i.e., that $S = R \times R$ and write $x = (r_1, r_2), s = (s_1, s_2) \in R \times R = S$. The canonical involution is given by $\overline{(s_1, s_2)} = (s_2, s_1)$. Let $x \in S$ such that $x(s - \bar{s}) = 0$ for all $s \in S$. Then $x(s - \bar{s}) = (r_1, r_2)(s_1 - s_2, s_2 - s_1) = (r_1(s_1 - s_2), -r_2(s_1 - s_2)) = 0$ for all $s \in S$. Put $s_2 = 1$, $s_1 = 2$ to obtain $r_2 = 0$, $r_1 = 0$ and thus $x = 0$.
(ii) If S is a quadratic étale algebra, then $S = \mathrm{Cay}(R, L, \alpha)$ since $2 \in R^\times$. Write $x = (r_1, r_2), s = (s_1, s_2) \in \mathrm{Cay}(R, L, \alpha)$. The canonical involution is given by $\overline{(s_1, s_2)} = (s_1, -s_2)$. Let $x \in S$ such that $x(s - \bar{s}) = 0$ for all $s \in S$. Then $x(s - \bar{s}) = (r_1, r_2)(0, 2s_2) = (\alpha(2s_2, r_2), 2s_2 r_1) = 0$ for all $s \in S$. Hence $s_2 r_1 = 0$ and $\alpha(2s_2, r_2) = 0$ for all $s_2 \in L$. If R is a domain, L is torsion

free. Therefore this implies $r_1 = 0$. Since $\alpha : L \times L \to R$ is a nondegenerate quadratic form, we also obtain $r_2 = 0$ this way. □

Corollary 20. *Let $A = (S, P, h)$. If $2 \in R^\times$ and if R is a domain, then $P = \tilde{S}$ and, in particular, $A = S \oplus \widetilde{S}$.*

The proof is straightforward and uses the above lemma.

Lemma 21. *Let A be an R-algebra which contains a quadratic étale subalgebra S. Then $(\widetilde{S \otimes_R R'}) = \widetilde{S} \otimes_R R'$, for any flat ring extension R' of R.*

Proof. The quadratic étale R-algebra S is generated by a finite set of elements Σ over R, thus $\widetilde{S} = \{x \in A | xs = \bar{s}x \text{ for all } s \in \Sigma\}$ and the quadratic étale algebra $S \otimes_R R'$ is generated by the finite set $\{s \otimes 1 | s \in \Sigma\}$. We obtain $(\widetilde{S \otimes_R R'}) = \{x \in A \otimes_R R' \mid x(s \otimes 1) = \overline{(s \otimes 1)}x \text{ for all } s \in \Sigma\}$.
For $s \in S$ define ϕ_s by $\phi_s(x) = xs - \bar{s}x$ and $\phi : A \to \prod_{s\in\Sigma} A$ by $a \mapsto (\phi_s(a))_{s\in\Sigma}$. ϕ is an R-module homomorphism such that $\ker(\phi) = \tilde{S}$. Tensoring the exact sequence $0 \to \ker(\phi) \to A \xrightarrow{\phi} \mathrm{Im}(\phi) \to 0$ by R', it follows that $\tilde{S} \otimes R' = \ker(\phi) \otimes R' = \ker(\phi \otimes 1_{R'})$. So we just have to show that $\ker(\phi \otimes 1_{R'}) = (\widetilde{S \otimes R'})$, where $\ker(\phi \otimes 1_{R'}) = \{\sum a_i \otimes r_i \in A \otimes R' \mid \sum \phi(a_i) \otimes r_i = 0\}$.
Since $\sum \phi(a_i) \otimes r_i = \sum (a_i s - \bar{s}a_i)_{s\in\Sigma} \otimes r_i = (\sum (a_i s - \bar{s}a_i) \otimes r_i)_{s\in\Sigma}$ (the last equality comes from the canonical isomorphism between $(\prod_{s\in\Sigma} A) \otimes R'$ and $\prod_{s\in\Sigma}(A \otimes R')$, which holds since Σ is finite), we have
$\ker(\phi \otimes 1_{R'}) = \{\sum a_i \otimes r_i \in A \otimes R' \mid \forall s \in \Sigma \quad \sum (a_i s - \bar{s}a_i) \otimes r_i = 0\}$.
We now prove both inclusions in the equality $\ker(\phi \otimes 1_{R'}) = (\widetilde{S \otimes R'})$:

"$\subseteq$" Let $x = \sum_i a_i \otimes r_i \in \ker(\phi \otimes 1_{R'})$ with $r_i \in R'$, $a_i \in A$, and let $s \in S$. Then

$$\begin{aligned} x(s \otimes 1) - \overline{(s \otimes 1)}x &= (\textstyle\sum_i a_i \otimes r_i)(s \otimes 1) - \overline{(s \otimes 1)}(\sum_i a_i \otimes r_i) \\ &= \textstyle\sum_i (a_i \otimes r_i)(s \otimes 1) - \sum_i \overline{(s \otimes 1)}(a_i \otimes r_i) \\ &= \textstyle\sum_i ((a_i s) \otimes r_i - (\bar{s}a_i) \otimes r_i) \\ &= \textstyle\sum_i (a_i s - \bar{s}a_i) \otimes r_i \\ &= 0. \end{aligned}$$

"$\supseteq$" Suppose $x = \sum_i a_i \otimes r_i \in A \otimes R'$ is such that $x(s \otimes 1_{R'}) = \overline{(s \otimes 1_{R'})}x$ for every $s \in \Sigma$. Then $x(s \otimes 1_{R'}) - \overline{(s \otimes 1_{R'})}x = \sum_i a_i s \otimes r_i - \sum_i \bar{s}a_i \otimes r_i = \sum_i (a_i s - \bar{s}a_i) \otimes r_i = 0$, for every $s \in S$. This proves $x \in \ker(1_{R'} \otimes \phi)$.

Corollary 22. *Let A be an R-algebra which contains a quadratic étale subalgebra S. Then $(\widetilde{S_p}) = (\widetilde{S})_p$ for all $p \in \mathrm{Spec}R$.*

Proof. Since $S_p = S \otimes_R R_p$ and R_p is flat over R, the assertion follows from the last lemma. □

Proposition 23. *Let A be an S-associative algebra over R such that $A = S \oplus \tilde{S}$ as S-module. Suppose in addition that $\tilde{S}$ is a finitely generated projective right S-module of rank one. Then there exists a sesquilinear form $h : \tilde{S} \times \tilde{S} \to S$ such that $A \cong (S, \tilde{S}, h)$.*

Proof. We know that $\tilde{S}$ becomes a right S-module by the action

$$\tilde{S} \times S \longrightarrow \tilde{S}, \quad (w, s) \longmapsto sw,$$

where the right-hand side is the multiplication in A. By assumption we also know that $\tilde{S}$ is even a finitely generated projective right S-module. Define

$$\begin{array}{rcl} h : \tilde{S} \times \tilde{S} & \to & S \\ (x, y) & \mapsto & xy \ . \end{array}$$

then h is a sesquilinear form and $A \cong (S, \tilde{S}, h)$. □

4. Automorphisms

Let S and S' be two quadratic étale algebras over R. Let both A and A' be two S-associative respectively S'-associative algebras over R of rank 4 such that $N(A) = S$ and $N(A') = S'$. Since any isomorphism preserves the nucleus, the algebras A and A' can be isomorphic only if they have - up to isomorphism - the same quadratic étale algebra as their nucleus.

Suppose that $S = S'$. Let $\mathbb{Z}_2(R)$ be the group of idempotents of R with the operation $\lambda \dot{+} \lambda' : = \lambda + \lambda' - 2\lambda\lambda'$, then $\mathrm{Aut}(S) \cong \mathbb{Z}_2(R)$. The map induced on S by an isomorphism $\Phi \colon A \xrightarrow{\sim} A'$ is an automorphism of S. If $S \cong R \times R$ is split then $\mathrm{Aut}(R \times R) \cong \mathbb{Z}_2$ [Knu, p. 128 ff].

Suppose that S is a domain. Let $A = \mathrm{Cay}(S, \mu)$ and $A' = \mathrm{Cay}(S, \mu')$ be two "classical" Cayley-Dickson doublings, with $\mu, \mu' \in S^\times \setminus R$. Then $N(A) = S = N(A')$ and both A and A' are non-split nonassociative quaternion algebras. Define $J = (0, 1) \in A$, $J' = (0, 1) \in A'$. Then $A = S \oplus SJ$, $A' = S \oplus SJ'$ as in [W]. The subspace SJ must be mapped to SJ', since it is determined by its relation to the 2-sided multiplication by S. Thus, if $\Phi(J) = \varepsilon J'$ for some $\varepsilon \in S^\times$, $\Phi(\mu) = \Phi(J^2) = (\varepsilon J')(\varepsilon J') = \varepsilon\bar{\varepsilon} J'^2 = \varepsilon\bar{\varepsilon}\mu'$ [W, Theorem 2].

Conversely, any $\varphi \in \mathrm{Aut}(S)$ and any $\varepsilon \in S^\times$ such that $\varphi(\mu) = \varepsilon\bar{\varepsilon}\mu'$, gives a (unique) isomorphism

$$\begin{aligned} \Phi\colon S \oplus SJ &\longrightarrow S \oplus SJ', \\ u + vJ &\longmapsto \varphi(u) + \varphi(v)\varepsilon J'. \end{aligned}$$

In this more general setting, [W, Theorem 2] becomes

Theorem 24. *Let S be a domain. Suppose $A = \mathrm{Cay}(S, \mu)$, $A' = \mathrm{Cay}(S, \mu')$ are two algebras over R which are classical Cayley-Dickson doublings of S with invertible scalars $\mu, \mu' \in S$. Then $A \cong A'$ if and only if $\varphi(\mu) = \varepsilon\bar{\varepsilon}\mu'$ for some $\varphi \in \mathrm{Aut}(S)$, and some $\varepsilon \in S^\times$.*

Every $\varphi \in \mathrm{Aut}(S)$ such that there exists $\varepsilon \in S^\times$ with $\varphi(\mu) = \varepsilon\bar{\varepsilon}\mu'$ induces an isomorphism

$$\Phi\colon A \longrightarrow A', \quad \Phi((u, v)) = (\varphi(u), \varepsilon\varphi(v)).$$

Note that for R connected (or R local), $\mathrm{Aut}(S) = \{\mathrm{id}, \sigma_S\}$ [K, p.128, p. 301]. However, in general $\mathrm{Aut}(S)$ may contain more that these two maps.

Corollary 25. *Let $A = \mathrm{Cay}(S, \mu)$ be a classical Cayley-Dickson doubling of a domain S as in Theorem 24. Let $\varphi \in \mathrm{Aut}(S)$. Then:*

- *For all φ such that $\varphi(\mu) = \mu$, the map $\Phi((u, v)) = (\varphi(u), \varphi(v)\varepsilon)$ is an element of $\mathrm{Aut}(A)$, for any $\varepsilon \in S^\times$ satisfying $\varepsilon\bar{\varepsilon} = n_{S/R}(\varepsilon) = 1$.*
- *If there exists $\varepsilon \in S^\times$ s.t. $-1 = \varepsilon\bar{\varepsilon} = n_{S/R}(\varepsilon)$, then for all φ such that $\varphi(\mu) = \bar{\mu}$, the map $\Phi((u, v)) = (\varphi(u), \varphi(v)\varepsilon)$ is an element of $\mathrm{Aut}(A)$.*
- *If $\mu \equiv \varphi(\mu) \bmod n_{S/R}(S^\times)$, and $\varphi(\mu) \neq \mu, \bar{\mu}$, then the map $\Phi((u, v)) = (\varphi(u), \varphi(v)\varepsilon)$ is an element of $\mathrm{Aut}(A)$, for any $\varepsilon \in S^\times$ with $\mu = \varepsilon\bar{\varepsilon}\varphi(\mu)$.*

These are all the automorphisms of A.

Now let $A = \mathrm{Cay}(S, P, \mu h)$ and $A' = \mathrm{Cay}(S, P', \mu' h')$ be two S-associative algebras with h and h' nondegenerate 1-hermitian forms and with $\mu, \mu' \in S^\times \setminus R$. We still assume that S is a domain. Take an algebra isomorphism $\Phi\colon A \xrightarrow{\sim} A'$, then again $\varphi\colon = \Phi|_S \in \mathrm{Aut}(S)$, and $\Phi|_P\colon P \xrightarrow{\sim} P'$ is an R-module isomorphism (which is easy to check locally). Furthermore, for all $w \in P$, $s \in S$,

$$\Phi(w \cdot s) = \Phi(w)\varphi(s).$$

If $\varphi = \mathrm{id}$ or $\varphi = \sigma$ then $\Phi|_P$ is S-linear or σ-semilinear, thus $(P, n) \cong (P', \lambda n')$ for some $\lambda \in R^\times$ if $n(w) = h(w, w)$ denotes a norm on P, and

$n'(w)\colon = h'(w',w')$ a norm on P' in the sense of [P]. Moreover, $P' \cong P$ or $P' \cong \overline{P}$ as S-modules, where $\overline{P}$ is P with the action of S twisted through σ. If R is connected, then also $(P,h) \cong (P',\lambda h')$ or $(P,h) \cong (\overline{P'},\overline{\lambda h'})$ holds [Knu, V. (6.1.3), (6.1.4), p. 301]. Since Φ is multiplicative,

$$\Phi((0,v)(0,v')) = \Phi(\mu h(v',v),0) = (\varphi(\mu)\varphi h(v',v),0)$$

and

$$\Phi((0,v))\Phi((0,v')) = (\mu' h'(\Phi(v'),\Phi(v)),0)$$

imply that

$$\varphi(\mu)\varphi h(v',v) = \mu' h'(\Phi(v'),\Phi(v)).$$

For $\varphi = \mathrm{id}$ or $\varphi = \sigma$ this is equivalent to

$$\begin{array}{rcll} \mu h(v',v) & = & \mu' h'(\Phi(v'),\Phi(v)) & (\text{if } \varphi = \mathrm{id}), \text{or to} \\ \overline{\mu}\overline{h}(v',v) & = & \mu' h'(\Phi(v'),\Phi(v)) & (\text{if } \varphi = \sigma), \end{array}$$

with $\overline{h}(v',v)\colon = h(v',v)$ the hermitian form on $\overline{P}$. We observe that μh is an ε-hermitian form on P, with $\varepsilon = \frac{\overline{\mu}}{\mu}$, and $\mu' h'$ an ε'-hermitian form, with $\varepsilon' = \frac{\overline{\mu'}}{\mu'}$. If R is connected we know from above that $(P,h) \cong (P',\lambda h')$ if $\varphi = \mathrm{id}$, and $(P,h) \cong (\overline{P'},\overline{\lambda h'})$ if $\varphi = \sigma$. Hence $\mu' = \mu\lambda$ and μh and $\mu' h'$ are isometric ε-hermitian forms on P. Analogously, if $\varphi = \sigma$ then $\overline{\mu}\overline{h}$ and $\mu' h'$ are isometric ε'-hermitian forms on P. We have proved

Theorem 26. *Let S be a domain. Let $A = \mathrm{Cay}(S,P,\mu h)$ and $A' = \mathrm{Cay}(S,P',\mu' h')$ be two nonassociative quaternion algebras over a connected ring R with h and h' nondegenerate 1-hermitian forms and with $\mu,\mu' \in S^\times \setminus R$. If A and A' are isomorphic algebras then*

$(P,\mu h) \cong (P',\mu' h')$ *are isometric ε-hermitian forms in case* $\Phi|_S = \mathrm{id}$, *and*
$(\overline{P},\overline{\mu}\overline{h}) \cong (P',\mu' h')$ *are isometric ε-hermitian forms in case* $\Phi|_S = \sigma$.

5. Examples

Let k be a field of characteristic not 2.

1) Let R be the polynomial ring $k[t]$. By [P, 6.8], any quadratic étale algebra S and any quaternion algebra over R are defined over k. Therefore either S is a domain and we have $S = S_0 \otimes_k R$, where S_0 is a separable quadratic field extension of k, or $S = R \times R$ is split. Since R is a principal ideal domain, $\mathrm{Pic}(R) = 0$. Thus every nondegenerate ε-hermitian form can be obtained from a nondegenerate 1-hermitian form by multiplication with a suitable scalar $\mu \in S^\times$.

Every split nonassociative quaternion algebra A is isomorphic to an algebra of the kind $\mathrm{Cay}(S,\mu)$ for some invertible $\mu \in S \setminus R$.

If $S = k(\sqrt{c}) \otimes_k R$ with $k(\sqrt{c})$ a separable quadratic field extension of k, then $S \cong k(\sqrt{c})[t]$ is also a principal ideal domain, and we conclude that $\mathrm{Pic}\, S = 0$. Therefore every non-split quaternion algebra A is isomorphic to $\mathrm{Cay}(k(\sqrt{c}) \otimes_k R, \mu)$ with a suitable element $\mu \in k(\sqrt{c})^\times \setminus k$, and $N(A) = S$.

2) Let $R = k[t, \frac{1}{t}]$ be the ring of Laurent polynomials. Since this is again a principal ideal domain, every nondegenerate ε-hermitian form can be obtained from a nondegenerate 1-hermitian form by multiplication with a suitable scalar $\mu \in S^\times$. Moreover, any quadratic étale algebra over R and any quaternion algebra over R with zero divisors splits, the latter being isomorphic to $\mathrm{Mat}_2(R)$. This implies that any split nonassociative quaternion algebra A over R is isomorphic to $\mathrm{Cay}(S,\eta)$ with $S = R \times R$, $\eta \in S^\times \setminus R$.

A non-split quadratic étale algebra S is a domain and either isomorphic to $k(\sqrt{c}) \otimes_k R$ or to $\mathrm{Cay}(R, \mu t)$ with $\mu \in k^\times$.

In case $S = k(\sqrt{c}) \otimes_k R \cong k(\sqrt{c})[t, \frac{1}{t}]$, $\mathrm{Pic}\, S$ is trivial and $A \cong \mathrm{Cay}(S,\eta)$ with $\eta \in S^\times \setminus R$, that is $\eta = \eta_1 t^j + \sqrt{c}\eta_2 t^j$ with $\eta_1, \eta_2 \in k, \eta_2 \neq 0$. Each such algebra has S as its nucleus.

If $S = \mathrm{Cay}(R, \mu t)$ then a (classical) quaternion algebra C over R containing S as a subalgebra is isomorphic to $C = \mathrm{Cay}(S, P, n_P) \cong \mathrm{Cay}(k(\sqrt{c}) \otimes_k R, \lambda t)$, $\lambda \in k^\times$, provided that C is not defined over k. Since $n_P \cong \langle \alpha_1 t^{\varepsilon_1}, \alpha_2 t^{\varepsilon_2} \rangle$ by [Kn, 13.4.4] with $\varepsilon_i \in \{0,1\}$, $\alpha_i \in k^\times$, the norm of C is given by $\langle 1, -c, -\lambda t, \lambda c t\rangle \cong \langle 1, -\mu t, -\alpha_1 t^{\varepsilon_1}, -\alpha_2 t^{\varepsilon_2}\rangle$ with $\varepsilon_1 + \varepsilon_2 = 1$, $-\alpha_2 \equiv \mu\alpha_1 t^{1+\varepsilon_1+\varepsilon}$. Moreover, $k(\sqrt{c}) \otimes R \cong k(\sqrt{\alpha_1}) \otimes R$, since the quadratic étale algebra S contained in C is unique up to isomorphism [Pu1, 3.10]. It follows that $n_P \cong \langle \alpha_1, -\mu\alpha_1 t\rangle \cong \alpha_1 \langle 1, -\mu t\rangle$. By [Knu, III.(7.3.3)] this implies that P and S are isomorphic S-modules. Thus every nonassociative quaternion algebra A with $S = \mathrm{Cay}(R, \mu t)$ as its nucleus is isomorphic to $\mathrm{Cay}(S,\eta)$ for a suitable $\eta \in S^\times \setminus R$, that is, $\eta = (\eta_1, \varepsilon t^j) \in S = R \oplus R$ for some $\varepsilon \in k^\times$.

3) Let $\langle 1, -a, -b\rangle_k$ be an anisotropic quadratic form. Take R to be the ring $k[t, \sqrt{at^2+b}]$. R is a quadratic extension of $k[t]$ and $\mathrm{Quot}(R) = k(t, \sqrt{at^2+b})$. The units of R are exactly the units of k, i.e., $R^\times = k^\times$. R is a principal ideal domain [Pf, Proposition 1]. Thus every nondegenerate ε-hermitian form can be obtained from a nondegenerate 1-hermitian form by multiplication with a suitable scalar $\mu \in S^\times$. Furthermore, every quadratic étale algebra over R containing zero divisors is isomorphic to $R \times R$, and any quaternion algebra over R with zero divisors is isomorphic to $\mathrm{Mat}_2(R)$.

Every nondegenerate quadratic form over R can be viewed as an unimodular R-lattice in the sense of [Pf]. We choose k as well as $a, b \in k^\times$ in such a way that there exist orthogonally indecomposable binary unimodular R-lattices. In particular, this implies the existence of splitting fields $k(\sqrt{c})$ of $(a,b)_k$ not isomorphic to $k(\sqrt{a})$. If the choice of k and $a, b \in k^\times$ does not permit such lattices over R (e.g., $k = \mathbb{R}$, $a = b = 1$) it is obvious from [Pu2, 2.4] that every quaternion algebra over R is already defined over k.

Results from Kneser [Kn] show that every quadratic étale algebra over R is defined over k. Every quaternion algebra over R which is not defined over k is isomorphic to $\mathrm{Cay}(k(\sqrt{c}) \otimes_k R, I_c, L(\alpha))$ with $\det L(\alpha) = -c$ by [Pu2, 2.4].

In particular, $\mathrm{Pic}\,(k(\sqrt{c}) \otimes_k R)$ is trivial if $k(\sqrt{c})$ is not a splitting field of $(a,b)_k$, or if $k(\sqrt{c}) \cong k(\sqrt{a})$, otherwise it contains only one non-trivial element, denoted by I_c. For the definition of the indecomposable binary R-lattice $L(\alpha)$ the reader is referred to [Pf]. These results can again be used to list all nonassociative quaternion algebras A over R.

A split nonassociative quaternion algebra A is isomorphic to an algebra of the kind $\mathrm{Cay}(S, \eta)$ for some invertible $\eta \in S = R \times R, \eta \notin R$.

If a non-split nonassociative quaternion algebra A has $S = k(\sqrt{c}) \otimes_k R$ as nucleus, we know that $\mathrm{Pic}\, S = 0$ if $k(\sqrt{c})$ is not a splitting field of $(a,b)_k$, or if $k(\sqrt{c}) \cong k(\sqrt{a})$. In this case, the algebra is isomorphic to $\mathrm{Cay}(S, \eta)$ for some $\eta \in S^\times \setminus R$.

If $k(\sqrt{c})$ is a quadratic splitting field of $(a,b)_k$ which is not isomorphic to $k(\sqrt{a})$, a nonassociative quaternion algebra A with nucleus $S = k(\sqrt{c}) \otimes_k R$ is either isomorphic to the classical Cayley-Dickson doubling $\mathrm{Cay}(S, \eta)$ with an invertible $\eta \in S \setminus R$, or it is isomorphic to $\mathrm{Cay}(S, I_c, \eta L(\alpha))$ where $L(\alpha) = \langle \begin{pmatrix} \alpha & \beta \\ \beta & \gamma \end{pmatrix} \rangle$ is an orthogonally indecomposable binary R-lattice with $\det L(\alpha) = -c$, and $\eta \in S^\times \setminus R$ is a suitable element. Note that in the second case, $S \cong k(\sqrt{c})[x_0, x_1]_{(x_0^2 - a x_1^2)}$, with x_0, x_1 indeterminates over $k(\sqrt{c})$.

4) Every point $P_0 \in \mathbb{P}^1_k = \mathrm{Proj}\, k[x_0, x_1]$ of degree two is represented by the principal ideal generated by a monic polynomial $f(t) = t^2 - a \in k[t]$ of degree two. Removing P_0 from $\mathbb{P}^1_k$ results in the affine scheme $\mathrm{Spec}\, R = \mathbb{P}^1_k - \{P_0\}$ of a ring R, with

$$\begin{aligned} R &= \{g(t)/f(t)^j \in k(t) \mid j \geq 0,\ y(t) \in k[t] \text{ with } \deg g \leq 2j\} \\ &\cong k[x_0, x_1]_{(x_0^2 - a x_1^2)}. \end{aligned}$$

Let R be this ring of f-fractions. R is a Dedekind domain with $\mathrm{Quot}(R) = k(t)$, the set of invertible elements of R is given by $k^\times$. The ideal class group of R, which is isomorphic to $\mathrm{Pic}\, R$ here, contains two elements, i.e., is isomorphic to $\mathbb{Z}_2$.

In this case we are not able to say that each ε-hermitian form can be obtained by scaling a 1-hermitian form, hence we are only able to describe all those nonassociative quaternion algebras which arise by taking a 1-hermitian form and scaling it with an invertible element $\mu \in S$. There still might be others.

Let L denote the non-trivial element in $\mathrm{Pic}\, R$, then L is isomorphic to the ideal $\left(\frac{1}{(t^2-a)}, \frac{t}{(t^2-a)}\right)$ in R as an R-module [Pu3, 2.1]. L has norm one, and $N_0\colon L \to R$ given by

$$\begin{aligned} &N_0\left(\tfrac{1}{t^2-a}\right) = \tfrac{1}{t^2-a},\ N_0\left(\tfrac{t}{t^2-a}\right) = \tfrac{t^2}{t^2-a},\\ &N_0\left(\tfrac{1}{t^2-a}, \tfrac{t}{t^2-a}\right) = \tfrac{2t}{t^2-a} \end{aligned}$$

is a norm on L satisfying $N_0 \otimes_k k(t) \cong \langle t^2 - a\rangle_{k(t)}$. Every quadratic étale algebra over R is either defined over k, or isomorphic to $\mathrm{Cay}(R, L, \mu N_0)$ for a suitably $\mu \in k^\times$ [Pu3, 2.5].

If $S = k(\sqrt{c}) \otimes_k R$ is a non-split quadratic étale algebra defined over k, then $\mathrm{Pic}\, S$ is trivial in case $k(\sqrt{c}) \cong k(\sqrt{a})$, and isomorphic to $\mathbb{Z}_2$ otherwise. In the latter case let E_c be the nontrivial element in this Picard group, then E_c is isomorphic to the ideal $\left(\frac{1}{t^2-a}, \frac{t}{t^2-a}\right)$ in S as an S-module [Pu3, 2.6]. E_c has norm one, $E_c = L \oplus L$ as an R-module, and $N(c)\colon E_c \to R$ given by $N(c)\colon = N_0 \oplus (-c)N_0$ is a norm on E_c by [Pu3, 2.7].

Thus we can list all split and non-split nonassociative quaternion algebras A over R where the ε-hermitian forms used in their construction are obtained by scaling a 1-hermitian form with some $\mu \in S^\times$: every such split nonassociative quaternion algebra A is isomorphic to an algebra of the kind $\mathrm{Cay}(S, \eta)$ or $\mathrm{Cay}(S, L \oplus L^\vee, \eta h_0)$ with an invertible $\eta \in S \setminus R$ and $S = R \times R$.

Let A be non-split with nucleus containing $S = k(\sqrt{a}) \otimes_k R$. Here,

$$S = k(\sqrt{a})\left[\frac{t+\sqrt{a}}{t-\sqrt{a}}, \frac{t-\sqrt{a}}{t+\sqrt{a}}\right]$$

is a ring of Laurent-polynomials over $k(\sqrt{a})$. It follows that $A \cong \mathrm{Cay}(S, \eta)$ with a suitable $\eta \in S \setminus R$ invertible.

Let A be non-split with nucleus containing $S = k(\sqrt{c}) \otimes_k R$, where $k(\sqrt{c})$ is not isomorphic to $k(\sqrt{a})$. Then $A \cong \mathrm{Cay}(S, \eta)$ or $A \cong \mathrm{Cay}(S, E_c, \eta N(c))$ with $\eta \in S^\times \setminus R$.

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