MODEL COMPLETENESS AND QUANTIFIER ELIMINATION FOR (ORDERED) CENTRAL SIMPLE ALGEBRAS WITH INVOLUTION

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ABSTRACT. We show that the theories of some (ordered) central simple algebras with involution over real closed fields are model-complete or admit quantifier elimination, and characterize positive cones in terms of morphisms in models of some of these theories.

CONTENTS

1. INTRODUCTION AND PRELIMINARIES

In a series of papers ([1, 2, 3, 4, 5]) the author and T. Unger investigated some properties of central simple algebras with involution that are linked to orderings on the base field and have strong similarities to classical notions in real algebra: Signatures of hermitian forms, "ideals" in the Witt group, "orderings" (positive cones) and valuations (gauges) on the central simple algebra with involution.

It is therefore natural to wonder if some model-theoretic properties, similar to the ones of ordered fields, could also be found in (ordered) central simple algebras with involution. This paper is a first investigation in this direction.

1.1. **Algebras with involution.** All fields in this paper will have characteristic different from 2. Our main reference for central simple algebras with involutions is [9], and we simply recall what will be needed in the paper.

By central simple algebra over a field K we mean an K-algebra A with 1 that is finite-dimensional over K and such that $K = Z(A)$. Such an algebra is isomorphic to a matrix algebra $M_\ell(D)$, for a unique $\ell \in \mathbb{N}$ and a K-division algebra D that is unique up to K-isomorphism. A splitting field L of A is an extension L of K such that $A \otimes_K L \cong M_m(L)$ for some m. Such a splitting field always exists (for instance the algebraic closure of K), and $\deg A := m = \sqrt{\dim_K A}$ is called the degree of A.

If A is a ring and σ is an involution on A, we denote by

$$\mathrm{Sym}(A,\sigma) := \{a \in A \mid \sigma(a) = a\}$$

the set of symmetric elements of A.

In this paper, F will always denote a field. By a central simple algebra with involution over F we mean a pair (A,σ) where A is a finite-dimensional F-algebra with 1, whose centre $Z(A)$ is a field, and where σ is an involution on A such that $F = Z(A) \cap \mathrm{Sym}(A,\sigma)$. Note that σ is then F-linear, and that $[Z(A) : F] \le 2$. We call F the base field of (A,σ).

The involution σ is said to be of the first kind if $F = Z(A)$, and of the second kind if $[Z(A) : F] = 2$. A finer classification of involutions is given by their type:

Involutions of the first kind can have two types, described as follows (with the notation $m := \deg A$): orthogonal if $\dim_F \mathrm{Sym}(A,\sigma) = \dfrac{m(m+1)}{2}$, or symplectic if $\dim_F \mathrm{Sym}(A,\sigma) = \dfrac{m(m-1)}{2}$), cf. [9, Proposition 2.6]. Involutions of the second kind are also called of unitary type.

Recall that, by the Skolem-Noether theorem:

Proposition 1.1 ([9, Propositions 2.7 and 2.18])**.** *If σ and γ are two F-linear involutions on A and are of the same kind, then there is $a \in A^\times$ such that $\sigma = \mathrm{Int}(a) \circ \gamma$.*

We will be particularly interested in central simple algebras with involution whose base field F is real closed. As recalled above, they are (up to isomorphism) of the form $M_\ell(D)$ where D is a finite-dimensional division algebra over F. Since F is real closed D is one of F, $F(\sqrt{-1})$, or $(-1,-1)_F$ (where $(-1,-1)_F$ denotes the quaternion algebra over F with usual basis $\{1, i, j, k\}$ such that $i^2 = j^2 = -1$ and $ij = -ji = k$). We will denote the canonical F-linear involutions on $F(\sqrt{-1})$ and $(-1,-1)_F$ by $-$ in both cases.

Remark 1.2. Recall that, for a field F:

- $F(\sqrt{-1})$ is a field if and only if $a^2 + b^2 = 0$ implies $a = b = 0$ for every $a, b \in F$, if and only if the quadratic form $\langle 1, 1\rangle$ is anisotropic.
- $(-1,-1)_F$ is a division algebra if and only if $a^2 + b^2 + c^2 + d^2 = 0$ implies $a = b = c = d = 0$ for every $a, b, c, d \in F$, if and only if the quadratic form $\langle 1, 1, 1, 1\rangle$ is anisotropic.

Let F be a field such that $(D, \vartheta) \in \{(F, \mathrm{id}), (F(\sqrt{-1}), -), ((-1,-1)_F, -)\}$ is a division algebra with involution. Let $n \in \mathbb{N}$. The involution ϑ^t on $M_n(D)$ is

$$\begin{cases} \text{orthogonal} & \text{if } (D,\vartheta) = (F, \mathrm{id}) \\ \text{symplectic} & \text{if } (D,\vartheta) = ((-1,-1)_F, -) \\ \text{unitary} & \text{if } (D,\vartheta) = (F(\sqrt{-1}), -). \end{cases}$$

In any of these three situations, $\mathrm{PSD}(M_n(D), \vartheta^t)$ will denote the set of symmetric positive-definite matrices. We will often simply write PSD if the algebra is clear from the context.

1.2. **Model-theoretic notation.** We will use the standard notation for L-structures (see for instance [12, p 8]), but will not distinguish between a structure and its base set: If $\mathscr{M}$ is an L-structure, we will denote the base set of $\mathscr{M}$ also by $\mathscr{M}$.

If S is a symbol in a language L and $\mathscr{M}$ is an L-structure, $S^{\mathscr{M}}$ will denote the interpretation of S in $\mathscr{M}$.

We will work with algebras with involution, and will be interested in various maps and relations that are naturally considered in this context, for instance the involution (often denoted σ), the base field (often denoted F), the reduced trace map (denoted Trd) etc. Our languages will contain symbols that will be interpreted by such an involution, field, reduced trace map... In order to make them easily recognizable, we will use the same symbols in the language, but underlined (so: $\underline{\sigma}$, $\underline{F}$, $\underline{\mathrm{Trd}}$, etc).

Frequently, an algebra with involution (A, σ) will be model of a theory whose language contains more symbols than just σ. We will still denote it by (A, σ), or even by A, when no confusion will seem likely to arise.

In this paper, "formula" means first-order formula and "theory" means first-order theory.

1.3. **Axiomatization of finite-dimensional central simple algebras.**

Lemma 1.3. *Let F be a field and let A be a finite-dimensional F-algebra whose centre is a field. Then A is a central simple algebra over its centre if and only if A is von Neumann regular.*

Proof. Assume that A is a central simple algebra. We know that $A \cong M_n(D)$ for some $n \in \mathbb{N}$ and some $Z(A)$-division algebra D. Then A is semisimple, and thus von Neumann regular (see for instance [10, Corollary 4.24]).

Conversely, let A be von Neumann regular. By [10, Theorem 4.25] A is semisimple (it is Noetherian since $\dim_F A < \infty$), so is a finite product of matrix rings over division rings ([10, Theorem 3.5]). Since its centre is a field and thus not a product of more than one field, A is a single matrix ring over a division ring, so is central simple. □

For $m \in \mathbb{N}$ we define the following two theories:

$$\begin{aligned} \mathrm{CSA}_m := {} & \text{the theory of von Neumann regular rings} \\ & \cup \{\text{the centre is a field}\} \\ & \cup \{\text{the dimension over the centre is } m\}, \end{aligned}$$

in the language L_R of rings and, in the language $L_{\text{CSA-I}} := L_R \cup \{\underline{F}, \underline{\sigma}\}$:

$$\begin{aligned} \text{CSA-I}_m := {} & \text{the theory of von Neumann regular rings} \\ & \cup \{\text{the centre is a field, } \underline{\sigma} \text{ is an involution}\} \\ & \cup \{\underline{F} \text{ is the field of all symmetric elements in the centre}\} \\ & \cup \{\text{the dimension over } \underline{F} \text{ is } m\} \end{aligned}$$

(CSA stands for central simple algebra and CSA-I for central simple algebra with involution). Lemma 1.3 immediately gives:

Corollary 1.4. *(1) The models of CSA_m are exactly the central simple algebras of dimension m over their centres.*

(2) The models of CSA-I_m are exactly the central simple algebras with involution over F of dimension m over F (where F denotes the interpretation of $\underline{F}$ in the model).

1.4. **The reduced trace, the $\star$ operation, and words of matrices.** We will consider two different traces.

(1) The usual matrix trace on $M_n(D)$, where D is a division algebra:

$$\operatorname{tr}((a_{ij})_{i,j=1,\ldots,n}) = \sum_{i=1}^{n} a_{ij}.$$

(2) The reduced trace, Trd_A, where A is a central simple algebra (see [9, Section 1A]; we will often simply write Trd instead of Trd_A if the algebra is clear from the context).

If A is a central simple algebra over $K := Z(A)$, the reduced trace is a K-linear map from A to K. It is obtained by extending the scalars to a splitting field L of A, (i.e., $K \subseteq L$ and there is an isomorphism of L-algebras $f : A \otimes_K L \to M_m(L)$ for some m), and then by taking the usual trace in $M_m(L)$. It can be shown that the result does not depend on the choice of L or of the isomorphism f (in particular the reduced trace is invariant under isomorphisms of F-algebras). So, for $a \in A$:

$$\operatorname{Trd}_A(a) := \operatorname{tr}(f(a \otimes 1)).$$

Remark 1.5. These two traces produce different results for a central simple algebra of the form $M_\ell(D)$ where D is a division algebra that is not a field (the reduced trace will have values in $Z(M_\ell(D)) = Z(D)$), but are equal if D is a field since there is no need to extend scalars to split the algebra.

Lemma 1.6. *Let A be a central simple algebra over K and let $f : A \to K$ be K-linear such that $f(xy) = f(yx)$ for every $x, y \in A$ and $f(1) = \deg A$. Then $f = \operatorname{Trd}_A$.*

Proof. Let L be a splitting field of A. A direct verification shows that $(f \otimes \mathrm{id})(xy) = (f \otimes \mathrm{id})(yx)$ for every $x, y \in A \otimes_K L \cong M_m(L)$ (for some $m \in \mathbb{N}$), and we still have $(f \otimes \mathrm{id})(1) = \deg A = \deg M_m(L)$. Therefore, by definition of the reduced trace, it suffices to show that $f \otimes \mathrm{id}$ is the reduced trace on $M_n(L)$, i.e., it suffices to show the result for $A = M_n(L)$.

For $r, s \in \{1, \ldots, m\}$, let E_{rs} be the matrix with 1 at entry (r, s) and 0 elsewhere. We have, for $r \neq s$:

$$f(E_{rs}) = f(E_{rr}E_{rs}) = f(E_{rs}E_{rr}) = f(0) = 0.$$

Furthermore, $f(xyx^{-1}) = f(y)$ by hypothesis, for every $y \in M_m(L)$ and $x \in M_m(L)^\times$. Therefore, if $P_{r,s}$ is the permutation matrix corresponding to the transposition $(r\ s)$, we have $P_{r,s}E_{rr}{P_{r,s}}^{-1} = E_{ss}$, so that

$$f(E_{ss}) = f(P_{r,s}E_{rr}{P_{r,s}}^{-1}) = f(E_{rr}).$$

In particular $f(I_m) = \sum_{r=1}^{m} E_{rr} = mE_{ss}$ for any $s \in \{1, \ldots, m\}$.

Since $f(I_m) = m = \sum_{r=1}^{m} f(E_{rr})$, it follows that $f(E_{rr}) = 1$ for every r, proving that $f = \operatorname{Trd}$ on $\{E_{rs}\}_{r,s=1,\ldots,m}$, which is a basis of $M_m(L)$ over L. □

Lemma 1.7. *Let A and B be two F-algebras such that A is central simple over $Z(A)$ and there is an isomorphism of F-algebras $f : A \to B$. Then, for every $a \in A$, $f(\operatorname{Trd}_A(a)) = \operatorname{Trd}_B(f(a))$.*

Proof. Let $f_0 := f \restriction Z(A) : Z(A) \to Z(B)$. Using the action of $Z(A)$ on $Z(B)$ induced by f_0, we can build the map $g : A' := A \otimes_{Z(A)} Z(B) \to B$, $a \otimes z \mapsto f(a)z$, which is an isomorphism of $Z(B)$-algebras (it is injective since A' is simple, and

$\dim_{Z(B)} A' = \dim_{Z(A)} A = \dim_{Z(B)} B)$. In particular, if $a \in A$ then $\mathrm{Trd}_{A'}(a \otimes 1) = \mathrm{Trd}_B(f(a))$.

We now consider the morphism of rings $f_0 : Z(A) \to Z(B)$. By [16, Theorem 4.3 e)] we have $f_0(\mathrm{Trd}_A(a)) = \mathrm{Trd}_{A'}(a \otimes 1)$. The result follows. □

We consider more closely the case of the reduced trace on $M_n(D)$, where $D = (-1,-1)_F$ is a quaternion division algebra over F. A splitting extension of D is given by $L := F(\sqrt{-1})$, and the map

$$f_0 : D \otimes_F L \to M_2(L),$$

$$i \otimes 1 \mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},\ j \otimes 1 \mapsto \begin{pmatrix} 0 & \sqrt{-1} \\ \sqrt{-1} & 0 \end{pmatrix},\ k \otimes 1 \mapsto \begin{pmatrix} \sqrt{-1} & 0 \\ 0 & -\sqrt{-1} \end{pmatrix}$$

is an isomorphism of L-algebras. It induces an isomorphism $f : M_\ell((-1,-1)_F) \to M_{2\ell}(L)$ such that, if $a = (a_{rs})$, then $f(a) = (f_0(a_{rs}))$. Therefore,

$$\mathrm{Trd}(a) = \sum_{r=1}^{\ell} \mathrm{tr}(f_0(a_{rr})).$$

Writing $a_{rr} = u_{r,1} + i u_{r,2} + j u_{r,3} + k u_{r,4}$ with $u_{r,1}, \ldots, u_{r,4} \in F$, we have

$$\begin{aligned} \mathrm{tr}(f_0(a_{rr})) &= u_{r,1}\,\mathrm{tr}(I_2) + u_{r,2}\,\mathrm{tr}(f(i)) + u_{r,3}\,\mathrm{tr}(f(j)) + u_{r,4}\,\mathrm{tr}(f(k)) \\ &= 2u_{r,1}. \end{aligned}$$

Thus:

$$\mathrm{Trd}(a) = 2\sum_{i=1}^{n} u_{r,1} = 2\,\mathrm{Re}(\mathrm{tr}(a)) = 2\,\mathrm{tr}(\mathrm{Re}(a)). \tag{1.1}$$

We now introduce the $*$ operation, following [11] and [19] (we use Wiegmann's notation $*$ from [19] since we will mostly refer to this paper; Lee denotes it by the function f): Let F be a field such that $(-1,-1)_F$ is a division algebra, and let $M \in M_n((-1,-1)_F)$, written as $M = M_1 + jM_2$ where $M_1, M_2 \in M_n(F(\sqrt{-1}))$. We define:

$$M^* := \begin{pmatrix} M_1 & -\overline{M_2} \\ M_2 & \overline{M_1} \end{pmatrix}$$

From [11, Section 4], we have

Proposition 1.8. *The map $X \mapsto X^*$ is an injective morphism of rings with involution from $(M_n((-1,-1)_F), -^t)$ to $(M_{2n}(F(\sqrt{-1})), -^t)$.*

Considering $a \in M_n((-1,-1)_F)$ written as $a = a_1 + ja_2$ with $a_1, a_2 \in M_n(F(\sqrt{-1}))$, a direct computation shows that

$$\mathrm{tr}(a^*) = \mathrm{tr}(a_1) + \mathrm{tr}(\overline{a_1}) = 2\,\mathrm{tr}(\mathrm{Re}(a)).$$

Putting this together with (1.1), we obtain:

Lemma 1.9. *Let F be a field such that $(-1,-1)_F$ is a division algebra, and let $a \in M_n((-1,-1)_F)$. Then*

$$\mathrm{Trd}(a) = \mathrm{tr}(a^*).$$

We use these observations to reformulate in a uniform way some results from several authors (see the proof for the references) on unitary similarity of tuples of matrices. These results are already recalled in [8, Theorem 2.2.2] for the real and complex cases.

Theorem 1.10. *Let $(D,\vartheta) \in \{(F,\mathrm{id}),(F(\sqrt{-1}),-),((-1,-1)_F,-)\}$ where F is a real closed field, and let $d \in \mathbb{N}$. Then the following are equivalent, for any $X,Y \in M_n(D)^d$:*

1. *There is $O \in M_n(D)$ such that $\vartheta(O)^t O = I_n$ and $\vartheta(O)^t X_i O = Y_i$ for $i = 1,\dots,d$.*
2. *For every word w in $x_1,\dots,x_d,\vartheta(x_1)^t,\dots,\vartheta(x_d)^t$ we have $\mathrm{Trd}(w(X,\vartheta(X)^t)) = \mathrm{Trd}(w(Y,\vartheta(Y)^t))$.*
3. *For every word w in $x_1,\dots,x_d,\vartheta(x_1)^t,\dots,\vartheta(x_d)^t$ of length at most n^2, we have $\mathrm{Trd}(w(X,\vartheta(X)^t)) = \mathrm{Trd}(w(Y,\vartheta(Y)^t))$.*

Proof. We first assume that $F = \mathbb{R}$. Recall from Remark 1.5 that $\mathrm{Trd}_{M_n(D)} = \mathrm{tr}$ when $D = \mathbb{R}$ or $\mathbb{C}$. The equivalence of (1) and (2) is [19, Theorem 4] for $D = \mathbb{C}$, and [18, Lemma 2] for $D = \mathbb{R}$. The degree bounds for $D = \mathbb{R}$ and $D = \mathbb{C}$ are from [13, Theorem 7.3], [14, Razmyslov's Theorem, p. 451].

If $(D,\vartheta) = ((-1,-1)_{\mathbb{R}},-)$, the result is a consequence of [19, Theorem 4] together with [19, Theorem 1], as briefly explained in the final paragraph on the same page as [19, Theorem 4], since $\mathrm{Trd} = 2(\mathrm{tr} \circ \mathrm{Re})$ in this case. We give some details since not many are given, and in order to make it clear that (3) is also covered:

(1)⇒(2): From $\bar{O}^t X_i O = Y_i$ we get $\bar{O}^t \bar{X}_i{}^t O = \bar{Y}_i{}^t$ for every i. Therefore we have $\bar{O}^t w(X,\bar{X}^t) O = w(\bar{O}^t X O, \bar{O}^t \bar{X}^t O) = w(Y,\bar{Y}^t)$, and the result follows since the reduced trace is invariant under F-algebra isomorphisms.

(2)⇒(3) is clear.

(3)⇒(1): By Lemma 1.9 we have $\mathrm{tr}(w(X,\vartheta(X)^t)^*) = \mathrm{tr}(w(Y,\vartheta(Y)^t)^*)$ for every word w of length at most n^2. Since $*$ is a morphism of algebras with involution (cf. Proposition 1.8) we get $\mathrm{tr}(w(X^*,\overline{X^*}^t)) = \mathrm{tr}(w(Y^*,\overline{Y^*}^t))$, i.e., $\mathrm{Trd}(w(X^*,\overline{X^*}^t)) = \mathrm{Trd}(w(Y^*,\overline{Y^*}^t))$ since we are taking the trace of complex matrices, cf. Remark 1.5. By the complex case, we get a unitary matrix U such that $\overline{U}^t X_i{}^* U = Y_i{}^*$ for every $i = 1,\dots,d$. By [19, Theorem 1] there is a unitary matrix V in $M_{2n}(\mathbb{C})$ such that $V = O^*$ for some $O \in M_n((-1,-1)_{\mathbb{R}})$ and $\overline{V}^t X_i{}^* V = Y_i{}^*$ for every $i = 1,\dots,d$ (the proof of [19, Theorem 1] shows that V only depends on U, so is the same for every i), i.e., $\overline{O^*}^t X_i{}^* O^* = Y_i{}^*$ for every $i = 1,\dots,d$. It is direct to check that O is unitary and, using that $*$ is an injective morphism of algebras with involution, we obtain first $(\overline{O}^t X_i O)^* = Y_i^*$ and then $\overline{O}^t X_i O = Y_i$ for every $i = 1,\dots,d$.

For the general case where F is a real closed field, observe that the equivalence of (1) and (3) can be expressed as a first-order formula in the language of fields, and thus follows from the case $F = \mathbb{R}$. And it is clear that (1) ⇒ (2) (with the same proof as above) and that (2) ⇒ (3). □

Remark 1.11. We need to use the reduced trace instead of the usual trace for matrices with quaternion coefficients, since there are matrices $A,B,U \in M_2((-1,-1)_{\mathbb{R}})$ such that $\overline{U}^t U = I_2$, $B = \overline{U}^t A U$ but $\mathrm{tr}(B) \neq \mathrm{tr}(A)$, see [20, Example 7.2].

2. Model completeness

2.1. **Matrix bases.** The matrix algebras $M_n(F)$, $M_n(F(\sqrt{-1}))$ and $M_n((-1,-1)_F)$, where F is formally real (or even real closed), will play an important role in this paper.

The essential properties of their canonical bases can be expressed by quantifier-free formulas. We present these formulas below, as well as two immediate consequences

(Lemmas 2.1 and 2.2). The results in this section are from [8, Section 2.1.2] for $M_n(F)$ and $M_n(F(\sqrt{-1}))$:

(1) Case 1: The algebra $M_n(F)$. Let $\mathscr{B} := \{E_{r,s}\}_{r,s=1}^n$ be its canonical basis (the matrix $E_{r,s}$ is the matrix with zeroes everywhere, except for a 1 at coordinates (r, s)). We have
$$M_n(F) \models \delta^{(1)}(E_{r,s})_{r,s\in\{1,\dots,n\}},$$
where
$$\delta^{(1)}(X_{r,s})_{r,s\in\{1,\dots,n\}} := \bigwedge_{r,s,t} X_{r,s} \cdot X_{s,t} = X_{r,t} \neq 0 \wedge \bigwedge_{r,s,t,\ell s\neq t} X_{r,s} \cdot X_{t,\ell} = 0.$$

(2) Case 2: The algebra $M_n(F(\sqrt{-1}))$. Writing $i := \sqrt{-1}$, the set $\mathscr{B} := \{E_{r,s}, E_{r,s}i\}_{r,s=1}^n$ is a basis of $M_n(F(\sqrt{-1}))$, and we have
$$M_n(F(\sqrt{-1}) \models \delta^{(2)}(E_{r,s}, E_{r,s}i)_{r,s\in\{1,\dots,n\}},$$
where
$$\delta^{(3)}(X^{(1)}_{r,s}, X^{(i)}_{r,s})_{r,s\in\{1,\dots,n\}} := \bigwedge_{\substack{x,y,z\in\{1,i\}\\ \delta\in\{-1,1\},\ xy=\delta z}} \bigwedge_{r,s,t} X^{(x)}_{r,s} \cdot X^{(y)}_{s,t} = \delta X^{(z)}_{r,t} \neq 0 \wedge \bigwedge_{r,s,t,\ell,s\neq t} X^{(x)}_{r,s} \cdot X^{(y)}_{t,\ell} = 0.$$

(3) Case 3: The algebra $M_n((-1,-1)_F)$. We denote by $\{1, i, j, k\}$ the usual basis of $(-1,-1)_F$. The set $\mathscr{B} := \{E_{r,s}, E_{r,s}i, E_{r,s}j, E_{r,s}k\}_{r,s=1}^n$ is a basis of $M_n((-1,-1)_F)$ and we have
$$M_n((-1,-1)_F) \models \delta^{(3)}(E_{r,s}, E_{r,s}i, E_{r,s}j, E_{r,s}k)_{r,s\in\{1,\dots,n\}},$$
where
$$\delta^{(3)}(X^{(1)}_{r,s}, X^{(i)}_{r,s}, X^{(j)}_{r,s}, X^{(k)}_{r,s})_{r,s\in\{1,\dots,n\}} := \bigwedge_{\substack{x,y,z\in\{1,i,j,k\}\\ \delta\in\{-1,1\},\ xy=\delta z}} \bigwedge_{r,s,t} X^{(x)}_{r,s} \cdot X^{(y)}_{s,t} = \delta X^{(z)}_{r,t} \neq 0 \wedge \bigwedge_{r,s,t,\ell,s\neq t} X^{(x)}_{r,s} \cdot X^{(y)}_{t,\ell} = 0.$$

Lemma 2.1. *Let A be an F-algebra.*

(1) If $\{e_{r,s}\}_{r,s\in\{1,\dots,n\}} \subseteq A$ is such that $A \models \delta^{(1)}(e_{r,s})_{r,s\in\{1,\dots,n\}}$. Then the set $\{e_{r,s}\}_{r,s\in\{1,\dots,n\}}$ is linearly independent over F.

(2) Assume that $F(\sqrt{-1})$ is a field (cf. Remark 1.2) and that we have elements $e^{(1)}_{r,s}, e^{(i)}_{r,s} \in A$ (for $r, s \in \{1, \dots, n\}$) such that $A \models \delta^{(2)}(e^{(1)}_{r,s}, e^{(i)}_{r,s})_{r,s\in\{1,\dots,n\}}$. Then the set $\{e^{(1)}_{r,s}, e^{(i)}_{r,s}\}_{r,s\in\{1,\dots,n\}}$ is linearly independent over F.

(3) Assume that $(-1,-1)_F$ is a division algebra (cf. Remark 1.2) and that we have elements $e^{(1)}_{r,s}$, $e^{(i)}_{r,s}$, $e^{(j)}_{r,s}$, $e^{(k)}_{r,s} \in A$ (for $r, s \in \{1, \dots, n\}$) such that $A \models \delta^{(3)}(e^{(1)}_{r,s}, e^{(i)}_{r,s}, e^{(j)}_{r,s}, e^{(k)}_{r,s})_{r,s\in\{1,\dots,n\}}$. Then the set $\{e^{(1)}_{r,s}, e^{(i)}_{r,s}, e^{(j)}_{r,s}, e^{(k)}_{r,s}\}_{r,s\in\{1,\dots,n\}}$ is linearly independent over F.

Proof. We prove the third statement, since it is the most involved. Assume that, for some $u_{r,s}, v_{r,s}, w_{r,s}, z_{r,s} \in F$, we have
$$\sum_{r,s} u_{r,s}e^{(1)}_{r,s} + v_{r,s}e^{(i)}_{r,s} + w_{r,s}e^{(j)}_{r,s} + z_{r,s}e^{(k)}_{r,s} = 0.$$

Let $s_0, t \in \{1, \dots, n\}$. Multiplying on the right by $e^{(1)}_{s_0,t}$ we obtain

$$\sum_r u_{r,s_0} e^{(1)}_{r,t} + v_{r,s_0} e^{(i)}_{r,t} + w_{r,s_0} e^{(j)}_{r,t} + z_{r,s_0} e^{(k)}_{r,t} = 0.$$

Multiplying this line on the left by $e^{(1)}_{t,r_0}$, $e^{(i)}_{t,r_0}$, $e^{(j)}_{t,r_0}$, or $e^{(k)}_{t,r_0}$, we obtain the following four equations

$$u_{r_0,s_0} e^{(1)}_{t,t} + v_{r_0,s_0} e^{(i)}_{t,t} + w_{r_0,s_0} e^{(j)}_{t,t} + z_{r_0,s_0} e^{(k)}_{t,t} = 0 \tag{2.1}$$

$$u_{r_0,s_0} e^{(i)}_{t,t} - v_{r_0,s_0} e^{(1)}_{t,t} + w_{r_0,s_0} e^{(k)}_{t,t} - z_{r_0,s_0} e^{(j)}_{t,t} = 0 \tag{2.2}$$

$$u_{r_0,s_0} e^{(j)}_{t,t} - v_{r_0,s_0} e^{(k)}_{t,t} - w_{r_0,s_0} e^{(1)}_{t,t} + z_{r_0,s_0} e^{(i)}_{t,t} = 0 \tag{2.3}$$

$$u_{r_0,s_0} e^{(k)}_{t,t} + v_{r_0,s_0} e^{(j)}_{t,t} - w_{r_0,s_0} e^{(i)}_{t,t} - z_{r_0,s_0} e^{(1)}_{t,t} = 0. \tag{2.4}$$

Computing $u_{r_0,s_0}(2.1) - v_{r_0,s_0}(2.2) - w_{r_0,s_0}(2.3) - z_{r_0,s_0}(2.4)$, we obtain

$$u^2_{r_0,s_0} + v^2_{r_0,s_0} + w^2_{r_0,s_0} + w^2_{r_0,s_0} = 0,$$

and the result follows by hypothesis on F. □

Lemma 2.2. *Let $D \in \{F, F(\sqrt{-1}), (-1,-1)_F\}$ with F a field such that D is a division algebra (see Remark 1.2), and let $A = M_n(D)$. Let B be an L-algebra for some field L, such that $\dim_F A = \dim_L B$ and A is a subring of B. Then $Z(A) = Z(B) \cap A$.*

Proof. We prove the case $D = F$, the other two are similar. With notation as at the start of this section, we have $A \models \delta^{(1)}(E_{r,s})_{r,s\in\{1,\dots,n\}}$ and therefore $B \models \delta^{(1)}(E_{r,s})$, since $\delta^{(1)}$ is quantifier-free. By Lemma 2.1(1) and since $\dim_L B = \dim_F A$, $\{E_{r,s}\}_{r,s\in\{1,\dots,n\}}$ is a basis of B over L. In particular, for $x \in A$ we have

$$\begin{aligned} x \in Z(A) &\Leftrightarrow \forall r, s\ xE_{r,s} = E_{r,s}x \text{ in } A \\ &\Leftrightarrow \forall r, s\ xE_{r,s} = E_{r,s}x \text{ in } B \\ &\Leftrightarrow x \in Z(B). \end{aligned}$$ □

2.2. **Model completeness.** The objective of this section is Proposition 2.6, which states that the theories CSA_m and $\mathrm{CSA\text{-}I}_m$ are model-complete if we ask that the centre, respectively the base field, is real closed.

Lemma 2.3. *Let $D \in \{F, F(\sqrt{-1}), (-1,-1)_F\}$ where F is a formally real field. Let A be an F-algebra and $f : A \to M_n(D)$ be an isomorphism of F-algebras. Let B be an L-algebra for some formally real field L, such that $F \subseteq L$, A is a subring of B, and $\dim_F A = \dim_L B$. Then:*

(1) There is an isomorphism of L-algebras g such that the following diagram is commutative:

$$\begin{array}{ccc} A & \xrightarrow{f} & M_n(D) \\ \subseteq \downarrow & & \downarrow \subseteq \\ B & \xrightarrow[g]{} & M_n(E) \end{array}$$

where $E := \begin{cases} L & \text{if } D = F \\ L(\sqrt{-1}) & \text{if } D = F(\sqrt{-1}) \\ (-1,-1)_F & \text{if } D = (-1,-1)_F \end{cases}$ *is a division algebra over L, and the inclusion on the right is the canonical one induced by $F \subseteq L$.*

(2) If $F \prec L$ as fields, then the inclusion of A in B is elementary in the language $L_R \cup \{\underline{\mathrm{Trd}}\}$ (where $\underline{\mathrm{Trd}}$ is interpreted by the reduced trace in A and B).

Proof. (1) Let E be defined as in the statement, and let $i = 1$ if $D = F$, $i = 2$ if $D = F(\sqrt{-1})$, and $i = 3$ if $D = (-1, -1)_F$. Since L is formally real, E is a division algebra. We consider the basis $\mathscr{B}$ of $M_n(D)$ introduced in cases 1, 2, and 3 at the start of Section 2.1. Then $A \models \delta^{(i)}(f^{-1}(\mathscr{B}))$, and thus $B \models \delta^{(i)}(f^{-1}(\mathscr{B}))$ since $\delta^{(i)}$ is quantifier-free. By Lemma 2.1, $f^{-1}(\mathscr{B})$ is linearly independent in B over L and is thus a basis of B over L (since $\dim_F A = \dim_L B$).

The structure constants of $M_n(D)$ for the basis $\mathscr{B}$ over F, $M_n(E)$ for the basis $\mathscr{B}$ over L, A for the basis $f^{-1}(\mathscr{B})$ over F, and B for the basis $f^{-1}(\mathscr{B})$ over L are all specified by the formula $\delta^{(i)}$, so are all the same. Since B and $M_n(E)$ are L-algebras, it follows that the map $g : B \to M_n(E)$, $f^{-1}(X) \mapsto X$ for every $X \in \mathscr{B}$, is an isomorphism and makes the diagram of the statement commutative.

(2) Since the L_R-structures $M_n(D)$ and $M_n(E)$ are interpretable in the same way in F and in L, we have $M_n(D) \prec M_n(E)$. The result follows because of the commutativity of the diagram. □

We need a version of Lemma 2.3 for algebras with involution when the base field is real closed. It requires first a preliminary lemma.

Lemma 2.4. *Let (A, σ) be a central simple algebra with involution over F real closed, and let $n \in \mathbb{N}$ and $(D, \vartheta) \in \{(F, \mathrm{id}), (F(\sqrt{-1}), -), ((-1, -1), -)\}$ be such that $A \cong M_n(D)$. Then:*

(1) σ is of the first kind if and only if $A \cong M_n(F)$ or $A \cong M_n((-1, -1)_F)$;
(2) σ is of the second kind if and only if $A \cong M_n(F(\sqrt{-1}))$.

In particular σ is of the same kind as ϑ and ϑ^t.

Proof. (1) Assume that σ is of the first kind, so that $Z(A) = F$ and thus $Z(A)$ is ordered. Therefore $Z(A) \not\cong F(\sqrt{-1})$, and $A \not\cong M_n(F(\sqrt{-1}))$.

Assume that $A \cong M_n(F)$ or $A \cong M_n((-1, -1)_F)$. Then $Z(A) = F$. If σ is of the second kind, then F has a subfield of index 2, which is impossible since F is real closed.

(2) It is a reformulation of (1). □

Lemma 2.5. *Let $(D, \vartheta) \in \{(F, \mathrm{id}), (F(\sqrt{-1}), -), ((-1, -1)_F, -)\}$ with F real closed. Let (A, σ) be an F-algebra with involution and $f : A \to M_n(D)$ be an isomorphism of F-algebras. Let (B, τ) be an L-algebra with involution for some real closed field L, such that (A, σ) is an $L_{CSA\text{-}I}$-substructure of (B, τ) (i.e., $F \subseteq L$, and $(A, \sigma) \subseteq (B, \tau)$), and $\dim_F A = \dim_L B$. Then:*

(1) The involutions σ and τ are of the same kind.
(2) There are an isomorphism of L-algebras g and $a \in M_n(D)^\times$ such that, if

$$(E, \vartheta') := \begin{cases} (L, \mathrm{id}) & \text{if } (D, \vartheta) = (F, \mathrm{id}) \\ (L(\sqrt{-1}), -) & \text{if } (D, \vartheta) = (F(\sqrt{-1}), -) \\ ((-1, -1)_F, -) & \text{if } (D, \vartheta) = ((-1, -1)_F, -) \end{cases}$$

then the following diagram is commutative

$$\begin{array}{ccc} (A,\sigma) & \xrightarrow{f} & (M_n(D), \operatorname{Int}(a)\circ\vartheta^t) \\ \subseteq\big\downarrow & & \big\downarrow\subseteq \\ (B,\tau) & \xrightarrow[g]{} & (M_n(E), \operatorname{Int}(a)\circ(\vartheta')^t) \end{array}$$

where the inclusion on the right is the canonical one induced by $F \subseteq L$*, and all maps respect the reduced trace* Trd.

(3) *The inclusion of* (A,σ) *in* (B,τ) *is elementary in the language* $L_{CSA\text{-}I}\cup\{\underline{\mathrm{Trd}}\}$ *(where* $\underline{\mathrm{Trd}}$ *is interpreted in each structure by the reduced trace).*

Proof. (1) By Lemma 2.2, $Z(A) \subseteq Z(B)$. If τ is of the first kind, then $Z(B) \subseteq \operatorname{Sym}(B,\tau)$ and thus $Z(A) \subseteq \operatorname{Sym}(A,\sigma)$, so that σ is of the first kind.

If σ is of the first kind, we have $A \cong M_n(F)$ or $M_n((-1,-1)_F)$ by Lemma 2.4, and thus $\dim_F A = n^2$ (in the first case) or $\dim_F A = 4n^2$ (in the second case). Assume that τ is of the second kind, so that $B \cong M_\ell(F(\sqrt{-1}))$ and thus $\dim_L B = 2\ell^2$. If $A \cong M_n(F)$, and since $\dim_F A = \dim_L B$, we get $n^2 = 2\ell^2$, impossible. If $A \cong M_n((-1,-1)_F)$ we get $4n^2 = 2\ell^2$, also impossible.

(2) By Lemma 2.4, σ is of the same kind as ϑ. Let (E,ϑ') be the central simple algebra with involution over L defined in the statement. Note that the canonical inclusion of $M_n(D)$ in $M_n(E)$ respects the reduced trace Trd.

By Lemma 2.3, we know that there is an isomorphism g of L-algebras such that the diagram without the involutions is commutative. In particular τ is of the same kind as ϑ' by Lemma 2.4.

Let σ' be the involution $M_n(D)$ such that $f : (A,\sigma) \to (M_n(D),\sigma')$ is an isomorphism of algebras with involution (i.e., $\sigma'\circ f = f\circ\sigma$), and τ' be the involution on $M_n(E)$ such that $g : (B,\tau) \to (M_n(E),\tau')$ is an isomorphism of algebras with involution (i.e., $\tau'\circ g = g\circ\tau$). Note that σ' is of the same kind as σ (and thus as ϑ) and that τ' is of the same kind as τ (and thus as ϑ'). Using that g extends f, we obtain that τ' extends σ', so that $(M_n(D),\sigma') \subseteq (M_n(E),\tau')$.

By the Skolem-Noether theorem (cf. Proposition 1.1), $\sigma' = \operatorname{Int}(a)\circ\vartheta^t$ for some $a \in M_n(D)^\times$, and $\tau' = \operatorname{Int}(b)\circ\vartheta'^t$ for some $b \in M_n(E)^\times$. Since τ' extends σ' we have $\operatorname{Int}(a)\circ\vartheta^t = \operatorname{Int}(b)\circ(\vartheta')^t$ on $M_n(D)$, so that $\operatorname{Int}(a) = \operatorname{Int}(b)$ on $M_n(D)$. If $\mathscr{B}$ is the basis of $M_n(D)$ over F from the start of Section 2.1, then $\mathscr{B}$ is a basis of $M_n(E)$ over L, and $\operatorname{Int}(a) = \operatorname{Int}(b)$ on $\mathscr{B}$, so that $\operatorname{Int}(a) = \operatorname{Int}(b)$ on $M_n(E)$. In particular we can take $a = b$, and the diagram indicated in the statement is commutative in the language $L_{\mathrm{CSA\text{-}I}}$.

We still need to check that the maps respect the reduced trace: It is the case for f and g by Lemma 1.7, it is clear for the canonical inclusion of $M_n(D)$ in $M_n(E)$, and it is therefore also the case for the inclusion of (A,σ) into (B,τ) since the diagram commutes.

(3) Since the $L_{\mathrm{CSA\text{-}I}}$-structures $(M_n(D), \operatorname{Int}(a)\circ\vartheta^t)$ and $(M_n(E), \operatorname{Int}(a)\circ(\vartheta')^t)$ are interpretable in the same way in F and in L, and $F \prec L$, we have $(M_n(D), \operatorname{Int}(a)\circ\vartheta^t) \prec (M_n(E), \operatorname{Int}(a)\circ(\vartheta')^t)$ in $L_{\mathrm{CSA\text{-}I}}$. The result follows because of the commutativity of the diagram. □

Let

$$\mathrm{CSA}_{m,\mathrm{rcf}} := \mathrm{CSA}_m \cup \{\text{the centre is real closed}\}$$

in the language L_R, and

$$\text{CSA-I}_{m,\text{rcf}} := \text{CSA-I}_m \cup \{\underline{F} \text{ is real closed}\}$$

in the language $L_{\text{CSA-I}}$.

Proposition 2.6. *The theories $CSA_{m,rcf}$ and $CSA\text{-}I_{m,rcf}$ are model-complete in the languages L_R and $L_{CSA\text{-}I}$, respectively.*

More precisely, if $\mathscr{M}$, $\mathscr{N}$ are models of $CSA_{m,rcf}$ (respectively $CSA\text{-}I_{m,rcf}$) and $\mathscr{M} \subseteq \mathscr{N}$ in L_R (respectively $L_{CSA\text{-}I}$), then $\mathscr{M} \prec \mathscr{N}$ in $L_R \cup \{\underline{P}, \underline{\text{Trd}}\}$ (respectively $L_{CSA\text{-}I} \cup \{\underline{P}, \underline{\text{Trd}}\}$), where $\underline{P}$ is interpreted by the unique ordering on $\underline{F}$ and $\underline{\text{Trd}}$ is interpreted by the reduced trace map in all models.

Proof. Let $\mathscr{M}, \mathscr{N} \models \text{CSA}_{m,\text{rcf}}$ in L_R, with $\mathscr{M} \subseteq \mathscr{N}$ in L_R. Then $\underline{F}^{\mathscr{M}} \prec \underline{F}^{\mathscr{N}}$ as fields since both are real closed. The result is then Lemma 2.3(2) (since the ordering on $\underline{F}$ is definable, as the set of all squares in $\underline{F}$). If $\mathscr{M} \subseteq \mathscr{N}$ in $L_{\text{CSA-I}}$, with $\mathscr{M}, \mathscr{N} \models \text{CSA-I}_{m,\text{rcf}}$, then $\underline{F}^{\mathscr{M}} \prec \underline{F}^{\mathscr{N}}$ and the result is Lemma 2.5(3). □

Let

$$\text{CSA}_{m,\text{of}} := \text{CSA}_m \cup \{\underline{P} \text{ is an ordering on the centre}\}$$

in the language $L_R \cup \{\underline{P}\}$, and

$$\text{CSA-I}_{m,\text{of}} := \text{CSA-I}_m \cup \{(\underline{F}, \underline{P}) \text{ is an ordered field}\}$$

in the language $L_{\text{CSA-I}} \cup \{\underline{P}\}$.

Corollary 2.7. *The theory $CSA_{m,rcf}$ is the model-companion of $CSA_{m,of}$.*

Proof. It follows from Proposition 2.6, since $\text{CSA}_{m,\text{of}}$ and $\text{CSA}_{m,\text{rcf}}$ are clearly cotheories ($(A, \sigma) \subseteq (A \otimes_{Z(A)} L, \sigma \otimes \text{id})$ where L is a real closure of $Z(A)$ at its ordering). □

Remark 2.8. The theory $\text{CSA-I}_{m,\text{rcf}}$ is not the model-companion of $\text{CSA-I}_{m,\text{of}}$ because not every model of $\text{CSA-I}_{m,\text{of}}$ can be embedded in a model of $\text{CSA-I}_{m,\text{rcf}}$ in the language $L_{\text{CSA-I}} \cup \{\underline{P}\}$. The problem arises when the centre is a proper ordered extension of the field of all symmetric elements of the centre (i.e., the interpretation of $\underline{F}$):

Let $\mathscr{A} := (A, \sigma) \models \text{CSA-I}_{m,\text{of}}$ with $F \subsetneq Z(A) = F(\alpha_0)$, where $F := \underline{F}^{\mathscr{A}}$. We can assume that α_0 is a root of a polynomial $X^2 - d \in F[X]$. The problem occurs when $d \in P := \underline{P}^{\mathscr{A}}$ (the ordering on F). Since $\alpha_0 \notin F$, we have $\sigma(\alpha_0) \neq \alpha_0$, and thus $\sigma(\alpha_0) = -\alpha_0$ since α_0 is a root of $X^2 - d$. We show that (A, σ) cannot be an $L_{\text{CSA-I}} \cup \{\underline{P}\}$-substructure of a model $\mathscr{B} := (B, \tau)$ of $\text{CSA-I}_{m,\text{rcf}}$:

Assume it is the case, where $L := \underline{F}^{\mathscr{B}}$ is real closed. Then (L, L^2) is an ordered extension of (F, P). Since $d \in P$, it has a square root α_1 in L. In particular $\tau(\alpha_1) = \alpha_1$. We consider two cases:

(1) If $\alpha_0 \in Z(B)$. Since $\alpha_1 \in Z(B)$ and both are roots of $X^2 - d$, we have $\alpha_1 = \varepsilon\alpha_0$ for some $\varepsilon \in \{-1, 1\}$, which implies $\tau(\alpha_1) = \varepsilon\tau(\alpha_0) = \varepsilon\sigma(\alpha_0) = -\varepsilon\alpha_0 = -\alpha_1$, contradiction.

(2) If $\alpha_0 \notin Z(B)$. Then α_0 and $-\alpha_0$ are different from α_1, and thus $\alpha_0, -\alpha_0, \alpha_1$ are three different roots of $X^2 - d$ in the field $Z(B)(\alpha_0)$, impossible.

Remark 2.9. This situation ($Z(A) = F(\alpha_0)$ where α_0 is a root of $X^2 - d$ with $d \in P$) leads to another problem when extending the scalars in A from F to a real closure L of (F, P). Since d has a square root in L, we obtain

$$A \otimes_F L \cong A \otimes_{Z(A)} Z(A) \otimes_F L \cong A \otimes_{Z(A)} L[X]/(X^2 - d)$$

$$\cong (A \otimes_{Z(A)} L) \times (A \otimes_{Z(A)} L),$$

which is not a simple algebra anymore. This situation does not occur when (A, σ) is equipped with a positive cone over P, cf. Lemma 3.13.

3. Quantifier elimination

We turn our attention to quantifier elimination with the help of a simple extension of [8, Theorem 2.2.4] (our Proposition 3.3). We then point out some potential problems, which can be avoided by introducing "orderings" (positive cones) on algebras with involution, and specifying the type of the involution (Proposition 3.17).

Definition 3.1. Let (D, ϑ) be a division algebra with involution over F formally real. We say that (D, ϑ) is

(1) of real type if $(D, \vartheta) = (F, \mathrm{id})$;
(2) of complex type if $(D, \vartheta) = (F(\sqrt{-1}), -)$;
(3) of quaternion type if $(D, \vartheta) = ((-1, -1)_F, -)$.

Lemma 3.2. *Let $(D, \vartheta) \in \{(F, \mathrm{id}), (F(\sqrt{-1}), -), ((-1, -1)_F, -)\}$ where F is a real closed field. Let $\mathscr{M}$ be elementarily equivalent to $(M_n(D), \vartheta^t)$ in $L := L_{CSA\text{-}I} \cup \{\underline{P}, \underline{\mathrm{Trd}}\}$, where $\underline{P}$ is interpreted in $M_n(D)$ by the order on F, and $\underline{\mathrm{Trd}}$ by the reduced trace map.*

Then there is an L-isomorphism $\phi_{\mathscr{M}} : \mathscr{M} \to (M_n(D_{\mathscr{M}}), (\vartheta_{\mathscr{M}})^t, \mathrm{Trd})$, where $(D_{\mathscr{M}}, \vartheta_{\mathscr{M}})$ is a sub-division algebra with involution of $\mathscr{M}$ over $F^{\mathscr{M}}$ that is of the same type as (D, ϑ) (note that $\underline{F}^{\mathscr{M}}$ is real closed).

Proof. Let $\mathscr{N}$ be the $L_{\text{CSA-I}} \cup \{\underline{P}, \underline{\mathrm{Trd}}\}$-structure $(M_n(D), \vartheta^t)$. It follows from Lemma 1.6 that "$\underline{\mathrm{Trd}}$ is the reduced trace" can be expressed by a first-order formula (depending on n and D). We define three $L_{\text{CSA-I}} \cup \{\underline{P}, \underline{\mathrm{Trd}}\}$-formulas (with reference to Section 2.1 for the formula $\delta^{(1)}$):

$$\begin{aligned}\Omega_{\text{orthogonal}} := {} & (\text{centre} = \underline{F}) \wedge \underline{\mathrm{Trd}} \text{ is the reduced trace} \wedge \\ & \exists e_{11}, \dots, e_{nn} \Big\{ \delta^{(1)}(e_{11}, \dots, e_{nn}) \wedge \{e_{11}, \dots, e_{nn}\} \text{ is an } \underline{F}\text{-basis} \wedge \\ & \qquad \underline{\sigma} = {}^t \Big\}\end{aligned}$$

$$\begin{aligned}\Omega_{\text{unitary}} := {} & \exists i \Big\{ i^2 = -1 \wedge (\text{centre} = \underline{F}(i)) \wedge \underline{\mathrm{Trd}} \text{ is the reduced trace} \wedge \\ & \exists e_{11}, \dots, e_{nn}\ \delta^{(1)}(e_{11}, \dots, e_{nn}) \wedge \\ & \{e_{11}, \dots, e_{nn}\} \text{ is a basis over the centre } \wedge \underline{\sigma} = -{}^t \Big\}\end{aligned}$$

$$\begin{aligned}\Omega_{\text{symplectic}} := {} & (\text{centre} = \underline{F}) \wedge \underline{\mathrm{Trd}} \text{ is the reduced trace} \wedge \\ & \exists i, j, k \exists e_{11}, \dots, e_{nn} \Big\{ i^2 = j^2 = -1 \wedge ij = -ji = k \wedge \\ & \mathrm{Span}_{\underline{F}}\{1, i, j, k\} \text{ is a division algebra } \wedge \delta^{(1)}(e_{11}, \dots, e_{nn}) \wedge \\ & i, j, k \text{ commute with } e_{11}, \dots, e_{nn} \wedge \\ & \{e_{11}, \dots, e_{nn}\} \text{ is a basis over } \mathrm{Span}_{\underline{F}}\{1, i, j, k\} \wedge \underline{\sigma} = -{}^t \Big\}\end{aligned}$$

Let Ω be the only one of the above three formulas such that $\mathscr{N} \models \Omega$. Then $\mathscr{M} \models \Omega$ and the map $\phi_{\mathscr{M}}$ is $\mathscr{M} \to M_n(D_{\mathscr{M}})$, $d \in D_{\mathscr{M}} \mapsto d$, $e_{rs} \mapsto E_{rs}$ (the matrix with 1 at entry (r, s) and 0 elsewhere). The map $\phi_{\mathscr{M}}$ is an isomorphism of $\underline{F}^{\mathscr{M}}$-algebras since the formula Ω specifies the structure constants. It therefore respects the reduced trace, and it is clear that it respects the ordering on $\underline{F}^{\mathscr{M}}$. Finally, $\underline{F}^{\mathscr{M}}$ is real closed since $\underline{F}^{\mathscr{M}} \equiv \underline{F}^{\mathscr{N}}$. □

The following result is a special case of [8, Theorem 2.2.4] in the case of matrices over real closed fields and algebraically closed fields, but also covers the additional case of matrices over real quaternions. While the proof is the same as that of [8, Theorem 2.2.4] (using Theorem 1.10 instead of [8, Theorem 2.2.2] in order to include the quaternion case), we still reproduce the relevant parts of it for the convenience of the reader, since it is reasonably short.

Proposition 3.3. *Let $(D,\vartheta) \in \{(F,\mathrm{id}), (F(\sqrt{-1}), -), ((-1,-1)_F, -)\}$, where F is a real closed field. Let T be the theory (without parameters) of $(M_n(D), \vartheta^t)$ in $L := L_{CSA\text{-}I} \cup \{\underline{P}, \underline{\mathrm{Trd}}\}$, where $\underline{P}$ is interpreted as the order on F, and $\underline{\mathrm{Trd}}$ as the reduced trace map.*

Then T has quantifier elimination in L.

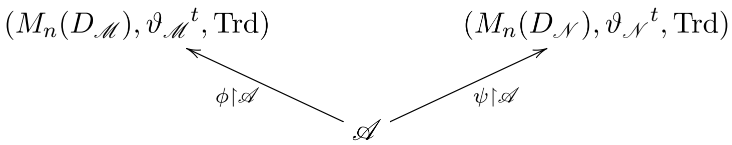
Proof. We first observe that T is model-complete: Let $\mathscr{M}$, $\mathscr{N}$ be models of T such that $\mathscr{M} \subseteq \mathscr{N}$ as L-structures. Then $\mathscr{M}$, $\mathscr{N}$ are models of CSA-$\mathrm{I}_{m,\mathrm{rcf}}$ for some m and thus $\mathscr{M} \prec \mathscr{N}$ in $L_{\text{CSA-I}} \cup \{\underline{P}, \underline{\mathrm{Trd}}\}$ by Proposition 2.6.

Therefore, it suffices to show that it has the amalgamation property over finitely generated substructures (this follows from [12, Theorem 3.1.4], where the proof of ii) implies i) shows that the common substructure can be taken finitely generated). Let $\mathscr{M}, \mathscr{N} \models T$ and let $\mathscr{A}$ be a finitely generated L-substructure of $\mathscr{M}$ and $\mathscr{N}$.

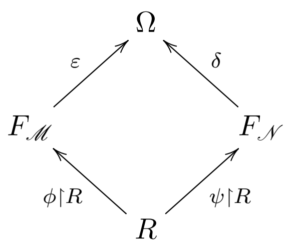
By Lemma 3.2, there are L-isomorphisms

$$\phi : \mathscr{M} \to (M_n(D_{\mathscr{M}}), {\vartheta_{\mathscr{M}}}^t, \mathrm{Trd}) \text{ and } \psi : \mathscr{N} \to (M_n(D_{\mathscr{N}}), {\vartheta_{\mathscr{N}}}^t, \mathrm{Trd})$$

where $(D_{\mathscr{M}}, \vartheta_{\mathscr{M}})$ and $(D_{\mathscr{N}}, \vartheta_{\mathscr{N}})$ are over the real closed fields $F_{\mathscr{M}}$ and $F_{\mathscr{N}}$, respectively, and are of the same type as (D,ϑ). Therefore, we have



Let R be the subring of $\mathscr{A}$ generated by the image of $\underline{\mathrm{Trd}}^{\mathscr{A}} = \underline{\mathrm{Trd}}^{\mathscr{M}} \restriction \mathscr{A}$. It is commutative since $\phi(R)$ is included in the field $Z(\mathscr{M})$. Since the theory of real closed fields has quantifier elimination, it has the amalgamation property over substructures ([7, Proposition 3.5.19]), so there are a real closed field Ω and L_R-embeddings $\varepsilon : F_{\mathscr{M}} \to \Omega$, $\delta : F_{\mathscr{N}} \to \Omega$ such that the following diagram commutes:



Let $(D_\Omega, \vartheta_\Omega)$ be the division algebra with involution over Ω of the same type as (D,ϑ) (and thus as $(\mathscr{M}, \vartheta_{\mathscr{M}})$ and $(\mathscr{N}, \vartheta_{\mathscr{N}})$). Both ε and δ induce canonical maps from $(D_{\mathscr{M}}, \vartheta_{\mathscr{M}})$ to $(D_\Omega, \vartheta_\Omega)$ and from $(D_{\mathscr{N}}, \vartheta_{\mathscr{N}})$ to $(D_\Omega, \vartheta_\Omega)$, and thus canonical maps

$$\varepsilon_n : (M_n(D_{\mathscr{M}}), {\vartheta_{\mathscr{M}}}^t) \to (M_n(D_\Omega), {\vartheta_\Omega}^t) \text{ and } \delta_n : (M_n(D_{\mathscr{N}}), {\vartheta_{\mathscr{N}}}^t) \to (M_n(D_\Omega), {\vartheta_\Omega}^t).$$

It is clear that ε_n and δ_n are L-embeddings.

Since $\mathscr{A}$ is finitely generated, there are $X_1, \dots, X_d \in \mathscr{A}$ such that $\mathscr{A}$ is generated by $X_1, \dots, X_d$ as L-structure.

Claim: There is $O \in M_n(D_\Omega)$ such that $\vartheta_\Omega(O)^t O = I_n$ and $\vartheta_\Omega(O)^t \varepsilon_n(\phi(X_i))O = \delta_n(\psi(X_i))$ for $i = 1, \dots, d$.

Proof of the claim: Let $Y_i := \varepsilon_n(\phi(X_i))$ and $Z_i := \delta_n(\phi(X_i))$. By Theorem 1.10 it suffices to show that

$$\begin{aligned}\mathrm{Trd}_{M_n(D_\Omega)}(w(Y_1, \dots, Y_d, \vartheta_\Omega(Y_1)^t, \dots, \vartheta_\Omega(Y_d)^t))) = \\ \mathrm{Trd}_{M_n(D_\Omega)}(w(Z_1, \dots, Z_d, \vartheta_\Omega(Z_1)^t, \dots, \vartheta_\Omega(Z_d)^t)))\end{aligned} \tag{3.1}$$

for every word $w(x_1, \dots, x_d, x'_1, \dots, x'_d)$. Consider

$$X := w(X_1, \dots, X_d, \underline{\sigma}^{\mathscr{M}}(X_1), \dots, \underline{\sigma}^{\mathscr{M}}(X_d)) \in \mathscr{A}.$$

We have

$$\begin{aligned}\mathrm{Trd}_{M_n(D_\Omega)}(\varepsilon_n(\phi(X))) &= \varepsilon(\mathrm{Trd}_{M_n(D_{\mathscr{M}})}(\phi(X))) \qquad \text{(by definition of } \varepsilon_n) \\ &= \varepsilon(\phi(\underline{\mathrm{Trd}}^{\mathscr{A}}(X))) \qquad \text{(since } \phi \text{ is an } L\text{-morphism)} \\ &= \delta(\psi(\underline{\mathrm{Trd}}^{\mathscr{A}}(X))) \qquad \text{(since } \varepsilon \circ \phi \restriction R = \delta \circ \psi \restriction R),\end{aligned}$$

while

$$\begin{aligned}\mathrm{Trd}_{M_n(D_\Omega)}(\delta_n(\psi(X))) &= \delta(\mathrm{Trd}_{M_n(D_{\mathscr{N}})}(\psi(X))) \qquad \text{(by definition of } \delta_n) \\ &= \delta(\psi(\underline{\mathrm{Trd}}^{\mathscr{A}}(X))) \qquad \text{(since } \psi \text{ is an } L\text{-morphism).}\end{aligned}$$

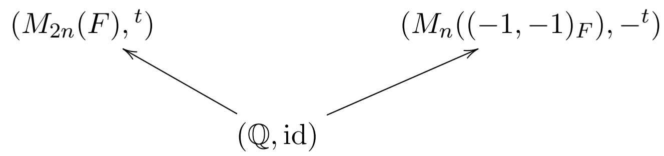
Therefore $\mathrm{Trd}_{M_n(D_\Omega)}(\varepsilon_n(\phi(X))) = \mathrm{Trd}_{M_n(D_\Omega)}(\delta_n(\psi(X)))$, proving (3.1) since ϕ, ψ, ε_n and δ_n are L-embeddings. **End of the proof of the Claim.**

Taking O as in the claim, the map $\xi : M_n(D_\Omega) \to M_n(D_\Omega)$, $X \mapsto \vartheta_\Omega(O)^t XO$ is an L-automorphism of $M_n(D_\Omega)$ (since the reduced trace is invariant under isomorphisms of $Z(D_\Omega)$-algebras). Using the Claim, we have $\xi \circ \varepsilon_n \circ \phi = \delta_n \circ \psi$ on $\mathscr{A}$, i.e., the maps $\xi \circ \varepsilon_n \circ \phi \restriction \mathscr{A}$ and $\delta_n \circ \psi \restriction \mathscr{A}$ form an amalgamation of $\mathscr{M}$ and $\mathscr{N}$ over $\mathscr{A}$. □

We will now consider quantifier elimination for some theories of central simple algebras with involution. We are interested in situations where the base field is ordered, so we will naturally have these theories specify that it is real closed. The main two difficulties are presented in the following remark.

Remark 3.4. Assume that T is a theory of central simple algebra with involution, such that $\text{CSA-I}_{m,\mathrm{rcf}} \subseteq T$ and T admits quantifier elimination:

(1) If structures like $(M_{2n}(F), {}^t)$ and $(M_n((-1,-1)_F), -{}^t)$ are models of T. In this case the amalgamation property applied to



would give a common elementary extension of the structures $(M_{2n}(F), {}^t)$ and $(M_n((-1,-1)_F), -{}^t)$, which is not possible since the two involutions are of different types (which can be expressed by a first-order property in the language $L_R \cup \{\underline{\sigma}\}$). So we need to avoid such (or similar) situations.

(2) A model $\mathscr{M}$ of T with $\underline{F}^{\mathscr{M}}$ real closed will be isomorphic to $(M_n(D), \mathrm{Int}(a) \circ \vartheta^t)$ (using the notation of Lemma 2.5) for some $a \in M_n(D)^\times$. In order to use Proposition 3.3, we would like to be able to go back to an algebra of the type $(M_n(D'), \vartheta'^t)$ (for some division algebra with involution (D', ϑ') of the same type as (D, ϑ)), so we need a way to get some control on how the involution is scaled by $\mathrm{Int}(a)$.

We will see in Section 3.3 that specifying the type of the involution (logically) solves the first problem. The second will be solved by introducing a positive cone, which will (on top of specifying an ordering on the base field) give some control over the involution, due to the links between positive cones and positive involutions (Remark 3.6(2) and Corollary 3.10(1)).

3.1. **Positive cones and positive involutions.** Positive cones on algebras with involution have been introduced in [4] as an attempt to define a notion of ordering that corresponds to signatures of hermitian forms. They are also closely linked to positive involutions.

Definition 3.5 ([4, Definition 3.1])**.** Let (A, σ) be a central simple algebra with involution over F. A *prepositive cone $\mathscr{P}$ on* (A, σ) is a subset $\mathscr{P}$ of $\mathrm{Sym}(A, \sigma)$ such that

(P1) $\mathscr{P} \neq \varnothing$;
(P2) $\mathscr{P} + \mathscr{P} \subseteq \mathscr{P}$;
(P3) $\sigma(a) \cdot \mathscr{P} \cdot a \subseteq \mathscr{P}$ for every $a \in A$;
(P4) $\mathscr{P}_F := \{u \in F \mid u\mathscr{P} \subseteq \mathscr{P}\}$ is an ordering on F.
(P5) $\mathscr{P} \cap -\mathscr{P} = \{0\}$ (we say that $\mathscr{P}$ is proper).

A prepositive cone $\mathscr{P}$ is over $P \in X_F$ (the set of all orderings on F) if $\mathscr{P}_F = P$, and a positive cone is a prepositive cone that is maximal with respect to inclusion.

We recall the following long list of results about positive cones. Most of them are direct or appear in some other papers.

Remark 3.6. (1) If $\mathscr{P}$ is a positive cone on (A, σ) and $a \in \mathscr{P} \setminus \{0\}$, then $\mathscr{P}_F = \mathscr{P}'_F := \{u \in F \mid ua \in \mathscr{P}\}$. Indeed, we clearly have $\mathscr{P}_F \subseteq \mathscr{P}'_F$, and if $u \in \mathscr{P}'_F \setminus \mathscr{P}_F$, then $-u \in \mathscr{P}_F$ and thus $ua, -ua \in \mathscr{P}$, contradicting (P5).

(2) It is possible for a positive cone not to contain the element 1. This depends on the involution (see Proposition 3.8(1)). For instance, if $a \in \mathrm{Sym}(A, \sigma) \cap A^\times$, then $\mathscr{P}$ is a positive cone on (A, σ) if and only if $a\mathscr{P}$ is a positive cone on $(A, \mathrm{Int}(a) \circ \sigma)$ ([4, Proposition 4.4]). This makes it easy to produce positive cones that contain the element 1 and positive cones that do not.

(3) If $P \in X_F$ there may not be a positive cone over P. The set of orderings over which there are no positive cones is the set $\mathrm{Nil}[A, \sigma]$ of orderings in X_F for which the signature of all hermitian forms is zero ([4, Proposition 6.6]). The set $\mathrm{Nil}[A, \sigma]$ is actually a clopen subset of X_F ([1, Corollary 6.5]).

(4) For $P \in X_F \setminus \mathrm{Nil}[A, \sigma]$, there are exactly two positive cones over P. If $\mathscr{P}$ is one of them, the other one is $-\mathscr{P}$ ([4, Theorem 7.5]). This freedom to choose the sign comes from the link between signatures of hermitian forms and positive cones, and corresponds to the fact that the signature of hermitian forms at P is only determined up to sign ([1, Start of Section 3.3]).

(5) Let $(D, \vartheta) \in \{(F, \mathrm{id}), (F(\sqrt{-1}), -), ((-1, -1)_F, -)\}$ with F real closed. The only two positive cones on $(M_n(D), \vartheta^t)$ over the unique ordering of F are PSD and NSD, the sets of positive semidefine, respectively negative semidefinite

matrices. This follows from the previous item since it can be checked that the set of PSD matrices is a positive cone over the unique ordering of F (it is clearly a pre-positive cone; if PSD $\subsetneq \mathscr{P}$ with $\mathscr{P}$ positive cone, using the fact that symmetric matrices can be diagonalized by congruences, we can assume by (P3) that $\mathscr{P}$ contains a diagonal matrix with at least one negative entry. Using (P3) to only keep this negative entry we then obtain a non-zero NSD matrix in $\mathscr{P}$, contradicting (P5)).

(6) For (D, ϑ) as in (5), the set of positive semidefinite matrices in $M_n(D)$ is equal to $\mathrm{HS}(M_n(D), \vartheta^t) := \{\vartheta(a)^t a \mid a \in M_n(D)\}$, the set of hermitian squares in $(M_n(D).\vartheta^t)$. This is due to the principal axis theorem (which also holds for quaternions, cf. [20, Corollary 6.2]).

We will also more generally consider hermitian squares in an algebra with involution (A, σ) and write

$$\mathrm{HS}(A, \sigma) := \{\sigma(a)a \mid a \in A\}.$$

Definition 3.7 ([15, Definition 1.1])**.** Let (A, σ) be a central simple algebra with involution over F, and let $P \in X_F$. The involution σ is called positive at P if the form $A \times A \to Z(A),\ (x, y) \mapsto \mathrm{Trd}(\sigma(x)y)$ is positive semidefinite at P (hence positive definite at P since it is nonsingular). For more details we refer to [3, Section 4].

We recall the following, which is obtained out of various results in [3, 4]:

Proposition 3.8. *Let F be a formally real field.*

(1) Let (A, σ) be a central simple algebra with involution over F and let $P \in X_F$. Then σ is positive at P if and only if there is a positive cone $\mathscr{P}$ on (A, σ) over P such that $1 \in \mathscr{P}$.

Let $(D, \vartheta) \in \{(F, \mathrm{id}), (F(\sqrt{-1}), -), ((-1, -1)_F, -)\}$.

(2) The involution ϑ^t on $M_n(D)$ is positive at every $P \in X_F$.

(3) Let $a \in \mathrm{Sym}(M_n(D), \vartheta^t)^\times$ be such that $\mathrm{Int}(a) \circ \vartheta^t$ is positive at P. Then a is a PSD or NSD matrix in $M_n(D)$ (and, up to replacing a by $-a$, we can assume that a is PSD).

Proof. (1) By [3, Corollary 4.6] σ is positive at P if and only if $|\mathrm{sign}^\eta_P \langle 1\rangle_\sigma| = n_P$. And by [4, final line of Theorem 7.5], $|\mathrm{sign}^\eta_P \langle 1\rangle_\sigma| = m_P$ if and only if 1 belongs to some positive cone over P. The link between both statements comes from the fact that $m_P = n_P$, cf. [4, Proposition 6.7].

(2) Let $P \in X_F$. The result follows from the first item, since $1 \in$ PSD which is a positive cone on $(M_n(D), \vartheta^t)$ over P by Remark 3.6(5).

We make two observations for the proof of the final item:

(i) In the special case of $(M_n(D), \vartheta^t)$ with (D, ϑ) as indicated in the statement, the signature of a hermitian form $\langle a\rangle_{\vartheta^t}$ is (up to sign) equal to the usual Sylvester signature of the matrix a (this is presented in [3], page 343: identifying symmetric matrices in $M_n(D)$ with hermitian forms, the signature is –up to sign– the map sign_P in equation (2.2), and the link with Sylvester signatures in the cases considered in this proposition is the final bullet point on that page).

(ii) The integer n_P defined in [4, (6.1)] is equal to n, again because of the precise form of the algebra $M_n(D)$ chosen in this proposition.

Using these, we have

(3) By [3, Proposition 4.8] (where σ_u is defined just before [3, Proposition 4.4]) we have $\mathrm{sign}^\eta_P \langle a^{-1}\rangle_{\vartheta^t} = \pm n_P$, which is equal to $\pm n$ by (ii) above. Since

$\langle a^{-1}\rangle_{\vartheta^t} \cong \langle a\rangle_{\vartheta^t}$, we have $\operatorname{sign}^\eta_P\langle a^{-1}\rangle_{\vartheta^t} = \operatorname{sign}^\eta_P\langle a\rangle_{\vartheta^t}$, and thus $\operatorname{sign}^\eta_P\langle a\rangle_{\vartheta^t} = \pm n$. By (i), it follows that a is PSD if this signature is equal to n, and NSD if it is equal to $-n$. □

Lemma 3.9. *(1) Let (A,σ) be an algebra with involution over F, and let $a \in A^\times$ be such that $a^{-1} = \sigma(b)b$ for some $b \in A$. Then $\operatorname{Int}(b)$ is an isomorphism of F-algebras with involution from $(A, \operatorname{Int}(a)\circ\sigma)$ to (A,σ).*

(2) Let $(D,\vartheta) \in \{(F,\mathrm{id}), (F(\sqrt{-1}), -), ((-1,-1)_F, -)\}$ where F is a real closed field, and let a be an invertible PSD matrix in $M_n(D)$. Then the F-algebras with involution $(M_n(D), \operatorname{Int}(a)\circ\vartheta^t)$ and $(M_n(D),\vartheta^t)$ are isomorphic.

Proof. (1) We simply have to check that $\operatorname{Int}(b)\circ\operatorname{Int}(a)\circ\sigma = \sigma\circ\operatorname{Int}(b)$, but this is a direct verification.

(2) The matrix a^{-1} is also PSD and, by the principal axis theorem (which holds in $M_n(D)$; see [20, Corollary 6.2] for the quaternion case), there is $b \in M_n(D)$ such that $a^{-1} = \vartheta(b)^t b$. The result now follows from the previous item. □

The next result shows how positive cones give us the control on the involution that we would like to have in order to get quantifier elimination (see Remark 3.4(2)).

Corollary 3.10. *Let (A,σ) be a central simple algebra with involution over F real closed, and let $\mathscr{P}$ be a positive cone on (A,σ) over the unique ordering of F, such that $1 \in \mathscr{P}$. Then*

(1) There are $(D,\vartheta) \in \{(F,\mathrm{id}), (F(\sqrt{-1}), -), ((-1,-1)_F, -)\}$ and an isomorphism $f : (A,\sigma) \to (M_n(D),\vartheta^t)$ of F-algebras with involution such that $f(\mathscr{P}) = PSD$.

(2) $\mathscr{P}$ is equal to $HS(A,\sigma) := \{\sigma(a)a \mid a \in A\}$, the set of hermitian squares in (A,σ).

Proof. (1) Since F is real closed there is an isomorphism $f : A \to M_n(D)$, where $(D,\vartheta) \in \{(F,\mathrm{id}), (F(\sqrt{-1}), -), ((-1,-1)_F, -)\}$. Under this isomorphism, the involution σ becomes $\operatorname{Int}(a)\circ\vartheta^t$, so that f is an isomorphism of algebras with involution from (A,σ) to $(M_n(D), \operatorname{Int}(a)\circ\vartheta^t)$.

By Proposition 3.8(1) and then (3), the involution σ is positive at P, and thus we can take for a a PSD matrix. We then obtain from Lemma 3.9 an isomorphism of F-algebras with involution $g : (A,\sigma) \to (M_n(D),\vartheta^t)$. In particular $g(\mathscr{P})$ is a positive cone on $(M_n(D),\vartheta^t)$ over F and $1 \in g(\mathscr{P})$, so that $g(\mathscr{P}) = \mathrm{PSD}$ (cf. Remark 3.6(5)).

(2) This is clear since any element of PSD is of the form $\vartheta(a)^t a$ in $M_n(D)$ by the principal axis theorem (see [20, Corollary 6.2] for the quaternion case). □

For the purpose of obtaining a first-order theory, we check that the fact that a unary relation is a positive cone can be expressed by a first-order formula in models of CSA-I_m:

Lemma 3.11. *Let $m \in \mathbb{N}$.*

(1) There is an $L_R \cup \{\underline{\sigma}, \underline{\mathscr{P}}\}$-formula PC_m such that for every $\mathscr{A} \models CSA\text{-}I_m$ and every interpretation $\underline{\mathscr{P}}^{\mathscr{A}}$ of $\underline{\mathscr{P}}$ in $\mathscr{A}$: $\mathscr{A} \models PC_m$ if and only if $\underline{\mathscr{P}}^{\mathscr{A}}$ is a positive cone on (A,σ).

(2) There is an $L_R \cup \{\underline{\sigma}, \underline{\mathscr{P}}, \underline{P}\}$-formula PC'_m such that for every $\mathscr{A} \models CSA\text{-}I_{m,of}$ and every interpretation $\underline{\mathscr{P}}^{\mathscr{A}}$ of $\underline{\mathscr{P}}$ in $\mathscr{A}$: $\mathscr{A} \models PC'_m$ if and only if $\underline{\mathscr{P}}^{\mathscr{A}}$ is a positive cone on (A,σ) over $\underline{P}^{\mathscr{A}}$.

Proof. We prove the first statement (the second is obtained by taking the conjunction of PM_m and "$\underline{P} = \underline{\mathscr{P}}_F$", which is first-order in $L_R \cup \{\underline{P}, \underline{\mathscr{P}}\}$). We introduce some notation first:

Let $\mathscr{A} = (A, \sigma)$ as algebra with involution over F. If $\mathscr{P}$ is a prepositive cone on (A, σ) (over some ordering $P := \mathscr{P}_F$), and $a \in \mathrm{Sym}(A, \sigma)$, we define

$$\mathscr{P}[a] := \{p + \sum_{i=1}^{k} u_i \sigma(x_i) a x_i \mid p \in \mathscr{P},\ k \in \mathbb{N},\ u_i \in \mathscr{P}_F,\ x_i \in A\}.$$

It is easily seen that $\mathscr{P}[a]$ satisfies properties (P1) up to (P4) of the definition of prepositive cone, and will be the smallest prepositive cone over P containing both $\mathscr{P}$ and a if it is proper (i.e., satisfies (P5)).

The statement "$\underline{\mathscr{P}}$ is a prepositive cone" is clearly first-order, so we only need to express that it is a maximal prepositive cone. This can be done by expressing:

$$\forall a \in \mathrm{Sym}(A, \sigma) \setminus \mathscr{P} \quad \underline{\mathscr{P}}[a] = \mathrm{Sym}(A, \sigma),$$

which itself can be expressed by a first-order formula if "$z \in \underline{\mathscr{P}}[a]$" can. By definition, $\mathscr{P}[a]$ is the convex cone over $\mathscr{P}_F$ generated by the set

$$\mathscr{P} \cup \{\sigma(x) a x \mid x \in A\}.$$

This is a convex cone in A (with respect to the ordering $\mathscr{P}_F$), and $\dim_F A = m$, so by Carathéodory's theorem (which holds for ordered fields, see for instance the proof of [6, Chapter I, Theorem 2.3]), for $z \in A$ we have

$$z \in \mathscr{P}[a]$$

$$\Leftrightarrow$$

$$\exists p \in \mathscr{P} \exists u_1, x_1, \ldots, u_{m+1}, x_{m+1} \ \bigwedge_{i=1}^{m+1} u_i \in \mathscr{P}_F \wedge z = p + \sum_{i=1}^{m+1} u_i \sigma(x_i) a x_i,$$

which is first-order since "$u \in \mathscr{P}_F$" clearly is. □

3.2. **Model-completeness.** We define the theory of ordered central simple algebras with involution of dimension m, and the same theory over a real closed field, in the language $L_{\text{OCSA-I}} := L_{\text{CSA-I}} \cup \{\underline{\mathscr{P}}\}$, to be:

$$\text{OCSA-I}_m := \text{CSA-I}_m \cup \{\underline{\mathscr{P}} \text{ is a positive cone}\}, \text{ and}$$

$$\text{OCSA-I}_{m,\mathrm{rcf}} := \text{CSA-I}_{m,\mathrm{rcf}} \cup \{\underline{\mathscr{P}} \text{ is a positive cone}\}$$

(with reference to Lemma 3.11 for the axiomatization of "$\underline{\mathscr{P}}$ is a positive cone").

Proposition 3.12. *The theory OCSA-I$_{m,rcf}$ is model-complete in $L_{\text{CSA-I}} \cup \{\underline{\mathscr{P}}\}$, and the theory OCSA-I$_{m,rcf} \cup \{\underline{\mathrm{Trd}}$ is the reduced trace$\}$ (cf. Lemma 1.6) is model-complete in $L_{\text{CSA-I}} \cup \{\underline{\mathscr{P}}, \underline{\mathrm{Trd}}\}$.*

Proof. Let $\mathscr{M} \subseteq \mathscr{N}$ be two models of $\text{OCSA-I}_{m,\mathrm{rcf}}$. By Lemma 2.5, we have a diagram as in statement (2) of this Lemma. We know that $\pm\mathrm{PSD}(M_n(D), \vartheta^t) = \pm\mathrm{HS}(M_n(D), \vartheta^t)$ are the only positive cones on $(M_n(D), \vartheta^t)$ over the unique ordering of F (see Remark 3.6(6) and (5)), and thus that $\pm a\mathrm{HS}(M_n(D), \vartheta^t)$ are the only positive cones on $(M_n(D), \mathrm{Int}(a) \circ \vartheta^t)$ over the unique ordering of F (Remark 3.6(2)), so that $f(\underline{\mathscr{P}}^{\mathscr{M}}) = \varepsilon a\mathrm{HS}(M_n(D), \vartheta^t)$ for some $\varepsilon \in \{-1, 1\}$. Similarly, $g(\underline{\mathscr{P}}^{\mathscr{N}}) = \delta a\mathrm{HS}(M_n(E), (\vartheta')^t)$ for some $\delta \in \{-1, 1\}$. Since $f(\underline{\mathscr{P}}^{\mathscr{M}}) \subseteq g(\underline{\mathscr{P}}^{\mathscr{N}})$, we have $\delta = \varepsilon$. Therefore $\underline{\mathscr{P}}^{\mathscr{M}} = f^{-1}(a)\mathrm{HS}(\mathscr{M}, \underline{\sigma}^{\mathscr{M}})$ and $\underline{\mathscr{P}}^{\mathscr{N}} = f^{-1}(a)\mathrm{HS}(\mathscr{N}, \underline{\sigma}^{\mathscr{N}})$ are defined by the same $L_{\text{CSA-I}}$-formula (with parameter a) in $\mathscr{M}$ and $\mathscr{N}$. Both statements then follow from Lemma 2.5(3). □

Lemma 3.13. *Let (A,σ) be a central algebra with involution over F, and let $\mathscr{P}$ be a positive cone on (A,σ) over $P \in X_F$. Let (L,Q) be an ordered extension of (F,P). Then $A \otimes_F L$ is a central simple algebra.*

Proof. If $F = Z(A)$ the result is clear (see for instance [9, Theorem 1.1(3) and (4)]), so we can assume that $F \neq Z(A)$, i.e., that σ is of the second kind. In particular $Z(A) = F(\sqrt{d})$ for some $d \in F$. By hypothesis $P \in X_F \setminus \text{Nil}[A,\sigma]$ (see [4, Proposition 6.6], where $\widetilde{X_F} := X_F \setminus \text{Nil}[A,\sigma]$), so that $d \notin P$ by [4, Proposition 8.4]. Therefore $\sqrt{d} \notin L$ (since $\sqrt{d}$ is not in the real closure of (F,P)) and

$$A \otimes_F L = A \otimes_{Z(A)} Z(A) \otimes_F L = A \otimes_{Z(A)} L[X]/(X^2 - d) = A \otimes_{Z(A)} L(\sqrt{d}),$$

which is central simple (see again [17, Chapter 8, Corollary 5.1]). □

Corollary 3.14. *The theory OCSA-$I_{m,rcf}$ is the model-companion of OCSA-I_m, and the theory OCSA-$I_{m,rcf} \cup \{\underline{\text{Trd}}$ is the reduced trace$\}$ is the model-companion of OCSA-$I_m \cup \{\underline{\text{Trd}}$ is the reduced trace$\}$.*

Proof. We only prove the second statement, since the first one is similar.

By Proposition 3.12 it suffices to show that OCSA-$\text{I}_m \cup \{\underline{\text{Trd}}$ is the reduced trace$\}$ and OCSA-$\text{I}_{m,\text{rcf}} \cup \{\underline{\text{Trd}}$ is the reduced trace$\}$ are cotheories. Let $\mathscr{M} = (A,\sigma)$ be a model of OCSA-I_m, let $F := \underline{F}^{\mathscr{M}}$, let $P \in X_F$ be such that $\underline{\mathscr{P}}^{\mathscr{M}}$ is over P, and let F_P be a real closure of F at P. By Lemma 3.13, the algebra $(A \otimes_F L, \sigma \otimes \text{id})$ is a central simple algebra with involution, and by [4, Proposition 5.8], it is equipped with a positive cone containing $\mathscr{P} \otimes 1$. It is thus a model of OCSA-$\text{I}_{m,\text{rcf}}$ and the inclusion $A \to A \otimes_F L$, $a \mapsto a \otimes 1$ is a morphism in the language $L_{\text{CSA-I}} \cup \{\underline{\mathscr{P}}, \underline{\text{Trd}}\}$, since this map preserves the reduced trace by definition of the reduced trace. □

Recall from Remark 3.4(1) that OCSA-$I_{m,rcf}$ does not have the amalgamation property over substructures (take the PSD matrices for positive cone on $(M_n(\mathbb{R}), t)$ and on $(M_n((-1,-1)_{\mathbb{R}}, -^t)$ to turn them into models of OCSA-$\text{I}_{m,\text{rcf}}$), and in particular does not have quantifier elimination (see [7, Proposition 3.5.19]).

3.3. Quantifier elimination with an involution of specified type.

Definition 3.15. We introduce a new constant symbol $\underline{a}$ and define the following theories in the language $L_{\text{CSA-I}} \cup \{\underline{\mathscr{P}}, \underline{\text{Trd}}, \underline{a}\}$, where we specify the type of the involution:

OCSA-$\text{I}^+_{m,\text{rcf}} :=$ OCSA-$\text{I}_{m,\text{rcf}} \cup \{\underline{\text{Trd}}$ is the reduced trace$\} \cup \{\underline{a} \in \underline{\mathscr{P}},\ \underline{a}$ is invertible$\}$

and

OCSA-$\text{OI}^+_{m,\text{rcf}} :=$ OCSA-$\text{I}^+_{m,\text{rcf}} \cup \{\underline{\sigma}$ is an orthogonal involution$\}$

OCSA-$\text{SI}^+_{m,\text{rcf}} :=$ OCSA-$\text{I}^+_{m,\text{rcf}} \cup \{\underline{\sigma}$ is a symplectic involution$\}$

OCSA-$\text{UI}^+_{m,\text{rcf}} :=$ OCSA-$\text{I}^+_{m,\text{rcf}} \cup \{\underline{\sigma}$ is a unitary involution$\}$

(with reference to Lemma 1.6 for the axiomatization of the reduced trace). Note that every positive cone contains an invertible element by [4, Lemma 3.6].

Remark 3.16. We can replace $\underline{a}$ by 1 in the theories above, so that the models will have positive cones that contain 1 (and will correspond to situations where the involution is positive, see Proposition 3.8(1). In this case there is no need to add the new constant $\underline{a}$ to the language.

Proposition 3.17. *The theories OCSA-OI$^+_{m,rcf}$, OCSA-SI$^+_{m,rcf}$ and OCSA-UI$^+_{m,rcf}$ each have quantifier elimination in the language $L_{CSA\text{-}I} \cup \{\underline{\mathrm{Trd}}, \underline{\mathscr{P}}\}$.*

Proof. We only prove it for OCSA-OI$^+_{m,\mathrm{rcf}}$, since the others are similar.

By Proposition 3.12, the theory OCSA-OI$^+_{m,\mathrm{rcf}}$ is model-complete in the language $L_{\mathrm{CSA\text{-}I}} \cup \{\underline{\mathscr{P}}, \underline{\mathrm{Trd}}\}$, so we only need to show that it has the amalgamation property over substructures. Let $\mathscr{M}, \mathscr{N} \models$ OCSA-OI$^+_{m,\mathrm{rcf}}$ and let $\mathscr{A}$ be a common $L_{\mathrm{CSA\text{-}I}} \cup \{\underline{\mathrm{Trd}}, \underline{\mathscr{P}}\}$-substructure of $\mathscr{M}$ and $\mathscr{N}$. Let $\iota_{\mathscr{M}}$ and $\iota_{\mathscr{N}}$ be the inclusions of $\mathscr{A}$ in $\mathscr{M}$ and $\mathscr{N}$, respectively.

In order to simplify the notation, we write a for the element $a^{\mathscr{M}} = a^{\mathscr{A}} = a^{\mathscr{N}}$, F for the field $\underline{F}^{\mathscr{M}}$, and L for the field $\underline{F}^{\mathscr{N}}$.

We turn $\mathscr{M}$ and $\mathscr{N}$ into $L_{\mathrm{CSA\text{-}I}} \cup \{\underline{P}, \underline{\mathrm{Trd}}, \underline{\mathscr{P}}\}$-structures by interpreting $\underline{P}$ in $\mathscr{M}$ and $\mathscr{N}$ by the set defined by the quantifier-free $L_{\mathrm{CSA\text{-}I}} \cup \{\underline{\mathscr{P}}\}$-formula (cf. Remark 3.6(1)):

$$x \in \underline{F} \wedge x\underline{a} \in \underline{\mathscr{P}}.$$

Since this formula is quantifier-free, $\mathscr{A}$ is an $L_{\mathrm{CSA\text{-}I}} \cup \{\underline{P}, \underline{\mathrm{Trd}}, \underline{\mathscr{P}}\}$-substructure of $\mathscr{M}$ and $\mathscr{N}$, i.e., the maps $\iota_{\mathscr{M}}$ and $\iota_{\mathscr{N}}$ are $L_{\mathrm{CSA\text{-}I}} \cup \{\underline{P}, \underline{\mathrm{Trd}}, \underline{\mathscr{P}}\}$-morphisms.

Scaling the involutions by a, we obtain the following diagram (we only indicate the base set, the involution, and the positive cone in each entry):

$$\begin{array}{ccc} \mathscr{M}' := (\mathscr{M}, \mathrm{Int}(a) \circ \underline{\sigma}^{\mathscr{M}}, a\underline{\mathscr{P}}^{\mathscr{M}}) & & \mathscr{N}' := (\mathscr{N}, \mathrm{Int}(a) \circ \underline{\sigma}^{\mathscr{N}}, a\underline{\mathscr{P}}^{\mathscr{N}}) \\ \nwarrow{\scriptstyle \iota_{\mathscr{M}}} & & \nearrow{\scriptstyle \iota_{\mathscr{N}}} \\ & \mathscr{A}' := (\mathscr{A}, \mathrm{Int}(a) \circ \underline{\sigma}^{\mathscr{A}}, a\underline{\mathscr{P}}^{\mathscr{A}}) & \end{array}$$

It is clear that $\iota_{\mathscr{M}}$ and $\iota_{\mathscr{N}}$ are still $L_{\mathrm{CSA\text{-}I}} \cup \{\underline{P}, \underline{\mathrm{Trd}}, \underline{\mathscr{P}}\}$-morphisms. Moreover, $a\underline{\mathscr{P}}^{\mathscr{M}}$ is a positive cone on $(\mathscr{M}, \mathrm{Int}(a) \circ \underline{\sigma}^{\mathscr{M}})$ by [4, Proposition 4.4], and contains 1 (indeed $a^2 \in a\underline{\mathscr{P}}^{\mathscr{M}}$ so that, using that $\underline{\sigma}^{\mathscr{M}}(a) = a$ and property (P3) of positive cones, $\underline{\sigma}^{\mathscr{M}}(a^{-1})a^2a^{-1} = 1 \in a\underline{\mathscr{P}}^{\mathscr{M}}$). Similarly, $1 \in a\underline{\mathscr{P}}^{\mathscr{N}}$.

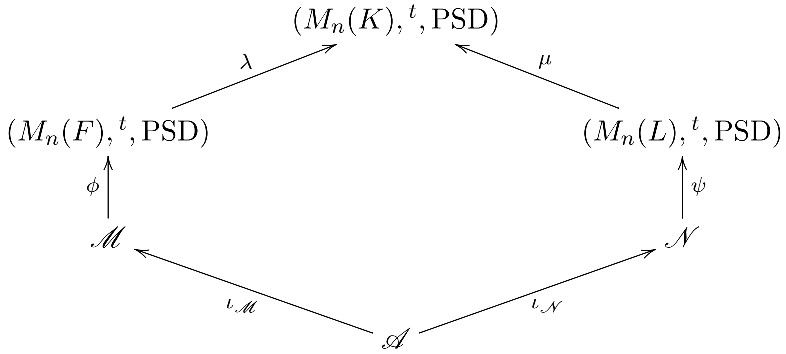
Observe that the involution $\mathrm{Int}(a) \circ \sigma$ is of the same type as σ since a is symmetric (it follows from $a \in \mathscr{P}$), by [9, Proposition 2.7(3)]. Therefore, by Corollary 3.10, we have two $L_{\mathrm{CSA\text{-}I}} \cup \{\underline{P}, \underline{\mathrm{Trd}}, \underline{\mathscr{P}}\}$-isomorphisms (whose images are $M_n(F)$ and $M_n(E)$ since the involution is orthogonal, see the end of Section 1.1):

$$\phi : \mathscr{M}' \to (M_n(F), {}^t, \mathrm{PSD}) \text{ and } \psi : \mathscr{N}' \to (M_n(L), {}^t, \mathrm{PSD})$$

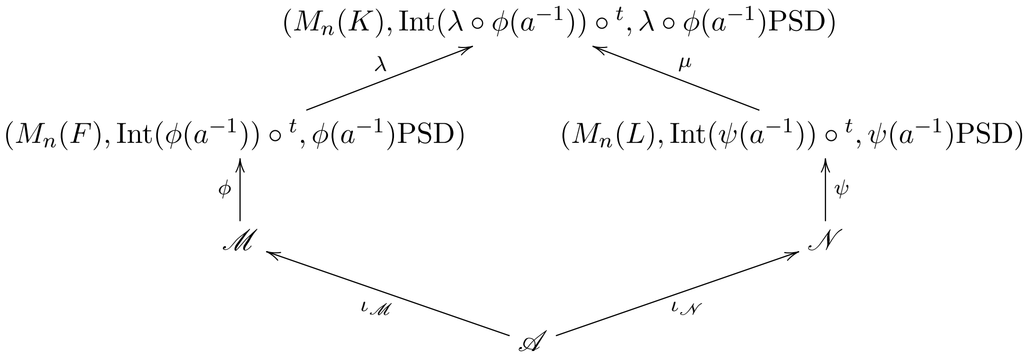
where $\underline{\mathrm{Trd}}$ is interpreted by the reduced trace in $M_n(F)$ and $M_n(L)$, and the maps ϕ and ψ respect $\underline{\mathrm{Trd}}$ by Lemma 1.7.

The two $L_{\mathrm{CSA\text{-}I}} \cup \{\underline{P}, \underline{\mathrm{Trd}}\}$-structures $(M_n(F), {}^t, F, \mathrm{Trd})$ and $(M_n(L), {}^t, L, \mathrm{Trd})$ are models of the theory of $(M_n(F), -^t, F, \mathrm{Trd})$ ($F \equiv L$ and both structures are interpretable in the same way in F and L) and $\phi \circ \iota_{\mathscr{M}}$ as well as $\psi \circ \iota_{\mathscr{N}}$ are $L_{\mathrm{CSA\text{-}I}} \cup \{\underline{P}, \underline{\mathrm{Trd}}\}$-morphisms.

By Proposition 3.3, we can amalgamate this diagram using two $L_{\mathrm{CSA\text{-}I}} \cup \{\underline{P}, \underline{\mathrm{Trd}}\}$-morphisms $\lambda : M_n(F) \to M_n(K)$ and $\mu : M_n(L) \to M_n(K)$. Since PSD is the set of hermitian squares in all three structures (Corollary 3.10(2)), λ and μ respect PSD, so that the following diagram consists of $L_{\text{CSA-I}} \cup \{\underline{P}, \underline{\text{Trd}}, \underline{\mathscr{P}}\}$-morphisms:



We now "scale back" and check that the morphisms (which are the same at the level of the elements) are still morphisms in the language $L_{\text{CSA-I}} \cup \{\underline{P}, \underline{\text{Trd}}, \underline{\mathscr{P}}\}$:



The morphisms fix the elements of F pointwise, so are clearly morphisms for $\{\underline{F}, \underline{P}\}$. We check for the involutions and the positive cones:

- The map ϕ. For the positive cones, we want $\phi(\underline{\mathscr{P}}^{\mathscr{M}}) \subseteq \phi(a^{-1})\text{PSD}$, which is clear since $\phi(a\underline{\mathscr{P}}^{\mathscr{M}}) \subseteq \text{PSD}$ by construction of ϕ. For the involutions, we want $\phi \circ \sigma(x) = \text{Int}(\phi(a^{-1})) \circ {}^t \circ \phi(x)$ for every $x \in \mathscr{M}$. We successively have, for every $x' \in \mathscr{M}$:
$$\begin{aligned} \phi \circ \text{Int}(a) \circ \underline{\sigma}^{\mathscr{M}}(x') = {}^t \circ \phi(x') \quad &(\text{by definition of } \phi) \\ \Leftrightarrow \phi(a\underline{\sigma}^{\mathscr{M}}(x')a^{-1}) = \phi(x')^t& \\ \Leftrightarrow \phi(\underline{\sigma}^{\mathscr{M}}(a^{-1}x'a)) = \phi(x')^t \quad &(\text{using that } \underline{\sigma}^{\mathscr{M}}(a) = a) \\ \Leftrightarrow \phi \circ \underline{\sigma}^{\mathscr{M}}(x) = \phi(axa^{-1})^t \quad &(\text{where } x := a^{-1}x'a) \\ \Leftrightarrow \phi \circ \underline{\sigma}^{\mathscr{M}}(x) = \phi(a^{-1})^t\phi(x)^t\phi(a)^t& \\ \Leftrightarrow \phi \circ \underline{\sigma}^{\mathscr{M}}(x) = \text{Int}(\phi(a^{-1})^t) \circ {}^t \circ \phi(x)& \end{aligned}$$
and the result follows since $\phi(a)^t = \phi(a)$ (indeed, using that $\phi \circ \text{Int}(a) \circ \underline{\sigma}^{\mathscr{M}} = {}^t \circ \phi$ by definition of ϕ, we obtain $\phi(a) = \phi(a\underline{\sigma}^{\mathscr{M}}(a)a^{-1}) = \phi(a)^t$).
- The map ψ: It is the exact same argument as for ϕ.
- The map λ. It is clear for the positive cones. For the involutions, we want $\lambda \circ \text{Int}(\phi(a^{-1})) \circ {}^t = \text{Int}(\lambda \circ \phi(a^{-1})) \circ {}^t \circ \lambda$. Let $x \in M_n(F)$. Then, using that ${}^t \circ \lambda = \lambda \circ {}^t$ (by definition of λ), we get
$$\begin{aligned} \text{Int}(\lambda \circ \phi(a^{-1})) \circ {}^t \circ \lambda(x) &= \lambda \circ \phi(a^{-1})\lambda(x)^t\lambda \circ \phi(a) \\ &= \lambda(\phi(a^{-1})x^t\phi(a)) \end{aligned}$$

$$= \lambda \circ \mathrm{Int}(\phi(a^{-1})) \circ {}^t(x).$$

- The map μ. Recall that by choice of λ and μ we have $\mu \circ \psi(a) = \lambda \circ \phi(a)$ and thus $\mu \circ \psi(a^{-1}) = \lambda \circ \phi(a^{-1})$. For the positive cones, and since $\mu(\mathrm{PSD}) \subseteq \mathrm{PSD}$, we have $\mu(\psi(a^{-1})\mathrm{PSD}) = \mu \circ \psi(a^{-1})\mu(\mathrm{PSD}) \subseteq \lambda \circ \phi(a^{-1})\mathrm{PSD}$. For the involutions, we want $\mu \circ \mathrm{Int}(\psi(a^{-1})) \circ {}^t = \mathrm{Int}(\lambda \circ \phi(a^{-1})) \circ {}^t \circ \mu$, i.e., $\mu \circ \mathrm{Int}(\psi(a^{-1})) \circ {}^t = \mathrm{Int}(\mu \circ \psi(a^{-1})) \circ {}^t \circ \mu$. Using that ${}^t \circ \mu = \mu \circ {}^t$ (by definition of μ), we get

$$\begin{aligned}\mathrm{Int}(\mu \circ \psi(a^{-1})) \circ {}^t \circ \mu(x) &= \mu \circ \psi(a^{-1})\mu(x)^t\mu \circ \psi(a)\\ &= \mu(\psi(a^{-1})x^t\psi(a))\\ &= \mu \circ \mathrm{Int}(\psi(a^{-1})) \circ {}^t(x).\end{aligned}$$

□

Corollary 3.18. *Each of the final three theories from Definition 3.15 is the model-companion of the same theory where we do not specify that $\underline{F}$ is real closed.*

Proof. We prove it for $T := \mathrm{OCSA\text{-}SI}^+_{m,\mathrm{rcf}}$, the argument is the same for the other two. Let T_0 be this theory whithout specifying that $\underline{F}$ is real closed. We know that T is model-complete by Proposition 3.17, so we only have to show that T and T_0 are cotheories: Let (A, σ) be a central simple algebra with involution over F that is a model of T_0, so that there is a positive cone $\mathscr{P}$ on (A, σ). Let L be a real closure of F at the ordering $\mathscr{P}_F$. Then $(A \otimes_F L, \sigma \otimes \mathrm{id})$ is a central simple algebra with involution over L (see Lemma 3.13), is equipped with a positive cone $\mathscr{Q}$ containing $\mathscr{P} \otimes 1$ ([4, Proposition 5.8]), and is thus a model of T that contains $(A, \sigma, \mathscr{P}, \mathrm{Trd}, a)$ as an $L_{\mathrm{CSA\text{-}I}} \cup \{\underline{\mathscr{P}}, \underline{\mathrm{Trd}}, \underline{a}\}$-substructure (the reduced trace of an element of a remains the same after scalar extension, by definition of the reduced trace). □

4. Correspondence between positive cones and morphisms

The model-completeness of $\mathrm{OCSA\text{-}I}_{m,\mathrm{rcf}}$ makes it interesting to point out that positive cones on algebras with involution are in bijection with morphisms of $L_{\mathrm{CSA\text{-}I}}$-structures into models of $\mathrm{OCSA\text{-}I}_{m,\mathrm{rcf}}$.

Lemma 4.1. *Let (A, σ) be a central simple algebra with involution over F and let $P \in X_F$. Let F_P be a real closure of F at $P \in X_F$. Then:*

1. *If σ is symplectic,* $\mathrm{Nil}[A, \sigma] = \{P \mid A \otimes_F F_P \cong M_n(F_P)\}$.
2. *If σ is orthogonal,* $\mathrm{Nil}[A, \sigma] = \{P \mid A \otimes_F F_P \cong M_n((-1, -1)_{F_P})\}$.
3. *If σ is unitary,*

$$\begin{aligned}\mathrm{Nil}[A, \sigma] = \{P \mid A \otimes_F F_P \cong M_n(F_P) \times M_n(F_P) \textit{ or}\\ A \otimes_F F_P \cong M_{n/2}((-1, -1)_{F_P}) \otimes M_{n/2}((-1, -1)_{F_P})\}.\end{aligned}$$

Proof. The original definition of $\mathrm{Nil}[A, \sigma]$ ([1, Definition 3.7]) is the list given in the Lemma (where the case $M_{n/2}((-1, -1)_{F_P}) \otimes M_{n/2}((-1, -1)_{F_P})$ for σ unitary was missing, an omission corrected in [2, p. 499]). The fact that this original definition coincides with the one used in this paper (the set of orderings at which the signatures of all hermitian forms are zero) is [1, Theorem 6.1]. □

Proposition 4.2. *Let $(A, \sigma) \models \mathrm{CSA\text{-}I}_m$, $(B, \tau) \models \mathrm{OCSA\text{-}I}_{m,rcf}$ and let $f : A \to B$ be a morphism of $L_{\mathrm{CSA\text{-}I}}$-structures, i.e., f is a morphism of rings with involution such that $f(\underline{F}^A) \subseteq \underline{F}^B$. Then $f^{-1}(\underline{\mathscr{P}}^B)$ is a positive cone on (A, σ) over $\underline{P}^A$.*

Proof. We write $(F, P) := (\underline{F}^A, \underline{P}^A)$ and $L := \underline{F}^B$, with Q the unique ordering on L. We first prove that $P \notin \mathrm{Nil}[A, \sigma]$. For this we consider two cases:

(1) If σ is of the first kind, i.e., $F = Z(A)$. We extend f to $f' : A \otimes_F L \to B$, $f'(a \otimes \ell) = f(a)\ell$ (using the action of F on L via f for the tensor product). Since $F = Z(A)$, $A \otimes_F L$ is simple, and f' is injective and therefore bijective (recall that $\dim_L A \otimes_F L = \dim_F A = \dim_L B$). Since f' is easily seen to respect the involutions, f' is an isomorphism of algebras with involution from $(A \otimes_F L, \sigma \otimes \mathrm{id})$ to (B, τ). Therefore, by Lemma 4.1, and since $Q \notin \mathrm{Nil}[B, \tau]$, we have $P \notin \mathrm{Nil}[A, \sigma]$.

(2) If σ is of the second kind, i.e., $Z(A) = F(\sqrt{d})$ for some $d \in F$. Therefore $\sigma(\sqrt{d}) = -\sqrt{d}$, and thus $\tau(f(\sqrt{d})) = -f(\sqrt{d})$. Assume that $P \in \mathrm{Nil}[A, \sigma]$, so that $d \in P$ ([4, Proposition 8.4]). Since $\sqrt{d}$ is invertible in A, $f(\sqrt{d})$ is invertible in B. Moreover $f(\sqrt{d})^2 = f(d) \in L$. Since f is a morphism of ordered fields from (F, P) to (L, Q) real closed, we have $f(\sqrt{d}) \in Q$ and there is $\alpha \in L$ such that $\alpha^2 = f(d)$. Therefore, in the field $L(f(\sqrt{d}))$ the elements $\alpha, -\alpha, f(\sqrt{d})$ are roots of $X^2 - f(d)$, so that $f(\sqrt{d}) = \pm\alpha \in L$ and thus $\tau(f(\sqrt{d})) = f(\sqrt{d})$, contradiction.

Since $P \notin \mathrm{Nil}[A, \sigma]$, there is a positive cone $\mathscr{P}$ on (A, σ) over P, and $A \otimes_F L$ is simple by Lemma 3.13. Going back to the argument presented in (1) above, the map f' is then an isomorphism of algebras with involutions, even if σ is of the second kind.

By [4, Proposition 5.8] there is a positive cone on $(A \otimes_F L, \sigma \otimes \mathrm{id})$ over Q containing $\mathscr{P} \otimes 1$ (the Proposition is written for an inclusion of fields, but applies also here: replace A by $A \otimes_F f(F)$, then use the inclusion from $f(F)$ into L), and thus there is a positive cone $\mathcal{S}$ on (B, τ) over Q such that $f'(\mathscr{P} \otimes 1) \subseteq \mathcal{S}$. Since $\mathscr{Q}$ and $-\mathscr{Q}$ are the only positive cones on (B, τ) over Q, up to replacing $\mathscr{P}$ by $-\mathscr{P}$ we must have $f'(\mathscr{P} \otimes 1) \subseteq \mathscr{Q}$. It follows that $\mathscr{P} \subseteq f^{-1}(\mathscr{Q})$, and thus that $\mathscr{P} = f^{-1}(\mathscr{Q})$ since $\mathscr{P}$ is a positive cone and $f^{-1}(\mathscr{Q})$ is easily seen to be a prepositive cone ((P1), (P2), (P3) are clear, (P5) holds since f is injective because A is simple, and for (P4) it suffices to check that $P \subseteq (f^{-1}(\mathscr{Q}))_F$, the other inclusion following from (P5)). □

Conversely, every positive cone on (A, σ) can be obtained in this way:

Proposition 4.3. *Let $(A, \sigma) \models$ CSA-I$_m$ and let $\mathscr{P}$ be a positive cone on (A, σ) over $P \in X_{\underline{F}^{\mathscr{A}}}$. Then there is a model (B, τ) of OCSA-I$_{m,rcf}$ and a morphism of $L_{CSA\text{-}I}$-structures $f : (A, \sigma) \to (B, \tau)$ such that $\mathscr{P} = f^{-1}(\underline{\mathscr{P}}^B)$.*

Proof. Let $F := \underline{F}^A$, and take for f the canonical map $A \to A \otimes_F F_P$, $a \mapsto a \otimes 1$, where F_P is a real closure of F at $\underline{P}^A$. The F_P-algebra $A \otimes_F F_P$ is central simple by Lemma 3.13 and by [4, Proposition 5.9] there is a positive cone $\mathscr{Q}$ on $(A \otimes_F F_P, \sigma \otimes \mathrm{id})$ over the unique ordering of F_P, so that $f(\mathscr{P}) \subseteq \mathscr{Q}$. Let $\mathscr{B}$ be the natural $L_{\text{CSA-I}}$-structure on $(A \otimes_F F_P, \sigma \otimes \mathrm{id})$, in which we also interpret the symbol $\underline{\mathscr{P}}$ by $\mathscr{Q}$, thus turning (B, τ) into a model of OCSA-I$_{m,\text{rcf}}$.

We check that $\mathscr{P} = f^{-1}(\mathscr{Q})$. By construction, we have $\mathscr{P} \subseteq f^{-1}(\mathscr{Q})$. It is easy to see that $f^{-1}(\mathscr{Q})$ is a prepositive cone on (A, σ) over P. Since $\mathscr{P}$ is a positive cone on (A, σ) over P, we must have $\mathscr{P} = f^{-1}(\mathscr{Q})$. □

References

[1] Vincent Astier and Thomas Unger. Signatures of hermitian forms and the Knebusch trace formula. *Math. Ann.*, 358(3-4):925–947, 2014.

[2] Vincent Astier and Thomas Unger. Signatures of hermitian forms and "prime ideals" of Witt groups. *Adv. Math.*, 285:497–514, 2015.

[3] Vincent Astier and Thomas Unger. Signatures of Hermitian forms, positivity, and an answer to a question of Procesi and Schacher. *J. Algebra*, 508:339–363, 2018.

[4] Vincent Astier and Thomas Unger. Positive cones on algebras with involution. *Adv. Math.*, 361:106954, 48, 2020.

[5] Vincent Astier and Thomas Unger. Positive cones and gauges on algebras with involution. *Int. Math. Res. Not. IMRN*, (10):7259–7303, 2022.

[6] Alexander Barvinok. *A course in convexity*, volume 54 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2002.

[7] C. C. Chang and H. J. Keisler. *Model theory*, volume 73 of *Studies in Logic and the Foundations of Mathematics*. North-Holland Publishing Co., Amsterdam, third edition, 1990.

[8] Igor Klep and Marcus Tressl. A model theoretic perspective on matrix rings. *Math. Z.*, 309(3):Paper No. 45, 2025.

[9] Max-Albert Knus, Alexander Merkurjev, Markus Rost, and Jean-Pierre Tignol. *The book of involutions*, volume 44 of *American Mathematical Society Colloquium Publications*. American Mathematical Society, Providence, RI, 1998. With a preface in French by J. Tits.

[10] T. Y. Lam. *A first course in noncommutative rings*, volume 131 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1991.

[11] H. C. Lee. Eigenvalues and canonical forms of matrices with quaternion coefficients. *Proc. Roy. Irish Acad. Sect. A*, 52:253–260, 1949.

[12] David Marker. *Model theory*, volume 217 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 2002. An introduction.

[13] C. Procesi. The invariant theory of $n \times n$ matrices. *Advances in Math.*, 19(3):306–381, 1976.

[14] Claudio Procesi. *Lie groups*. Universitext. Springer, New York, 2007. An approach through invariants and representations.

[15] Claudio Procesi and Murray Schacher. A non-commutative real Nullstellensatz and Hilbert's 17th problem. *Ann. of Math. (2)*, 104(3):395–406, 1976.

[16] David J. Saltman. *Lectures on division algebras*, volume 94 of *CBMS Regional Conference Series in Mathematics*. Published by American Mathematical Society, Providence, RI; on behalf of Conference Board of the Mathematical Sciences, Washington, DC, 1999.

[17] Winfried Scharlau. *Quadratic and Hermitian forms*, volume 270 of *Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1985.

[18] K. S. Sibirskij. Algebraic invariants for a set of matrices. *Sib. Math. J.*, 9:115–124, 1968.

[19] N. A. Wiegmann. Necessary and sufficient conditions for unitary similarity. *J. Austral. Math. Soc.*, 2:122–126, 1961/62.

[20] Fuzhen Zhang. Quaternions and matrices of quaternions. *Linear Algebra Appl.*, 251:21–57, 1997.