Elementary equivalence of lattices of open sets definable in o-minimal expansions of real closed fields

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1 Introduction

It is well-known that any two real closed fields R and S are elementarily equivalent. We can then consider some simple constructions of new structures out of real closed fields, and try to determine if these constructions, when applied to R and S, give elementarily equivalent structures. We can for instance consider $\mathrm{def}(R^n, R)$, the ring of definable functions from R^n to R, and we obtain without difficulty that the rings $\mathrm{def}(R^n, R)$ and $\mathrm{def}(S^n, S)$ are elementarily equivalent ([A]).

However, if we consider $\mathrm{cdef}(R^n, R)$, the ring of continuous definable functions from R^n to R, the situation becomes more complicated: Unpublished results of M. Tressl show that, for $n > 1$, $\mathrm{cdef}(R^n, R)$ defines the set of constant functions with integer value, by a formula that is independent of R and n. Therefore we may have $\mathrm{cdef}(R^n, R) \not\equiv \mathrm{cdef}(S^n, S)$, for instance if one field is Archimedean and the other not.

It shows that introducing conditions linked to the topology of the real closed field may present an obstacle to elementary equivalence. To understand the situation better it is natural to consider simpler structures than rings of continuous definable functions, but that still demand some topological information from the field. This is what we do in this paper, where we consider the lattices of open definable sets. We show in particular, in Corollary 2.16, that if R and S are elementarily equivalent o-minimal expansions of real closed fields, then the lattices of open definable subsets of R^n and of open definable subsets of S^n are $L_{\infty\omega}$-elementarily equivalent in the language of bounded lattices expanded by predicates for the dimension and Euler characteristic. The proof is done by a back-and-forth argument.

It is worth noting that by [G, Corollary 1] and for $n > 1$, the lattice of semi-algebraic open subsets of R^n (for R real closed field) is undecidable. In particular, there can be no description of the theory of such lattices in terms of "simpler" structures that would be constructive enough to give decidability results.

2 Boolean algebras of definable sets equipped with predicates for dimension, Euler characteristic and open sets

We follow the notation and definitions of [V], in particular we use the definition of complex that appears in this book. We work with o-minimal expansions of real closed fields, i.e. with real closed fields that are o-minimal in a fixed language containing L_{of}, the language of ordered fields.

Concerning the notation, we denote by $\mathbb{N}_+$ the set of positive integers, by $L_{BA} = \{\vee, \wedge, \neg, \top, \perp\}$ the language of boolean algebras, and by $\mathrm{cl}(A)$ the topological closure of a set A. If $\bar{a} = (a_1, \dots, a_n)$ and $\bar{i} = (i_1, \dots, i_k) \subseteq \{1, \dots, n\}$, we denote by $\bar{a}_{\bar{i}}$ the tuple $(a_{i_1}, \dots, a_{i_k})$.

Finally, by definable we mean definable with parameters, unless otherwise specified.

Definition 2.1. Let M be an ordered field and let $n \in \mathbb{N}_+$. Let K be a complex in M^n.

1. We denote by $V(K)$ the set of vertices of K. If $S = (a_0, \dots, a_k) \in K$ and $\bar{a} = (a_0, \dots, a_k)$, we denote by $(\bar{a})$ the simplex S.

2. Let $x_0, \dots, x_\ell \in M^n$. We denote by $AI(x_0, \dots, x_\ell)$ the formula in the language of fields expressing the fact that the points of coordinates $x_0, \dots, x_\ell$ are affine independent.

3. Let $\bar{a} = (a_1, \dots, a_m)$ be an enumeration of the vertices of K and let $\bar{x} = (x_1, \dots, x_m)$ (where each x_i is a tuple of n variables). We define $\Sigma_{K,\bar{a}}(\bar{x})$, the type of K with respect to the enumeration $\bar{a}$, to be the following set of L_{of}-sentences

$$\begin{aligned}
&\{\mathrm{AI}(\bar{x}_{\bar{i}}) \mid \bar{i} \subseteq \{1, \dots, m\} \wedge (\bar{a}_{\bar{i}}) \in K\} \cup \\
&\{\mathrm{cl}((\bar{x}_{\bar{i}})) \cap \mathrm{cl}((\bar{x}_{\bar{j}})) = \mathrm{cl}((\bar{x}_{\bar{k}})) \mid \bar{i}, \bar{j}, \bar{k} \subseteq \{1, \dots, m\} \wedge \\
&\qquad (\bar{a}_{\bar{i}}), (\bar{a}_{\bar{j}}), (\bar{a}_{\bar{k}}) \in K \wedge \mathrm{cl}((\bar{a}_{\bar{i}})) \cap \mathrm{cl}((\bar{a}_{\bar{j}})) = \mathrm{cl}((\bar{a}_{\bar{k}}))\} \cup \\
&\{\mathrm{cl}((\bar{x}_{\bar{i}})) \cap \mathrm{cl}((\bar{x}_{\bar{j}})) = \emptyset \mid \bar{i}, \bar{j} \subseteq \{1, \dots, m\} \wedge (\bar{a}_{\bar{i}}), (\bar{a}_{\bar{j}}) \in K \wedge \\
&\qquad \mathrm{cl}((\bar{a}_{\bar{i}})) \cap \mathrm{cl}((\bar{a}_{\bar{j}})) = \emptyset\}.
\end{aligned}$$

Lemma 2.2. *Let M be an ordered field and let $n \in \mathbb{N}_+$. Let K be a complex in M^n with $\bar{a} = (a_1, \ldots, a_m)$, an enumeration of $V(K)$. Let S be an ordered field and let $\bar{s} \subseteq S$ be such that $S \models \Sigma_{K,\bar{a}}(\bar{s})$. We define the following set of simplices in S^n:*

$$W := \{(\bar{s}_{\bar{i}}) \mid \bar{i} \subseteq \{1, \ldots, m\} \wedge \mathrm{AI}(\bar{x}_{\bar{i}}) \in \Sigma_{K,\bar{a}}\}.$$

Then W is a complex in S^n and $\Sigma_{W,\bar{s}} = \Sigma_{K,\bar{a}}$. We say that the complex W is determined by $\Sigma_{K,\bar{a}}$.

Proof. We follow the definition of complex given in [V, Definition 1.5, Chapter 8]. It is clear that each $(\bar{s}_{\bar{i}}) \in W$ is a simplex since by hypothesis $\bar{s}_{\bar{i}}$ is affine independent, and the other conditions in the definition of complex are also satisfied since the definition of $\Sigma_{K,\bar{a}}$ mimics the definition of complex. The fact that $\Sigma_{W,\bar{s}} = \Sigma_{K,\bar{a}}$ follows from the definition of W. $\square$

Definition 2.3. Let M be an o-minimal expansion of a real closed field. A homeomorphism of complexes is a triple (ξ, K, W) such that

1. K and W are complexes in M^n for some $n \in \mathbb{N}_+$,
2. $\xi : |K| \to |W|$ is a homeomorphism,
3. $\xi \upharpoonright V(K)$ is a bijection from $V(K)$ to $V(W)$,
4. for every $C = (a_0, \ldots, a_k) \in K$, $\xi(C)$ is equal to $(\xi(a_0), \ldots, \xi(a_k))$, has dimension k, and belongs to W,
5. the map $\tilde{\xi} : K \to W$, which sends a simplex $C \in K$ to the simplex $\xi(C) \in W$, is a bijection.

We say that two complexes K and W in M^n are homeomorphic as complexes if there is a map ξ such that the triple (ξ, K, W) is a homeomorphism of complexes. We say that this homeomorphism of complexes is (M-)definable if the map ξ is (M-)definable.

Proposition 2.4. *Let M be an o-minimal expansion of a real closed field and let $n \in \mathbb{N}_+$. Let K and W be complexes in M^n. Then K and W are definably homeomorphic as complexes if and only if $\Sigma_{K,\bar{a}} = \Sigma_{W,\bar{b}}$ for some well-chosen enumerations $\bar{a}$ and $\bar{b}$ of the vertices of K and W.*

Proof. "$\Rightarrow$" Let (ξ, K, W) be a homeomorphism of complexes between K and W (it does not need to be definable). Let $\bar{a} = (a_1, \ldots, a_m)$ be an enumeration of $V(K)$. Then $\xi(\bar{a})$ is an enumeration of $V(W)$. By hypothesis $(a_{i_0}, \ldots, a_{i_k})$ is a simplex in K if and only if $(\xi(a_{i_0}), \ldots, \xi(a_{i_k}))$ is a simplex

in L, and using that ξ is a homeomorphism, it follows that $\Sigma_{K,\bar{a}} = \Sigma_{W,\xi(\bar{a})}$.
"$\Leftarrow$" Let $\bar{a} = (a_1, \dots, a_m)$ be an enumeration of $V(K)$ and let $\bar{b} = (b_1, \dots, b_m)$ be an enumeration of $V(W)$ such that $\Sigma_{K,\bar{a}} = \Sigma_{W,\bar{b}}$. For every tuple $i_0, \dots, i_k \in \{1, \dots, m\}$ we have $(a_{i_0}, \dots, a_{i_k}) \in K$ if and only if $(b_{i_0}, \dots, b_{i_k}) \in W$.

Let $C = (a_{i_0}, \dots, a_{i_k}) \in K$. For $k = 0$ we define $\xi_{(a_{i_0})}((a_{i_0})) = (b_{i_0})$ and for $k \geq 1$ we define

$$\begin{array}{cccc} \xi_C : & C & \to & (b_{i_0}, \dots, b_{i_k}) \\ & \sum_{r=0}^{k} t_r a_{i_r} & \mapsto & \sum_{r=0}^{k} t_r b_{i_r} \\ & (\text{all } t_r > 0,\ \sum_{r=0}^{k} t_r = 1) & & \end{array}$$

It is clear that ξ_C is a homeomorphism for every $C \in K$, and we define

$$\xi : |K| \to |W|,\ \xi = \bigcup_{C \in K} \xi_C.$$

It is easy to check that ξ is continuous. For instance using [V, Lemma 4.2, Chapter 6]: Let x belong to some $C \in K$ and let $\gamma : (0,1) \to |K|$ be definable continuous such that $\lim_{t \to 0^+} \gamma(t) = x$. Since we are only interested in the behaviour of γ at 0^+, we can assume by o-minimality that there is a simplex $T \in K$ such that $\gamma((0,1)) \subseteq T$. Say $C = (a_{i_0}, \dots, a_{i_k})$ and $T = (a_{j_0}, \dots, a_{j_\ell})$ with $\{i_0, \dots, i_k\} \subseteq \{j_0, \dots, j_\ell\}$ (since $C \subseteq \mathrm{cl}(T)$). By definition of ξ we have $\xi(C) = (b_{i_0}, \dots, b_{i_k})$ and $\xi(T) = (b_{j_0}, \dots, b_{j_\ell})$. We then see easily that $\lim_{t \to 0^+} \xi \circ \gamma(t) = \xi(x)$.
Conversely, $\xi^{-1} = \bigcup_{C \in K} \xi_C^{-1}$, and since ξ_C^{-1} is defined in a similar way to ξ_C, we see that ξ^{-1} is also continuous, and so that ξ is a homeomorphism. The other conditions in Definition 2.3 follow directly from the definition of ξ.

Observe that by construction of ξ we have $\xi(\bar{a}) = \bar{b}$. $\square$

Definition 2.5. Let $R \prec M$ be two ordered fields in some language L.

1. Let $\phi(\bar{x})$ be an L-formula. If $C = \phi(R^n)$ is a definable subset of R^n, we denote by C_M the subset $\phi(M^n)$ of M^n.

2. If $K = \{C_1, \dots, C_\ell\}$ is a complex in R^n, we denote by K_M the complex $\{(C_1)_M, \dots, (C_\ell)_M\}$ in M^n.

Definition 2.6. Let R and S be o-minimal expansions of real closed fields in the same language L, and let M be an elementary extension of R and S. Let $n \in \mathbb{N}_+$, let ϕ be a bounded definable subset of R^n, and let ψ be a bounded definable subset of S^n.

We denote by $I(\phi, \psi)$ the set of all bijections $f : \mathcal{A} \to \mathcal{B}$, where

1. $\mathcal{A}$ is a partition of ϕ into definable sets and $\mathcal{B}$ is a partition of ψ into definable sets.

2. there are

   - two complexes K in R^n and W in S^n,
   - a triangulation (F, K) of ϕ partitioning every element of $\mathcal{A}$,
   - a triangulation (G, W) of ψ partitioning every element of $\mathcal{B}$, and
   - an M-definable homeomorphism of complexes (ξ, K_M, W_M)

$$\begin{array}{ccc} |K_M| & \xrightarrow{\xi} & |W_M| \\ \uparrow & & \uparrow \\ |K| & & |W| \\ {\scriptstyle F}\uparrow & & \uparrow{\scriptstyle G} \\ \phi & & \psi \end{array}$$

   such that f is the map induced by the above diagram, i.e. for every $A \in \mathcal{A}$ such that $F(A) = C_1 \cup \cdots \cup C_\ell$ and $\xi((C_i)_M) = (E_i)_M$ with $C_i \in K$, $E_i \in W$ (for $i = 1, \ldots, \ell$), we have

$$f(A) = G^{-1}(E_1 \cup \cdots \cup E_\ell).$$

Every partition $\mathcal{A}$ of ϕ generates a boolean subalgebra $\mathrm{BA}(\mathcal{A})$ of the boolean algebra of subsets of ϕ, whose atoms are precisely the elements of $\mathcal{A}$. So if $f \in I(\phi, \psi)$, f induces an L_{BA}-isomorphism

$$\begin{array}{rccc} \mathrm{BA}(f): & \mathrm{BA}(\mathcal{A}) & \to & \mathrm{BA}(\mathcal{B}) \\ & A_1 \dot\cup \cdots \dot\cup A_r & \mapsto & f(A_1) \dot\cup \cdots \dot\cup f(A_r) \\ & (\text{where } A_1, \ldots, A_r \in \mathcal{A}) & & \end{array}$$

We denote by $I^{BA}(\phi, \psi)$ the set of all $\mathrm{BA}(f) : \mathrm{BA}(\mathcal{A}) \to \mathrm{BA}(\mathcal{B})$, for $f \in I(\phi, \psi)$.

Definition 2.7. We define the languages L^n and $\widetilde{L}^n$ by

$$L^n := L_{BA} \cup \{D_k \mid k = 0, \ldots, n\} \cup \{E_k \mid k \in \omega\} \cup \{\mathrm{Open}\},$$

$$\widetilde{L}^n := \{\vee, \wedge, \top, \bot\} \cup \{D_k \mid k = 0, \ldots, n\} \cup \{E_k \mid k \in \omega\},$$

where the D_k's, E_k's and Open are new unary predicates.
In a structure whose elements are definable subsets of R^n, where R is an o-minimal expansion of a real closed field, the new predicates will be interpreted

as follows

$$
\begin{gathered}
D_k(A) \Leftrightarrow \dim A = k \\
E_k(A) \Leftrightarrow E(A) = k \\
\text{(where } E \text{ denotes the Euler characteristic)} \\
\mathrm{Open}(A) \Leftrightarrow A \text{ is open in } R^n.
\end{gathered}
$$

Lemma 2.8. *With notation as in Definition 2.6, let $f \in I(\phi, \psi)$. Then* $\mathrm{BA}(f)$ *is an L^n-isomorphism from*

$$
\begin{aligned}
&(\mathrm{BA}(\mathcal{A}); \vee, \wedge, \neg, \top, \bot, (D_k)_{k=0}^n, (E_k)_{k\in\omega}, \mathrm{Open}) \\
&\textit{to} \\
&(\mathrm{BA}(\mathcal{B}); \vee, \wedge, \neg, \top, \bot, (D_k)_{k=0}^n, (E_k)_{k\in\omega}, \mathrm{Open}).
\end{aligned}
$$

Proof. Since $\mathrm{BA}(f)$ is an L_{BA}-isomorphism, we only have to show that $\mathrm{BA}(f)$ is a $(\{D_k \mid k = 0, \dots, n\} \cup \{E_k \mid k \in \omega\} \cup \{\mathrm{Open}\})$-isomorphism. Let $A \in \mathrm{BA}(\mathcal{A})$. Then $F(A) = C_1 \cup \dots \cup C_\ell$ for some $C_1, \dots, C_\ell \in K$, and if $E_1, \dots, E_\ell \in W$ are such that $\xi((C_i)_M) = (E_i)_M$ for $i = 1, \dots, \ell$, then $f(A) = G^{-1}(E_1 \dot\cup \dots \dot\cup E_\ell)$. The result for the predicates D_k and Open then follows from

$$
\begin{aligned}
\dim A = \dim F(A) &= \max\{\dim C_i \mid i = 1, \dots, \ell\} \\
&= \max\{\dim E_i \mid i = 1, \dots, \ell\} \\
&= \dim f(A)
\end{aligned}
$$

and

$$
\begin{aligned}
A \text{ open} &\Leftrightarrow C_1 \cup \dots \cup C_\ell \text{ open} \\
&\Leftrightarrow \xi((C_1)_M) \cup \dots \cup \xi((C_\ell)_M) \text{ open} \\
&\Leftrightarrow (E_1)_M \cup \dots \cup (E_\ell)_M \text{ open} \\
&\Leftrightarrow E_1 \cup \dots \cup E_\ell \text{ open} \\
&\Leftrightarrow f(A) \text{ open.}
\end{aligned}
$$

For the Euler characteristic, we first observe that $E(C_i) = E(E_i)$ for $i = 1, \dots, \ell$ (indeed, if C_i is a simplex of dimension k, then E_i is also a simplex of dimension k and $E(C_i) = (-1)^k = E(E_i)$). Since F and G are definable bijections, by [V, Proposition 2.4, Chapter 4], to prove that $E(A) = E(f(A))$ we only have to check that $E(C_1 \cup \cdots C_\ell) = E(E_1 \cup \dots \cup E_\ell)$. Since these unions are disjoint unions, this statement is equivalent to $\sum_{i=1}^{\ell} E(C_i) = \sum_{i=1}^{\ell} E(i)$ (see [V, 2.9, Chapter 4]), and the result is proved. $\square$

Definition 2.9. If M is an o-minimal structure, $n \in \mathbb{N}_+$ and Ω is a definable subset of M^n, we denote by

1. $\mathrm{def}_M(\Omega)$ the boolean algebra of subsets of Ω that are definable in M.
2. $\mathrm{odef}_M(\Omega)$ the lattice of open subsets of Ω that are definable in M.

We recall the following definition, which is a reformulation of [H, pp. 97-98].

Definition 2.10. Let L be a first-order language and let $\mathcal{M}$ and $\mathcal{N}$ be L-structures.

1. A partial L-isomorphism from $\mathcal{M}$ to $\mathcal{N}$ is an L-isomorphism between an L-substructure of $\mathcal{M}$ and L-substructure of $\mathcal{N}$.
2. A set I of partial L-isomorphisms from $\mathcal{M}$ to $\mathcal{N}$ is called a back-and-forth system if for every $f \in I$:
   (a) for every $a \in \mathcal{M}$ there is $g \in I$ such that $a \in \operatorname{dom} g$ and g extends f;
   (b) for every $b \in \mathcal{N}$ there is $g \in I$ such that $b \in \operatorname{Im} g$ and g extends f.
3. We say that $\mathcal{M}$ and $\mathcal{N}$ are back-and-forth equivalent if there is a non-empty set of partial L-isomorphisms from $\mathcal{M}$ to $\mathcal{N}$ that is a back-and-forth system.

Lemma 2.11. *With the same notation as in Definition 2.6, assume that $I(\phi, \psi)$ is non-empty. Then $I^{BA}(\phi, \psi)$ is a back-and-forth system between $\mathrm{def}_R(\phi)$ and $\mathrm{def}_S(\psi)$.*

Proof. Let $f \in I(\phi, \psi)$ and let U be a definable subset of ϕ such that $U \notin \mathrm{dom}(\mathrm{BA}(f)) = \mathrm{BA}(\mathcal{A})$. (The case of U being a definable subset of ψ, $U \notin \operatorname{Im} f$, is similar.)

By the triangulation theorem [V, Theorem 2.9, Chapter 8] there is a triangulation (F_1, K_1) of $|K|$ partitioning $F(U)$ and every element of K. By definition of triangulation, the map F_1 is definable in R, say for instance that the graph of F_1 is defined by the formula $F_1(\bar{r}, \bar{v})$ where $\bar{v}$ is a tuple of $2n$ variables, $\bar{r} \subseteq R$, and $F_1(\bar{u}, \bar{v})$ is a formula without parameters.

In such a case, i.e. if a formula $\theta(\bar{c}, \bar{v})$ defines the graph of a function (where $\bar{c}$ is a tuple of parameters), we will denote this function by $f_{\theta(\bar{c},\bar{v})}$. So for instance, in the situation described above we have $F_1 = f_{F_1(\bar{r},\bar{v})}$.

We fix an enumeration $\bar{a}$ of the vertices of K and an enumeration $\bar{a}'$ of the vertices of K_1. Then $\xi(\bar{a})$ is an enumeration of the vertices of W and

$\Sigma_{K,\bar{a}} = \Sigma_{W,\xi(\bar{a})}$. The following set of L-sentences describes how the simplices of K_1 are included in the image of the simplices of K by $f_{F_1(\bar{r},\bar{v})}$

$$\begin{aligned}\Sigma_{F_1}(\bar{x},\bar{y}) = &\{(y_{i_1},\dots,y_{i_t}) \subseteq f_{F_1(\bar{z},\bar{v})}((x_{j_1},\dots,x_{j_\ell})) \mid \\ &\quad \mathrm{AI}(x_{j_1},\dots,x_{j_\ell}) \in \Sigma_{K,\bar{a}}(\bar{x}) \wedge \mathrm{AI}(y_{i_1},\dots,y_{i_t}) \in \Sigma_{K_1,\bar{a}'}(\bar{y}) \wedge \\ &\quad (a'_{i_1},\dots,a'_{i_t}) \subseteq f_{F_1(\bar{r},\bar{v})}((a_{j_1},\dots,a_{j_\ell}))\} \\ &\cup \{(y_{i_1},\dots,y_{i_t}) \cap f_{F_1(\bar{z},\bar{v})}((x_{j_1},\dots,x_{j_\ell})) = \emptyset \mid \\ &\quad \mathrm{AI}(x_{j_1},\dots,x_{j_\ell}) \in \Sigma_{K,\bar{a}}(\bar{x}) \wedge \mathrm{AI}(y_{i_1},\dots,y_{i_t}) \in \Sigma_{K_1,\bar{a}'}(\bar{y}) \wedge \\ &\quad (a'_{i_1},\dots,a'_{i_t}) \cap f_{F_1}(\bar{r},\bar{v})((a_{j_1},\dots,a_{j_\ell})) = \emptyset\}.\end{aligned}$$

We have

$$\begin{aligned}R \models &\exists\bar{x}\exists\bar{y}\exists\bar{z} \quad \Sigma_{K,\bar{a}}(\bar{x}) \wedge \Sigma_{K_1,\bar{a}'}(\bar{y}) \wedge \\ &[F_1(\bar{z},\bar{v}) \text{ defines the graph of a triangulation from the} \\ &\text{complex determined by } \Sigma_{K,\bar{a}}(\bar{x}) \text{ to the complex} \\ &\text{determined by } \Sigma_{K_1,\bar{a}'}(\bar{y}),\ \text{partitioning the simplices} \\ &\text{in the complex determined by } \Sigma_{K,\bar{a}}(\bar{x})] \wedge \\ &\Sigma_{F_1}(\bar{x},\bar{y}).\end{aligned}$$

(The above sentence can be expressed as a first-order sentence in the language L.)
Since $R \equiv S$ and $\Sigma_{K,\bar{a}} = \Sigma_{W,\xi(\bar{a})}$, it follows that

$$\begin{aligned}S \models &\exists\bar{x}\exists\bar{y}\exists\bar{z} \quad \Sigma_{W,\xi(\bar{a})}(\bar{x}) \wedge \Sigma_{K_1,\bar{a}'}(\bar{y}) \wedge \\ &[F_1(\bar{z},\bar{v}) \text{ defines the graph of a triangulation from the} \\ &\text{complex determined by } \Sigma_{W,\xi(\bar{a})}(\bar{x}) \text{ to the complex} \\ &\text{determined by } \Sigma_{K_1,\bar{a}'}(\bar{y}),\ \text{partitioning the simplices} \\ &\text{in the complex determined by } \Sigma_{W,\xi(\bar{a})}(\bar{x})] \wedge \\ &\Sigma_{F_1}(\bar{x},\bar{y}).\end{aligned} \tag{1}$$

Let $\bar{\alpha}, \bar{b}$ and $\bar{s} \subseteq S$ be tuples realising the variables $\bar{x}, \bar{y}$ and $\bar{z}$ respectively in (1). Let W' be the complex in S determined by $\bar{\alpha}$ (as in Lemma 2.2) and let W_1 be the complex in S determined by $\bar{b}$. Since $\Sigma_{K_1,\bar{a}'} = \Sigma_{W_1,\bar{b}}$ and $\Sigma_{W',\alpha} = \Sigma_{W,\xi(\bar{a})}$, by Proposition 2.4 there is an M-definable homeomorphism of complexes $(\sigma, (K_1)_M, (W_1)_M)$ and an S-definable homeomorphism of complexes (ξ', W_M, W'_M), which yield the following (informal) diagram

$$\begin{tikzcd}
(K_1)_M \arrow[rr, "\sigma"] & & (W_1)_M \\
K_M \arrow[u, "f_{F_1(\bar{r},\bar{v})}"] \arrow[r, "\xi"] & W_M \arrow[r, "\xi'"] & W'_M \arrow[u, "f_{F_1(\bar{s},\bar{v})}"'] \\
\phi_M \arrow[u, "F"] & \psi_M \arrow[u, "G"'] &
\end{tikzcd} \tag{2}$$

This diagram need not be commutative at the level of maps (i.e. there is no reason why we should have $\sigma \circ f_{F_1(\bar{r},\bar{v})} = f_{F_1(\bar{s},\bar{v})} \circ \xi' \circ \xi$), but is actually commutative at the level of boolean algebras generated by the complexes, as proved in the following claim.

Claim 2.12. Let $C \in K$ and $E \in W'$ be such that $\xi' \circ \xi(C_M) = E_M$. Let $f_{F_1(\bar{r},\bar{v})}(C_M) = (C'_1)_M \cup \cdots \cup (C'_r)_M$ with $C'_i \in K_1$ and $\sigma((C'_i)_M) = (E'_i)_M$ for $E'_i \in W_1$ and $i = 1, \ldots, \ell$.
Then $f_{F_1(\bar{s},\bar{v})}(E_M) = (E'_1)_M \cup \cdots \cup (E'_r)_M$.

Proof of Claim 2.12: Let $C = (a_{j_1}, \ldots, a_{j_\ell})$, i.e. $E = (\xi' \circ \xi(a_{j_1}), \ldots, \xi' \circ \xi(a_{j_\ell}))$. If $a'_{i_1}, \ldots, a'_{i_t}$, taken from the tuple $\bar{a}'$, are such that $(a'_{i_1}, \ldots, a'_{i_t}) \in K_1$ (i.e. $(b_{i_1}, \ldots, b_{i_t}) \in W_1$), we have

$$\begin{aligned}
(a'_{i_1}, \ldots, a'_{i_t}) \subseteq f_{F_1(\bar{r},\bar{v})}(C) \Leftrightarrow & \\
&\Leftrightarrow (y_{i_1}, \ldots, y_{i_t}) \subseteq f_{F_1(\bar{z},\bar{v})}((x_{j_1}, \ldots, x_{j_\ell})) \in \Sigma_{F_1}(\bar{x}, \bar{y}) \\
&\Leftrightarrow (b_{i_1}, \ldots, b_{i_t}) \subseteq f_{F_1(\bar{s},\bar{v})}((\alpha_{j_1}, \ldots, \alpha_{j_\ell})) \\
&\Leftrightarrow \sigma((a'_{i_1}, \ldots, a'_{i_t})) \subseteq f_{F_1(\bar{s},\bar{v})}((\xi' \circ \xi(a_{j_1}), \ldots, \xi' \circ \xi(a_{j_\ell}))) \\
&\Leftrightarrow \sigma((a'_{i_1}, \ldots, a'_{i_t})_M) \subseteq f_{F_1(\bar{s},\bar{v})}(E_M)
\end{aligned}$$

which proves the statement. End of the proof of Claim 2.12.

We simply follow diagram (2) to find what set we associate to U in the back-and-forth process:
Let $C'_1, \ldots, C'_\ell$ be simplices in K_1 such that $(f_{F_1(\bar{r},\bar{v})} \circ F)(U_M) = (C'_1)_M \cup \cdots \cup (C'_\ell)_M$. Observe that for $i = 1, \ldots, \ell$, each $\sigma((C'_i)_M)$ is a simplex in $(W_1)_M$, i.e. is equal to $(E'_i)_M$, for some $E'_i \in W_1$. In particular the set $E := (f_{F_1(\bar{s},\bar{v})})^{-1}(E'_1 \cup \cdots \cup E'_\ell)$ is definable in S and if we define $V := (\xi' \circ G)^{-1}(E)$, then V is also definable in S. We define a map f' such that $\mathrm{BA}(f')$ extends $\mathrm{BA}(f)$ and sends U to V.

Let $\mathcal{A}' := (f_{F_1(\bar{r},\bar{v})} \circ F)^{-1}(K_1)$ and $\mathcal{B}' := (f_{F_1(\bar{s},\bar{v})} \circ \xi' \circ G)^{-1}(W_1)$. Since $F_1(\bar{r}, \bar{v})$ partitions the simplices of K and $F_1(\bar{s}, \bar{v}) \circ \xi'$ partitions the simplices of W, $\mathcal{A}'$ is a refinement of $\mathcal{A}$ and $\mathcal{B}'$ is a refinement of $\mathcal{B}$. In particular $\mathcal{A}'$ is a partition of ϕ, $\mathcal{B}'$ is a partition of ψ, $\mathrm{BA}(\mathcal{A}) \subseteq \mathrm{BA}(\mathcal{A}')$ and $\mathrm{BA}(\mathcal{B}) \subseteq \mathrm{BA}(\mathcal{B}')$. For each $(f_{F_1(\bar{r},\bar{v})} \circ F)^{-1}(C')$ with $C' \in K_1$, there is a unique $E' \in W_1$ such that $\sigma(C'_M) = E'_M$, and we define $f' : \mathcal{A}' \to \mathcal{B}'$ as follows:

$$f'((f_{F_1(\bar{r},\bar{v})} \circ F)^{-1}(C')) := (f_{F_1(\bar{s},\bar{v})} \circ \xi' \circ G)^{-1}(E').$$

It is clear that f' is a bijection from $\mathcal{A}'$ to $\mathcal{B}'$.

Claim 2.13. 1. $U \in \mathrm{BA}(\mathcal{A}')$;

2. $\mathrm{BA}(f')$ extends $\mathrm{BA}(f)$;

3. $f' \in I(\phi, \psi)$.

Proof of Claim 2.13:

1. This holds by definition of $\mathcal{A}'$ and the triangulation $(f_{F_1(\bar{r},\bar{v})}, K_1)$.

2. Let $A \in \mathcal{A}$. Since F partitions every element of $\mathcal{A}$ we have $F(A_M) = (C_1)_M \cup \cdots \cup (C_\ell)_M$ for some $C_1, \ldots, C_\ell \in K$, and since $f_{F_1(\bar{r},\bar{v})}$ partitions the simplices in K, we have $f_{F_1(\bar{r},\bar{v})}((C_i)_M) = (C'_{i,1})_M \cup \cdots \cup (C'_{i,r_i})_M$ for some $C'_{i,j} \in K_1$. Let $E_1, \ldots, E_\ell \in W$ be such that $\xi((C_i)_M) = (E_i)_M$ for $i = 1, \ldots, \ell$, and let $E'_{i,j} \in W_1$ be such that $\sigma((C'_{i,j})_M) = (E'_{i,j})_M$ for $i = 1, \ldots, \ell$, $j = 1, \ldots, r_i$.

   Since $(\xi' \circ \xi)((C_i)_M) = \xi'((E_i)_M)$, Claim 2.12 gives, for $i = 1, \ldots, \ell$,

   $$(f_{F_1(\bar{s},\bar{v})} \circ \xi')((E_i)_M) = (E'_{i,1})_M \cup \cdots \cup (E'_{i,r_i})_M. \tag{3}$$

   By definition of $\mathrm{BA}(f)$ and $\mathrm{BA}(f')$:

   $$\mathrm{BA}(f)(A_M) = G^{-1}((E_1)_M \cup \cdots \cup (E_\ell)_M),$$

   $$\begin{aligned}\mathrm{BA}(f')(A_M) &= \bigcup_{i=1}^{\ell} (f_{F_1(\bar{s},\bar{v})} \circ \xi' \circ G)^{-1}((E'_{i,1})_M \cup \cdots \cup (E'_{i,r_i})_M) \\ &= G^{-1}[\bigcup_{i=1}^{\ell} (f_{F_1(\bar{s},\bar{v})} \circ \xi')^{-1}((E'_{i,1})_M \cup \cdots \cup (E'_{i,r_i})_M)],\end{aligned}$$

   and the claim follows by (3).

3. To finish checking that $f' \in I(\phi, \psi)$, we have to verify the second item in Definition 2.6. For this we consider the complexes K_1 and W_1, and the maps $f_{F_1(\bar{r},\bar{v})} \circ F$ and $f_{F_1(\bar{s},\bar{v})} \circ \xi' \circ G$. We know that $(f_{F_1(\bar{r},\bar{v})} \circ F, K_1)$ is an R-definable triangulation, while $(f_{F_1(\bar{s},\bar{v})} \circ \xi' \circ G, W_1)$ is an S-definable triangulation. Finally, (σ, K_{1M}, W_{1M}) is an M-definable homeomorphism of complexes, and the rest of Definition 2.6 is fulfilled by definition of f'.

End of the proof of Claim 2.13.

Therefore $I^{BA}(\phi, \psi)$ is a back-and-forth system. $\square$

Our main result now follows from Karp's theorem, which we briefly recall (see [H, Corollary 3.5.3]).

Theorem 2.14 (Karp)**.** *Let L be a first-order language and let $\mathcal{M}$ and $\mathcal{N}$ be L-structures. Then $\mathcal{M}$ and $\mathcal{N}$ are back-and-forth equivalent if and only if they are $L_{\infty\omega}$-equivalent (i.e. satisfy the same $L_{\infty\omega}$-formulas).*

Theorem 2.15. *Let R and S be elementarily equivalent o-minimal L_0-structures that are expansions of real closed fields, and let $\phi, \theta_1, \ldots, \theta_k$ be L_0-formulas with n free variables such that $\theta_i(R^n) \subseteq \phi(R^n)$ for $i = 1, \ldots, k$. Then the structures*

$$(\mathrm{def}_R(\phi(R^n)); \vee, \wedge, \neg, \top, \bot, (D_\ell)_{\ell=0}^n, (E_\ell)_{\ell\in\omega}, \mathrm{Open}, \theta_1(R^n), \ldots, \theta_k(R^n))$$

and

$$(\mathrm{def}_S(\phi(S^n)); \vee, \wedge, \neg, \top, \bot, (D_\ell)_{\ell=0}^n, (E_\ell)_{\ell\in\omega}, \mathrm{Open}, \theta_1(S^n), \ldots, \theta_k(S^n))$$

are $L^n_{\infty\omega}$-equivalent.

Proof. We first observe that R^n and ${(0,1)_R}^n$ on the one hand, and S^n and ${(0,1)_S}^n$ on the other hand, are definably homeomorphic, using homeomorphisms that are defined by the same L_0-formula without parameters in R and S. Therefore, and up to applying these homeomorphisms, we can assume that ϕ defines a bounded subset of R^n and S^n.

By Lemma 2.11 and Theorem 2.14, it suffices to show that the set $I(\phi(R^n), \phi(S^n))$ is non-empty and contains a map sending $\theta_i(R^n)$ to $\theta_i(S^n)$ for $i = 1, \ldots, k$. Let (F_1, K) be a triangulation of $\phi(R^n)$ partitioning $\theta_1(R^n), \ldots, \theta_k(R^n)$. Let $F_1(\bar{y}, \bar{u})$ be an L_0-sentence such that $F_1(\bar{r}, \bar{u})$ defines the graph of F_1 for some $\bar{r} \in R$, and let $\bar{a}$ be an enumeration of $V(K)$. Then

$$\begin{aligned} R \models\, & \exists \bar{x} \exists \bar{z} \quad \Sigma_{K,\bar{a}}(\bar{x}) \wedge F_1(\bar{z}, \bar{u}) \text{ defines the graph} \\ & \text{of a triangulation from } \phi \text{ to the complex} \\ & \text{determined by } \Sigma_{K,\bar{a}}(\bar{x}), \text{ partitioning } \theta_1, \ldots, \theta_k. \end{aligned}$$

The above sentence can be expressed as a first-order L_0-sentence Ω and since $R \equiv S$, it follows that

$$S \models \Omega. \tag{4}$$

Let $\bar{b}, \bar{s} \subseteq S$ be realisations of the variables $\bar{x}$ and $\bar{z}$ in (4), let W be the complex in S^n determined by $\Sigma_{K,\bar{a}}(\bar{b})$ and let G_1 be the triangulation whose graph is defined by $F_1(\bar{s}, \bar{u})$. We have $\Sigma_{K,\bar{a}}(\bar{x}) = \Sigma_{W,\bar{b}}(\bar{x})$. Let $\mathcal{A} = F_1^{-1}(K)$ and $\mathcal{B} = G_1^{-1}(W)$.

By Proposition 2.4, if M is any common elementary extension of R and S, there is a homeomorphism of complexes (ξ, K_M, W_M) such that for every simplex $(a_{i_1}, \ldots, a_{i_\ell})$ of K, $\xi((a_{i_1}, \ldots, a_{i_\ell})_M) = (b_{i_1}, \ldots, b_{i_\ell})_M$. It follows that the map

$$\begin{array}{rccc} f: & \mathcal{A} & \to & \mathcal{B} \\ & F_1^{-1}(C) & \mapsto & G_1^{-1}(E) \end{array}$$

(where $C \in K$ and E is the unique element in W such that $E_M = \xi(C_M)$) is in $I(\phi(R^n), \phi(S^n))$, with $f(\theta_i(R^n)) = \theta_i(S^n)$ for $i = 1, \ldots, k$ and the result follows. $\square$

Corollary 2.16. *Let R and S be o-minimal L_0-structures that are expansions of real closed fields, and let $A \subseteq R \cap S$ be such that R and S are elementarily equivalent as $L_0(A)$-structures (where $L_0(A)$ is the language L_0 expanded by constants for the elements of A). Let ϕ be an $L_0(A)$-formula with n free variables. Then*

1. *the bounded lattices*

   $$(\mathrm{odef}_R(\phi(R^n)); \vee, \wedge, \top, \bot, (D_\ell)_{\ell=0}^n, (E_\ell)_{\ell\in\omega}, \theta_1(R^n), \ldots, \theta_k(R^n))$$
   and
   $$(\mathrm{odef}_S(\phi(S^n)); \vee, \wedge, \top, \bot, (D_\ell)_{\ell=0}^n, (E_\ell)_{\ell\in\omega}, \theta_1(S^n), \ldots, \theta_k(S^n))$$

   are $\widetilde{L}^n_{\infty\omega}$-equivalent, for every $\theta_1(R^n), \ldots, \theta_k(R^n)$ open subsets of $\phi(R^n)$ that are definable with parameters in A.

In particular, if $R \prec S$ then

2. $(\mathrm{odef}_R(\phi(R^n); \vee, \wedge, \top, \bot, (D_\ell)_{\ell=0}^n, (E_\ell)_{\ell\in\omega})$ *is an $\widetilde{L}^n$-elementary substructure of* $(\mathrm{odef}_S(\phi(S^n)); \vee, \wedge, \top, \bot, (D_\ell)_{\ell=0}^n, (E_\ell)_{\ell\in\omega})$.

Proof. Statement 1 follows immediately from Theorem 2.15, while statement 2 is a clear consequence of the first one. $\square$

3 Link with semilinear sets

We refer to [V, Chapter 1] for the notion of semilinear set, and recall [V, 2.14, Exercise 2, Chapter 8]: Let $S_1, \ldots, S_k$ be semilinear subsets of a bounded semilinear set $S \subseteq R^n$ (where R is an ordered field). Then there is a complex K in R^n such that $|K| = S$ and each S_i is a union of elements of K. For the sake of terminology, it is convenient to reformulate this as a triangulation result: If S is a bounded semilinear subset of R^n and $S_1, \ldots, S_k$ are semilinear subsets of S, then there is a complex K in R^n such that $(\mathrm{id} : S \to |K|, K)$ is a triangulation of S partitioning $S_1, \ldots, S_k$.

We say that a triangulation (F, K) of some definable set is semilinear if the homeomorphism F is semilinear. In this section, R denotes a fixed o-minimal expansion of a real closed field.

Much credit for this section goes to Marcus Tressl, who provided both the question and the reference to [B]. We recall Definition 1.3 and Theorem 1.4 from [B] (restated to follow our notational conventions).

Definition 3.1. Let K be a complex in R^n. A triangulation (f, K') of $|K|$ is a normal triangulation of the complex K if it satisfies the following conditions

1. (f, K') partitions every simplex in K,
2. K' is a subdivision of K and
3. for every $T \in K'$ and $S \in K$, if $T \subseteq S$ then $T \subseteq f(S)$.

Observe that in such a case we have $f(S) = S$ for every $S \in K$. Definition 1.3 in [B] asks that $\phi'(T) \subseteq S$ whenever $T \in K'$ and $S \in K$ are such that $T \subseteq S$ (and where ϕ' is the homeomorphism in the normal triangulation). This is due to a different notation for triangulations: a triangulation (F, W) of the set S would be denoted in [B] by (W, F^{-1}), i.e. the homeomorphism starts from the realisation of the complex.

Theorem 3.2 (Normal Triangulation Theorem)**.** *Let K be a complex in R^n and let $S_1, \ldots, S_\ell$ be definable subsets of $|K|$. Then there exists a normal triangulation of K partitioning $S_1, \ldots, S_\ell$.*

Definition 3.3. Let $n \in \mathbb{N}_+$ and let Ω be a semilinear subset of R^n. We denote by

1. $\mathrm{sl}_R(\Omega)$ the boolean algebra of semilinear subsets of Ω that are definable with parameters from R.
2. $\mathrm{osl}_R(\Omega)$ the lattice of open semilinear subsets (for the order topology on R) of Ω that are definable with parameters from R.

Methods similar to those of the previous section, together with Theorem 3.2, allow us to compare the structures $\mathrm{def}_R(\phi)$ and $\mathrm{sl}_R(\phi)$, and well as $\mathrm{odef}_R(\phi)$ and $\mathrm{osl}_R(\phi)$, when ϕ is a bounded semilinear subset of R^n.

Definition 3.4. Let $n \in \mathbb{N}_+$ and assume that ϕ is a bounded semilinear subset of R^n. We denote by $I(\phi)$ the set of all bijections $f : \mathcal{A} \to \mathcal{B}$ such that

1. $\mathcal{A}$ is a partition of ϕ into definable sets, and $\mathcal{B}$ is a partition of ϕ into semilinear sets.

2. There are

   - a complex K in R^n,
   - a triangulation (F, K) of ϕ partitioning every element of $\mathcal{A}$, and
   - a semilinear triangulation (G, K) of ϕ partitioning every element of $\mathcal{B}$

$$\begin{tikzcd} & {|K|} & \\ \phi \arrow[ur, "F"] & & \phi \arrow[ul, "G"'] \end{tikzcd}$$

   such that f is the map induced by the above diagram, i.e. for every $A \in \mathcal{A}$ such that $F(A) = C_1 \cup \cdots \cup C_\ell$ with $C_i \in K$, (for $i = 1, \ldots, \ell$), we have

$$f(A) = G^{-1}(C_1 \cup \cdots \cup C_\ell).$$

As in Lemma 2.8, such a map f induces an L^n-isomorphism $\mathrm{BA}(f)$ from $\mathrm{BA}(\mathcal{A})$ to $\mathrm{BA}(\mathcal{B})$, and, defining

$$I^{BA}(\phi) := \{\mathrm{BA}(f) \mid f \in I(\phi)\},$$

we have the following lemma.

Lemma 3.5. *With the same notation and hypotheses as in Definition 3.4, assume that $I(\phi)$ is non-empty. Then $I^{BA}(\phi)$ is a back-and-forth system between $\mathrm{def}_R(\phi)$ and $\mathrm{sl}_R(\phi)$.*

Proof. Let $f \in I(\phi)$. For this proof, we need to check both directions of the back-and-forth.

- Let $U \in \mathrm{sl}_R(\phi)$ be such that $U \notin \mathrm{Im}\,\mathrm{BA}(f) = \mathrm{BA}(\mathcal{B})$. As explained at the beginning of this section, by [V, 2.14, Exercise 2, Chapter 8] there is a semilinear triangulation (id, W) of $|K|$ partitioning the semilinear set $G(U)$ and every element of K, so we have the following maps.

$$\begin{tikzcd} & {|W|} & \\ & {|K|} \arrow[u, "\mathrm{id}"] & \\ \phi \arrow[ur, "F"] & & \phi \arrow[ul, "G"'] \end{tikzcd}$$

Let $\mathcal{A}' := F^{-1}(W)$ and $\mathcal{B}' := G^{-1}(W)$. The triangulations (F, W) and (G, W) define a map $f' : \mathcal{A}' \to \mathcal{B}'$ by $f'(F^{-1}(T)) := G^{-1}(T)$ for every $T \in W$. By definition we have $f' \in I(\phi)$, and $U \in \operatorname{Im} \mathrm{BA}(f')$. We only have to check that $\mathrm{BA}(f')$ extends $\mathrm{BA}(f)$. Let $A \in \mathrm{BA}(\mathcal{A})$ and write $F(A) = C_1 \cup \cdots \cup C_s$ with $C_1, \ldots, C_s \in K$. By definition we have $\mathrm{BA}(f)(A) = G^{-1}(C_1 \cup \cdots \cup C_s)$. Furthermore, each C_i is of the form $D_{i,1} \cup \cdots \cup D_{i,\ell_i}$ for some $D_{i,1}, \ldots, D_{i,\ell_i} \in W$, and

$$\begin{aligned}
\mathrm{BA}(f')(A) &= G^{-1}(\bigcup_{i=1}^{s} D_{i,1} \cup \cdots \cup D_{i,\ell_i}) \\
&= G^{-1}(\bigcup_{i=1}^{s} C_i) \\
&= \mathrm{BA}(f)(A).
\end{aligned}$$

- Let $U \in \mathrm{def}_R(\phi)$ be such that $U \notin \operatorname{dom} \mathrm{BA}(f) = \mathrm{BA}(\mathcal{A})$. Applying Theorem 3.2 we find a normal triangulation (H, K') of K partitioning $F(U)$. By definition of normal triangulation, K' is a subdivision of K and therefore the identity map from $|K|$ to $|K'|$ is a triangulation of $|K|$ partitioning every simplex in K. As observed after the definition of normal triangulation, we have $H(S) = S$ for every $S \in K$.

$$\begin{tikzcd}
& {|K'|} & \\
& {|K|} \arrow[u, bend left, "H"] \arrow[u, bend right, "\mathrm{id}"'] & \\
\phi \arrow[ur, "F"] & & \phi \arrow[ul, "G"']
\end{tikzcd}$$

We define $\mathcal{A}' := (H \circ F)^{-1}(K')$ and $\mathcal{B}' := (\mathrm{id} \circ G)^{-1}(K')$. The triangulations $(H \circ F, K')$ and $(\mathrm{id} \circ G, K')$ define a map $f' : \mathcal{A}' \to \mathcal{B}'$ by $f'((H \circ F)^{-1}(T)) := (\mathrm{id} \circ G)^{-1}(T)$ for every $T \in K'$. By definition we have $f' \in I(\phi)$ and $U \in \operatorname{dom} \mathrm{BA}(f')$. Observe that by construction $H \circ F(U) = S_1' \cup \cdots \cup S_r'$ for some $S_1', \ldots, S_r' \in K'$ and thus $(\mathrm{id} \circ G)^{-1}(H \circ F(U))$ is a semilinear subset of ϕ.

We only have to check that $\mathrm{BA}(f')$ extends $\mathrm{BA}(f)$. Let $A \in \mathrm{BA}(\mathcal{A})$ and write $F(A) = C_1 \cup \cdots \cup C_r$ with $C_1, \ldots, C_r \in K$. By definition of f we have $\mathrm{BA}(f)(A) = G^{-1}(C_1 \cup \cdots \cup C_r)$. To compute $\mathrm{BA}(f')(A)$ we write $H(C_i) = C_{i,1}' \cup \cdots \cup C_{i,r_i}'$ for some $C_{i,1}', \ldots, C_{i,r_i}' \in K'$. It follows that

$$C_{i,1}' \cup \cdots \cup C_{i,r_i}' = H(C_i) = C_i \tag{5}$$

since H is a normal triangulation of K and $C_i \in K$. We have $A = \bigcup_{i=1}^r \bigcup_{j=1}^{r_i} (H \circ F)^{-1}(C'_{i,j})$ and thus

$$\begin{aligned}
\mathrm{BA}(f')(A) &= \bigcup_{i=1}^{r} \bigcup_{j=1}^{r_i} (\mathrm{id} \circ G)^{-1}(C'_{i,j}) \\
&= \bigcup_{i=1}^{r} G^{-1}(\bigcup_{j=1}^{r_i} C'_{i,j}) \\
&= \bigcup_{i=1}^{r} G^{-1}(C_i) \text{ by (5)} \\
&= \mathrm{BA}(f)(A).
\end{aligned}$$

□

The following two results follow, as in the previous section.

Theorem 3.6. *Let ϕ be a bounded semilinear subset of R^n and let $\theta_1, \ldots, \theta_r$ be semilinear subsets of ϕ. Then the structures*

$$(\mathrm{def}_R(\phi); \vee, \wedge, \neg, \top, \bot, (D_\ell)_{\ell=0}^n, (E_\ell)_{\ell\in\omega}, \mathrm{Open}, \theta_1, \ldots, \theta_k)$$
and
$$(\mathrm{sl}_R(\phi); \vee, \wedge, \neg, \top, \bot, (D_\ell)_{\ell=0}^n, (E_\ell)_{\ell\in\omega}, \mathrm{Open}, \theta_1, \ldots, \theta_k)$$

are $L^n_{\infty\omega}$-equivalent.

Corollary 3.7. *Let ϕ be a bounded semilinear subset of R^n.*

1. *The bounded lattices*

$$(\mathrm{odef}_R(\phi); \vee, \wedge, \top, \bot, (D_\ell)_{\ell=0}^n, (E_\ell)_{\ell\in\omega}, \theta_1, \ldots, \theta_k)$$
and
$$(\mathrm{osl}_R(\phi); \vee, \wedge, \top, \bot, (D_\ell)_{\ell=0}^n, (E_\ell)_{\ell\in\omega}, \theta_1, \ldots, \theta_k)$$

are $\widetilde{L}^n_{\infty\omega}$-equivalent, for every $\theta_1, \ldots, \theta_k$ open semilinear subsets of ϕ.

2. *In particular $(\mathrm{osl}_R(\phi); \vee, \wedge, \top, \bot, (D_\ell)_{\ell=0}^n, (E_\ell)_{\ell\in\omega})$ is an elementary $\widetilde{L}^n$-substructure of $(\mathrm{odef}_R(\phi); \vee, \wedge, \top, \bot, (D_\ell)_{\ell=0}^n, (E_\ell)_{\ell\in\omega})$.*

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