SPECIAL GROUPS WHOSE ISOMETRY RELATION IS A FINITE UNION OF COSETS

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Abstract. $\aleph_0$-stable $\aleph_0$-categorical linked quaternionic mappings are studied and are shown to correspond (in some sense) to special groups which are $\aleph_0$-stable, $\aleph_0$-categorical, satisfy $AP(3)$ and have finite 2-symbol length. They are also related to special groups whose isometry relation is a finite union of cosets, which are then considered on their own, as well as their links with pseudofinite, profinite and weakly normal special groups.

The algebraic theory of quadratic forms is naturally divided into the reduced theory of quadratic forms (corresponding to the theory of quadratic forms over formally real Pythagorean fields) and the non-(necessarily) reduced theory. The former, with its links with the theory of orderings is much more developed, and a striking example of this is Marshall's classification of spaces of orderings of finite chain length ([20]). In the language of other axiomatisations of the algebraic theory of quadratic forms, it tells us that Witt rings, or special groups, or linked quaternionic mappings that are reduced and of finite chain length are completely classified. There is no corresponding result for the non-reduced theory. However, reduced as well as non-reduced special groups and linked quaternionic mappings are models of first-order theories, and Marshall's classification tells us that stable reduced special groups are exactly those that are reduced and of finite chain length, and are also $\aleph_0$-stable and $\aleph_0$-categorical (see the remark after theorem 4.3). In this paper we first consider $\aleph_0$-stable $\aleph_0$-categorical, not necessarily reduced, linked quaternionic mappings. We show that they correspond to $\aleph_0$-stable $\aleph_0$-categorical special groups satisfying $AP(3)$ and an extra condition (related, in the field case, to the generation of the 2-torsion part of the Brauer group by quaternion algebras), themselves related to special groups whose isometry relation is a finite union of cosets. We then investigate these special groups using the control this hypothesis gives us on their definable subsets and show some of their links with profinite, pseudofinite and weakly normal special groups (these results are gathered in theorem 4.3). We also consider some local-global principles (theorem 3.23 and its corollaries).

This paper relies on results originally coming from the model theory of modules, and of course on the axiomatic theory of quadratic forms, via the notions of linked quaternionic mapping and special group. The facts used from these topics are briefly recalled in the first section.

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1. Preliminaries

1.1. Notions from the model theory of modules.

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Definition 1.1. Let L be a first-order language.

(1) A positive-primitive L-formula (pp-formula for short) is a formula of the form
$$\exists \bar{x} \bigwedge_{i=1}^{n} \theta_i(\bar{x}),$$
where the $\theta_i(\bar{x})$ are atomic L-formulas and $n \in \mathbb{N}$.

(2) A (partial) positive-primitive type (pp-type for short) is a consistent set of positive-primitive formulas. If M is an L-structure and $\bar{a}$ is a tuple of elements of M, the pp-type of $\bar{a}$ in M, denoted $pp^M(\bar{a})$, is the set of pp-formulas belonging to the complete type of a over M.

(3) Let M and N be L-structures and let $f : M \to N$ be an L-morphism. f is called pure if for every positive-primitive formula $\phi(\bar{u})$ (without parameters) and every $\bar{m} \in M$
$$N \models \phi(f(\bar{m})) \Rightarrow M \models \phi(\bar{m}).$$

(4) An L-structure M is algebraically compact (see [26], theorem 2.8 p. 28) if every system of atomic formulas with parameters in M which is finitely satisfied in M actually has a solution in M. The system may be in any number (finite or infinite) of unknowns.

We use a largely standard notation for the space of types: If T is a complete theory in a first-order language L and A is a subset of a model of T, we denote by $S_n(T, A)$ the set of n-types over A with respect to T. We simply write $S_n(T)$ when A is empty, or $S_n(A)$ if T is clear from the context.

We will be interested in the case of vector spaces equipped with some predicates representing subspaces.

Let K be a finite field (which is fixed until section 1.2.1), let $L_K := \{0, +\} \cup \{a\}_{a \in K}$ be the language of vector spaces over K (the symbols a are unary function symbols which will represent the scalar product by elements of K, as is usual in the language of modules) and let T_K be the theory consisting of the axioms of vector spaces over K in the language L_K.
We fix $m, n_1, \ldots, n_m \in \mathbb{N}$ and we expand L_K to $L_0 := L_K \cup \{U_1, \ldots, U_m\}$ where U_i is an n_i-ary relation symbol for $i = 1, \ldots, m$. Let T_0 be T_K together with the axioms expressing that the interpretation of each U_i is a K-subvector space.
We recall the following property of T_0:

Proposition 1.2. *If V is an $\aleph_0$-categorical model of T_0 then V is $\aleph_0$-stable.*

Proof. By [13, theorem 2], $Th(V)$ admits quantifier elimination modulo pp-formulas in the language L_0. The characterization of $\aleph_0$-stability for modules (see for instance [26, theorem 3.1 (c)]) also holds for the L_0-structure V (with the same proof). It follows that $\aleph_0$-categoricity implies $\aleph_0$-stability. □

It is easy to check that the following version of [26, theorem 2.8] still holds (with the same proof, besides obvious modifications due to the slightly different context).

Theorem 1.3 ([26], theorem 2.8)**.** *Let L be any language. The following are equivalent, for any L-structure N:*

(1) *N is algebraically compact;*
(2) *Every partial pp-type (in one variable) over N which is finitely satisfied in N is actually realised in N;*
(3) *N is injective over pure embeddings, that is, if A, B are L-structures and $f : A \to N$, $g : A \to B$ are L-morphisms with g pure, then there is an L-morphism $h : B \to N$ such that $f = h \circ g$;*

(4) *If M is an L-structure, $\bar{a}$ is in M, $\bar{b}$ is in N (the tuples $\bar{a}$, $\bar{b}$ can be infinite) and $pp^M(\bar{a}) \subseteq pp^N(\bar{b})$ then there is an L-morphism $f : M \to N$ with $f(\bar{a}) = \bar{b}$.*

If N is an L_0-structure model of T_0 then the previous four properties are also equivalent to:

(5) *If N is purely embedded in another model M of T_0 then this embedding is split, i.e. $M = N \times M'$ (as L_0-structures) for some model M' of T_0.*

1.2. **Witt rings, linked quaternionic mappings and special groups.** We assume that the reader is somewhat familiar with abstract Witt rings, as defined in [19], even though we recall their definition and a few facts:

Definition 1.4. An abstract Witt ring is a pair (W, G) where W is a commutative ring with unity, G is a subgroup of exponent 2, containing -1, of the multiplicative group $W^\times$ of invertible elements, and such that:

1. G generates W additively;
2. The properties $AP(1)$ and $AP(2)$ hold in W (see the definition of $AP(k)$ below);
3. For every $n \in \mathbb{N}$ and every $a_1, \dots, a_n, b_1, \dots, b_n \in G$, if $a_1 + \cdots + a_n = b_1 + \cdots + b_n$ then there exist $a, b, c_3, \dots, c_n \in G$ such that $a_1 + a = b_1 + b$ and $a_2 + \cdots + a_n = a + c_3 + \cdots + c_n$ (and then $b_2 + \cdots + b_n = b + c_3 + \cdots + c_n$).

If (W, G) is an abstract Witt ring then I is its ideal of even-dimensional forms:

$$I := \{a_1 + \cdots + a_{2n} \mid n \in \mathbb{N},\ a_1, \dots, a_{2n} \in G\}.$$

If (W, G) is the Witt ring of some field, then (W, G) has the following property (proved by Arason and Pfister, see [2, Haupsatz]) for every $k \in \mathbb{N}$:

$$(AP(k)) \qquad \begin{array}{l} \text{If } a_1 + \cdots + a_n \in I^k \text{ for some } a_1, \dots, a_n \in G \text{ and } n < 2^k, \\ \text{then } a_1 + \cdots + a_n = 0. \end{array}$$

In general, it is an open question whether an abstract Witt ring (W, G) is always the Witt ring of some field, and we only know that (W, G) satisfies $AP(1)$ and $AP(2)$. It is not known whether $AP(1)$ and $AP(2)$ imply $AP(k)$ for other values of k in the non-reduced case (they do in the reduced case, in the sense that a reduced Witt ring satisfies $AP(k)$ for every $k \in \mathbb{N}$, see [19] corollary 4.15, or [8] theorem 7.31).

We now briefly present two other axiomatisations of the algebraic theory of quadratic forms. The first one, linked quaternionic mappings is due to Marshall and Yucas (see [21]) and is (à priori) stronger than abstract Witt rings, since it corresponds to abstract Witt rings satisfying $AP(3)$. The second axiomatisation, special groups, is due to Dickmann and Miraglia (see [8]), and is equivalent to abstract Witt rings in the sense that the category of special groups is isomorphic to the category of abstract Witt rings.

Definition 1.5.

1. A linked quaternionic mapping is a triple (G, B, q), where G and B are abelian groups of exponent 2, such that G has a distinguished element -1 and q is a map from $G \times G$ to B satisfying, for every $a, b, c, d \in G$:
   (a) q is symmetric and bilinear;
   (b) $q(a, a) = q(a, -1)$;
   (c) $q(a, b) = q(c, d) \Rightarrow \exists x \in G\ q(a, b) = q(a, x) \wedge q(c, d) = q(c, x)$.
2. A special group is a structure $(G, \cdot, 1, -1, \equiv)$, where $(G, \cdot, 1, -1)$ is a group of exponent 2 written multiplicatively (which we will often consider as a vector space over $\mathbb{F}_2$), with a distinguished element -1, and $\equiv$ is a binary relation between pairs of elements of G (so actually a 4-ary relation), called

the binary isometry relation, such that the following axioms are satisfied, for every $a, b, c, d, x \in G$:

(SG0) $\equiv$ is an equivalence relation;
(SG1) $\langle a, b\rangle \equiv \langle b, a\rangle$;
(SG2) $\langle a, -a\rangle \equiv \langle 1, -1\rangle$;
(SG3) $\langle a, b\rangle \equiv \langle c, d\rangle \Rightarrow ab = cd$;
(SG4) $\langle a, b\rangle \equiv \langle c, d\rangle \Rightarrow \langle a, -c\rangle \equiv \langle -b, d\rangle$;
(SG5) $\langle a, b\rangle \equiv \langle c, d\rangle \Rightarrow \langle ax, bx\rangle \equiv \langle cx, dx\rangle$;
(SG6) The isometry relation of forms of dimension 3 (see (1) below) is transitive.

Remark 1.6. (1) If G is a group with distinguished element -1 and $a \in G$, we write $-a$ for $-1 \cdot a$.
(2) These objects are all models of first-order theories in the appropriate languages (see (1) below for the expression of the special group axiom SG6 as a first-order sentence). The language of quaternionic mappings is $L_{QM} := \{G, B, \cdot, +, 1, 0, -1, q\}$, where $(G, \cdot, 1)$ and $(B, +, 0)$ are groups of exponent 2 together with the obvious interpretations for -1 and q, while the language of special groups is $L_{SG} := \{\cdot, 1, -1, \equiv\}$, once again with the obvious interpretations.

It is possible, starting with any one of these two structures, to define an abstract Witt ring associated to it, denoted by $W(G)$. The procedure is in both cases as follows: We first define a (diagonal) quadratic form to be a tuple $\langle a_1, \ldots, a_n\rangle$ of elements of G and we define the sum and tensor product of diagonal forms in the usual way:

$$\langle a_1, \ldots, a_n\rangle \oplus \langle b_1, \ldots, b_m\rangle = \langle a_1, \ldots, a_n, b_1, \ldots, b_m\rangle,$$
$$\langle a_1, \ldots, a_n\rangle \otimes \langle b_1, \ldots, b_m\rangle = \langle a_1b_1, \ldots, a_1b_m, a_2b_1, \ldots, a_nb_m\rangle.$$

The isometry of forms of dimension 2 is then defined, for linked quaternionic mappings by $\langle a, b\rangle \equiv \langle c, d\rangle \Leftrightarrow (ab = cd \wedge q(a, b) = q(c, d))$, and for special groups by the 4-ary relation of the special group. The isometry between forms of dimension $n \geq 3$ is defined by induction as follows:

$$\begin{gathered}\langle a_1, \ldots, a_n\rangle \equiv \langle b_1, \ldots, b_n\rangle \\ \Leftrightarrow \\ \exists x, y, c_3, \ldots, c_n \in G\ \langle a_1, x\rangle \equiv \langle b_1, y\rangle \wedge \langle a_2, \ldots, a_n\rangle \equiv \langle x, c_3, \ldots, c_n\rangle \wedge \\ \langle b_2, \ldots, b_n\rangle \equiv \langle y, c_3, \ldots, c_n\rangle.\end{gathered} \tag{1}$$

The notion of Witt-equivalence is defined from isometry in the usual way and the abstract Witt ring is the set of all forms modulo Witt equivalence (see [21] section 3 for linked quaternionic mappings, and [8] section 1.25 for special groups).
An important difference between these two axiomatisations is that in the case of special groups, the associated Witt ring satisfies the properties $AP(1)$ and $AP(2)$ (see [8, section 1.25] together with [19, p. 63]), while in the case of a quaternionic mapping it also satisfies $AP(3)$ (see [21, corollary 3.7]).

Conversely, if (W, G) is an abstract Witt ring, it is possible to define a special group from (W, G) as follows: the underlying group of G is that of the abstract Witt ring, and $\equiv$ is defined by $\langle a, b\rangle \equiv \langle c, d\rangle$ if and only if $a + b = c + d$ in the Witt ring (see [7]).
To define a quaternionic mapping from an abstract Witt ring, we need the abstract Witt ring to satisfy the extra axiom $AP(3)$. If this is the case, then the quaternionic mapping is obtained as follows: G is the group G given by the abstract Witt ring, B is the group I^2/I^3 and q is the map $(a, b) \mapsto (1 - a)(1 - b) + I^3$.

1.2.1. *On special groups.* Since most of the remainder of this paper will be concerned with special groups, we reformulate this last construction directly in terms of special groups, and recall some of the main results about them. If G is a special group, we denote by $I(G)$ the ideal of even dimensional forms in the Witt ring of G:

$$I(G) := \{\langle a_1, \dots, a_{2n}\rangle \mid n \in \mathbb{N},\ a_1, \dots a_{2n} \in G\} \subseteq W(G),$$

and consider the following map associated to G:

$$\begin{array}{rcl} q_G : G \times G & \to & I(G)^2/I(G)^3 \\ (a,b) & \mapsto & \langle 1, -a\rangle \otimes \langle 1, -b\rangle + I(G)^3. \end{array}$$

If G satisfies $AP(3)$ then $(G, I(G)^2/I(G)^3, q_G)$ is a linked quaternionic mapping.

Let $n \in \mathbb{N}$. Then $I(G)^n$ is additively generated by the Pfister forms of degree n, which by definition are the forms

$$\langle\langle a_1, \dots, a_n\rangle\rangle := \langle 1, a_1\rangle \otimes \dots \otimes \langle 1, a_n\rangle,$$

with $a_1, \dots, a_n \in G$. We also denote such a form by $\langle\langle \bar{a}\rangle\rangle$, where $\bar{a} = (a_1, \dots, a_n)$. Using this notation, we see that the property $AP(3)$ can be expressed by the following set of first-order L_{SG}-sentences:

$$\{\forall a_1, \dots, a_6 \in G\ \left(\exists \bar{b}_1, \dots, \bar{b}_n \in G^3\ \langle a_1, \dots, a_6\rangle = \langle\langle \bar{b}_1\rangle\rangle + \dots + \langle\langle \bar{b}_n\rangle\rangle\right) \\ \to \langle a_1, \dots, a_6\rangle \text{ is hyperbolic}\}_{n\in\mathbb{N}},$$

where the equality sign denotes Witt equivalence.

Recall that we have $I(G)^2 = \{\phi \in I(G) \mid d_\pm \phi = 1\}$, where $d_\pm\langle a_1, \dots, a_n\rangle := (-1)^{n(n-1)/2} a_1 \dots a_n$ is the signed discriminant of $\langle a_1, \dots, a_n\rangle$ (see [19, corollary 3.9]).

There are two important constructions used to build new special groups: product, which is the usual product of first-order structures, and extension (see [8] page 90 and example 1.10). A special group built up from finite special groups by applying a finite number of times the operations of product and extension is called a special group of finite type.

Definition 1.7. Let G be a special group.

(1) For a form $\phi = \langle a_1, \dots, a_n\rangle$ over G, the set of elements represented by ϕ is

$$D_G(\phi) := \{b \in G \mid \exists b_2, \dots, b_n\ \phi \equiv \langle b, b_2, \dots, b_n\rangle\}$$

(we also write $D(\phi)$ if the special group is clear from the context).

(2) G is called reduced if $-1 \neq 1$ and $D_G\langle 1, 1\rangle = \{1\}$.

(3) The chain length of G, $cl(G)$, is the largest integer n such that there exist $a_0, \dots, a_n \in G$ with $D_G\langle 1, a_0\rangle \subsetneq \dots \subsetneq D_G\langle 1, a_n\rangle$, if such an integer exists, and ∞ otherwise.

Remark 1.8. (1) For a special group G, it is equivalent to know its binary isometry relation or its binary representation relation. This follows from the equivalence:

$$\langle a, b\rangle \equiv \langle c, d\rangle \Leftrightarrow (ac \in D_G\langle 1, cd\rangle \wedge ab = cd).$$

(2) There is only one structure of reduced special group on the 2-element multiplicative group $\{-1, 1\}$ and it is characterised by $D\langle 1, 1\rangle = \{1\}$. We denote this reduced special group by $\mathbb{Z}_2$.

(3) Every special group of finite type has finite chain length. This is easily checked by induction on the construction of the special group.

Reduced special groups form a category that is isomorphic to the category of (abstract) spaces of orderings defined by Marshall (see [22] or [1] for the definition and [8] chapter 3 for the isomorphism) and the importance of the operations of product and extension comes from the following result, due to Marshall (see [20], where it is proved for spaces of orderings; this proof can also be found in [1, 22]):

Theorem 1.9 (Marshall)**.** *Let G be a reduced special group of finite chain length. Then G is built up from $\mathbb{Z}_2$ by applying a finite number of times the operations of product and extension (and is in particular of finite type).*

From a model-theoretic point of view, any special group of finite type is $\aleph_0$-categorical and $\aleph_0$-stable (see [3] corollary 4.5). Moreover, it is easy to check by induction on the construction of such a special group that its isometry relation (seen as a subset of G^4) is a finite union of cosets of subgroups of G^4, from which follows that every definable subset of G^n is a boolean combination of cosets of subgroups of G^n, for every $n \in \mathbb{N}$.

We need the notion of Pfister index of a quadratic form or a special group, which was introduced and developed in [9]:

Definition 1.10.
(1) For an integer $n \geq 0$ and a quadratic form ϕ over G, the Pfister index of degree n of ϕ in G, $I(n, \phi, G)$, is the least integer k such that ϕ is Witt-equivalent to a linear combination, with coefficients in G, of k Pfister forms of degree n, if $\phi \in I^n(G)$, and 0 otherwise.
(2) For each integer $m \geq 1$, the m-Pfister index of G in degree n is:

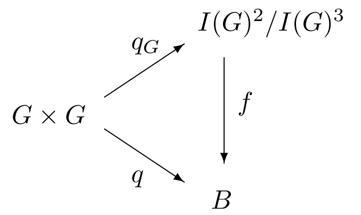
$$I(n, m, G) = \sup\{I(n, \phi, G) \mid \phi \text{ is a form of dimension } m\} \in \mathbb{N} \cup \{0, \infty\}.$$

Definition 1.11. Let (G, B, q) be a linked quaternionic mapping. The 2-symbol length of (G, B, q), denoted by $\lambda(G, B, q)$, is the least integer k such that every element of $\langle \operatorname{Im} q \rangle$ can be written as a sum of k elements of $\operatorname{Im} q$, if such an integer exists, and ∞ otherwise (where $\langle X \rangle$ denotes the subgroup of B generated by a set $X \subseteq B$).
Similarly, the 2-symbol length of a special group G, denoted by $\lambda(G)$, is the least integer k such that every element of $\langle \operatorname{Im} q_G \rangle$ can be written as a sum of k elements of $\operatorname{Im} q_G$, if such an integer exists, and ∞ otherwise.

This notion in the case of a field F, where $B = Br_2(F)$ and $q(a, b)$ is the class of the quaternion algebra $(a, b)_F$ in $Br(F)$ (where $Br(F)$ denotes the Brauer group of F and $Br_2(F)$ denotes its 2-torsion part) can be found in [5], but also in [24] (where it is called the 2-linkage number) or in [16].
If $(G, I(G)^2/I(G)^3, q_G)$ is the linked quaternionic mapping associated to a special group G satisfying $AP(3)$, then, of course, $\lambda(G, I(G)^2/I(G)^3, q_G) = \lambda(G)$, but it seems unclear whether $\lambda(G) = \lambda(G, B, q)$ when G is the special group of an arbitrary quaternionic mapping (G, B, q). The following lemma shows that one inequality always holds:

Lemma 1.12. *Let (G, B, q) be a quaternionic mapping and let G be the special group defined from it. Then $\lambda(G, B, q) \leq \lambda(G)$.*

Proof. We can assume that $\lambda(G)$ is finite. By [19, lemma 3.15 and theorem 3.16], the quaternionic mapping $q_G : G \times G \to I(G)^2/I(G)^3$ is the universal Steinberg symbol of G (see the definitions p. 50 and 51 in [19]). There is then a group homomorphism $f : I(G)^2/I(G) \to B$ such that the diagram



is commutative.
Consider now $w = \sum_{i=1}^{n} q(a_i, b_i) \in \langle \operatorname{Im} q \rangle$. We have $w = f(\sum_{i=1}^{n} q_G(a_i, b_i))$ and by hypothesis there are $c_1, d_1, \ldots, c_{\lambda(G)}, d_{\lambda(G)} \in G$ such that $\sum_{i=1}^{n} q_G(a_i, b_i) = \sum_{i=1}^{\lambda(G)} q_G(c_i, d_i)$, from which follows, by applying f, that $w = \sum_{i=1}^{\lambda(G)} q(c_i, d_i)$. $\square$

We now present some basic properties of the 2-symbol length.

Lemma 1.13. *Let $\{K_i\}_{i \in I}$ be a set of special groups. We have:*

(1) *If $\lambda(K_{i_0}) \geq \lambda_0$ for some $i_0 \in I$ then $\lambda(\prod_{i \in I} K_i) \geq \lambda_0$;*
(2) *If the following two conditions are satisfied*

$$\forall i \in I \ \lambda(K_i) \leq \lambda_0 \text{ and}$$
$$\forall n \in \mathbb{N} \ \exists N \in \mathbb{N} \ \forall i \in I \ I(3, n + 4\lambda_0, K_i) \leq N,$$

then $\lambda(\prod_{i \in I} K_i) \leq \lambda_0$.

Proof. We write K for $\prod_{i \in I} K_i$.

(1) By hypothesis there are $l \geq \lambda_0$ and $D_0 := \langle a_{i_0,1}, \ldots, a_{i_0,2n} \rangle + I(K_{i_0})^3 \in \sum_{i=1}^{l} \operatorname{Im} q_{K_{i_0}} \setminus \sum_{i=1}^{\lambda_0 - 1} \operatorname{Im} q_{K_{i_0}}$ (the notation $A \setminus B$ denotes the set-theoretic difference). Define $\langle a_{i,1}, \ldots, a_{i,2n} \rangle := n\mathbb{H} \in W(K_i)$ for $i \in I \setminus \{i_0\}$ and let $D := \langle (a_{i,1})_{i \in I}, \ldots, (a_{i,2n})_{i \in I} \rangle \in W(K)$. Suppose now that $\lambda(K) < \lambda_0$. Then D can be written as

$$D = \langle\langle (b_{i,1})_{i \in I}, (c_{i,1})_{i \in I} \rangle\rangle + \cdots + \langle\langle (b_{i,\lambda_0 - 1})_{i \in I}, (c_{i,\lambda_0 - 1})_{i \in I} \rangle\rangle + \phi, \tag{2}$$

for some $(b_{i,j})_{i \in I}, (c_{i,j})_{i \in I} \in K$ and $\phi \in I(K)^3$.
Applying to the equality (2) the map $\pi : W(K) \to W(K_{i_0})$ induced by the projection $K \to K_{i_0}$ we get

$$D_0 = \langle\langle b_{i_0,1}, c_{i_0,1} \rangle\rangle + \cdots + \langle\langle b_{i_0,\lambda_0 - 1}, c_{i_0,\lambda_0 - 1} \rangle\rangle + \pi(\phi),$$

and writing ϕ as a sum of Pfister forms of degree 3 over K we see that $\pi(\phi)$ is a sum of Pfister forms of degree 3 over K_{i_0} and thus belongs to $I(K_{i_0})^3$. From this follows that $D_0 \in \sum_{i=1}^{\lambda_0 - 1} \operatorname{Im} q_{K_{i_0}}$, a contradiction.

(2) Let $D = \langle (a_{i,1})_{i \in I}, \ldots, (a_{i,n})_{i \in I} \rangle + I(K)^3 \in I(K)^2/I(K)^3$. For $i \in I$ we have, by assumption:

$$\langle a_{i,1}, \ldots, a_{i,n} \rangle = \langle\langle b_{i,1}, c_{i,1} \rangle\rangle + \cdots + \langle\langle b_{i,\lambda_0}, c_{i,\lambda_0} \rangle\rangle + \alpha_{i,1} \langle\langle \bar{d}_{i,1} \rangle\rangle + \cdots + \alpha_{i,k_i} \langle\langle \bar{d}_{i,k_i} \rangle\rangle,$$

for some $b_{i,j}, c_{i,j}, \alpha_{i,j} \in K_i$ and $\bar{d}_{i,j} \in K_i^3$. By hypothesis, we can assume that $k_i = N$ for every $i \in I$ (because the form $\langle a_{i,1}, \ldots, a_{i,n} \rangle - \langle\langle b_{i,1}, c_{i,1} \rangle\rangle - \cdots - \langle\langle b_{i,\lambda_0}, c_{i,\lambda_0} \rangle\rangle$ has dimension $n + 4\lambda_0$), from which follows

$$\langle (a_{i,1})_{i \in I}, \ldots, (a_{i,n})_{i \in I} \rangle = \langle\langle (b_{i,1})_{i \in I}, (c_{i,1})_{i \in I} \rangle\rangle + \cdots + \langle\langle (b_{i,\lambda_0})_{i \in I}, (c_{i,\lambda_0})_{i \in I} \rangle\rangle + (\alpha_{i,1})_{i \in I} \langle\langle (\bar{d}_{i,1})_{i \in I} \rangle\rangle + \cdots + (\alpha_{i,N})_{i \in I} \langle\langle (\bar{d}_{i,N})_{i \in I} \rangle\rangle,$$

proving that $D \in \sum_{i=1}^{\lambda_0} \operatorname{Im} q_K$.

□

Lemma 1.14. *Let G and H be special groups. Then*

1. *If $f : G \to H$ is a surjective morphism of special groups then $\lambda(H) \leq \lambda(G)$.*
2. *If $f : G \to H$ is a pure morphism of special groups then $\lambda(G) \leq \lambda(H)$.*
3. *If G is finite then $\lambda(G) \leq |G|^2$.*

Proof. (1) We can assume that $\lambda(G)$ is finite. Let $w = \sum_{i=1}^n \langle\langle a_i, b_i\rangle\rangle$ be an element of $I(H)^2$. Since f is surjective, there are $c_i, d_i \in G$ such that $w = \sum_{i=1}^n \langle\langle f(c_i), f(d_i)\rangle\rangle$. By hypothesis we find $e_1, f_1, \ldots, e_{\lambda(G)}, f_{\lambda(G)} \in G$ such that $\sum_{i=1}^n \langle\langle c_i, d_i\rangle\rangle = \sum_{j=1}^{\lambda(G)} \langle\langle e_j, f_j\rangle\rangle \mod I(G)^3$, i.e. there exist $\bar{\alpha}_1, \ldots, \bar{\alpha}_k \in G^3$ and $r, s \in \mathbb{N}$ such that

$$G \models \bigoplus_{i=1}^{n} \langle\langle c_i, d_i\rangle\rangle \oplus r\langle -1, 1\rangle \equiv s\langle -1, 1\rangle \oplus \bigoplus_{j=1}^{\lambda(G)} \langle\langle e_j, f_j\rangle\rangle \oplus \bigoplus_{l=1}^{k} \langle\langle \bar{\alpha}_l\rangle\rangle.$$

Applying f to this existential positive formula shows that w is the sum of at most $\lambda(G)$ elements of $\operatorname{Im} q_H$ modulo $I(H)^3$.

(2) We can assume that $\lambda(H)$ is finite. Let $w \in \langle \operatorname{Im} q_G\rangle$, $w = w_0 + I(G)^3$ for some $w_0 \in I(G)^2$. We consider the map

$$\begin{array}{rcl} \xi : I(G)^2/I(G)^3 & \to & I(H)^2/I(H)^3 \\ \phi + I(G)^3 & \mapsto & f(\phi) + I(H)^3. \end{array}$$

It is well-defined since $I(G)^3 \subseteq I(H)^3$ and it is a morphism of groups. By hypothesis we have, for $k = \lambda(H)$, $\xi(w) = q_H(\bar{h}_1) + \cdots + q_H(\bar{h}_k)$ for some $\bar{h}_1, \ldots, \bar{h}_k \in H^2$, which means that, for some $l \in \mathbb{N}$ and $\bar{a}_1, \ldots, \bar{a}_l \in H^3$

$$f(w_0) = \langle\langle -\bar{h}_1\rangle\rangle + \cdots + \langle\langle -\bar{h}_k\rangle\rangle + \sum_{i=1}^{l} \langle\langle \bar{a}_i\rangle\rangle \text{ in the Witt ring of } H.$$

This in turn means that there are $r, s \in \mathbb{N}$ such that

$$\begin{array}{rl} H \models & \exists \bar{h}_1, \ldots, \bar{h}_k \in H^2 \ \exists \bar{a}_1, \ldots, \bar{a}_l \in H^3 \\ & f(w_0) \oplus r\langle 1, -1\rangle \equiv s\langle 1, -1\rangle \oplus \langle\langle -\bar{h}_1\rangle\rangle \oplus \cdots \oplus \langle\langle -\bar{h}_k\rangle\rangle \oplus \bigoplus_{i=1}^{l} \langle\langle \bar{a}_i\rangle\rangle. \end{array}$$

This formula is positive-existential in L_{SG} and using the purity of f, we get that G satisfies the same formula (with w_0 instead of $f(w_0)$), proving that w is a sum of at most $\lambda(H)$ elements of $\operatorname{Im} q_G$.

(3) $\langle \operatorname{Im} q_G\rangle$ is a vector space over $\mathbb{F}_2$ and is generated by $\{q_G(a_1, a_2) \mid a_1, a_2 \in G\}$ which has cardinality at most $|G|^2$.

□

We conclude this section by checking that Kahn's result [16, théorème 2] linking the u-invariant and the 2-symbol length still holds for special groups. The proofs work in exactly the same way, but we reproduce them here, due to the different context. We begin by checking the translation of lemma 5 from [23] to our setting. To do so we use the Witt invariant for abstract Witt rings, denoted by w, as defined in [19] p. 53 (note that the map denoted by s in [19] is the map we call q_G for a special group G).

Lemma 1.15. *Let G be a special group with associated quaternionic mapping q_G.*

1. *Let $f \in I(G)^2$ be of dimension $2m > 0$. Then $w(f)$ is a sum of $m - 1$ elements of $\operatorname{Im} q_G$;*
2. *Let $D \in I(G)^2/I(G)^3$ be a sum of $m-1$ elements of $\operatorname{Im} q_G$ for some $m \in \mathbb{N}$. Then there exists a quadratic form f over G, of dimension $2m$, such that $f \in I(G)^2$ and $w(f) = D$.*

Proof. (1) We proceed by induction on m. If $m = 1$ then (by AP(2)) f is the hyperbolic plane $\mathbb{H}$ and $w(f) = 0$ is the sum of 0 elements of $\operatorname{Im} q_G$. If $m > 1$, write $f = \langle a_1, \dots, a_{2m}\rangle = \langle a_1, a_2, a_3, a_1a_2a_3\rangle + \langle -a_1a_2a_3, a_4, \dots, a_{2m}\rangle$ in $W(G)$. We know $d_\pm f = d_\pm\langle a_1, a_2, a_3, a_1a_2a_3\rangle = 1$ and hence $d_\pm\langle -a_1a_2a_3, a_4, \dots, a_{2m}\rangle = 1$ so all three forms belong to $I(G)^2$. By [19, proposition 3.11] we get $w(f) = w(\langle a_1, a_2, a_3, a_1a_2a_3\rangle) + w(\langle -a_1a_2a_3, a_4, \dots, a_{2m}\rangle)$. Since $\langle a_1, a_2, a_3, a_1a_2a_3\rangle = a_1\langle\langle a_1a_2, a_1a_3\rangle\rangle$ we see that $w(\langle a_1, a_2, a_3, a_1a_3a_3\rangle = q_G(-a_1a_2, -a_1a_3)$ ([19, lemma 3.13]), while we know by induction that $w(\langle -a_1a_2a_3, a_4 \dots, a_{2m}\rangle)$ is a sum of $m-2$ elements of $\operatorname{Im} q_G$.

(2) The proof is by induction on m. If $m = 1$ then $D = 0 = w(\mathbb{H})$. Assume now that $D = D' + q_G(a, b)$ where D' is a sum of $m-2$ elements of $\operatorname{Im} q_G$. By induction $D' = w(\psi)$ for some form $\psi \in I(G)^2$ of dimension $2(m-1)$. Let $c \in D_G\psi$. Then $\psi \perp \langle -c, ac, bc, -abc\rangle$ is isotropic, so isometric to $\phi \perp \mathbb{H}$ for some form ϕ of dimension $2m$. Since $d_\pm\phi = d_\pm(\phi \perp \mathbb{H}) = d_\pm(\psi \perp \langle -c, ac, bc, -abc\rangle) = d_\pm\psi = 1$ we know that $\phi \in I(G)^2$. And $w(\phi) = w(\phi \perp \mathbb{H}) = w(\psi \perp \langle -c, ac, bc, -abc\rangle) = w(\psi) + w(-c\langle\langle -a, -b\rangle\rangle) = D' + q_G(a, b) = D$. □

Corollary 1.16 ([16], théorème 2)**.** *Let G be a special group and let*

$$u(G) := \sup\{\dim\phi \mid \phi \textit{ anisotopic form over } G\} \in \mathbb{N} \cup \{\infty\}.$$

Then $u(G) \geq 2(\lambda(G) + 1)$.

Proof. Assume first $\lambda(G)$ is finite and let $m = \lambda(G) + 1$. By definition of $\lambda(G)$ there is $D \in \sum_{i=1}^{m-1} \operatorname{Im} q_G \setminus \sum_{i=1}^{m-2} \operatorname{Im} q_G$. By the second part of lemma 1.15 there is $f \in I(G)^2$ of dimension $2m$ such that $D = w(f)$. Suppose now that f is isotropic. Then $f = f' \oplus \mathbb{H}$ for some form f' which is then in $I(G)^2$ and has dimension $2(m-1)$. By the first part of lemma 1.15, $w(f') \in \sum_{i=1}^{m-2} \operatorname{Im} q_G$, a contradiction since $w(f') = w(f) = D$. So f, of dimension $2m = 2(\lambda(G)+1)$, is anisotropic. This gives $u(G) \geq 2(\lambda(G) + 1)$. The case $\lambda(G) = \infty$ is similar. □

2. $\aleph_0$-stable $\aleph_0$-categorical linked quaternionic mappings

The idea is simply to apply [4, theorem 56]:

Definition 2.1. Let $b : U \times V \to W$ be a bilinear map of abelian groups. We say that b is trivial-by-finite if there are subgroups U_0 of U and V_0 of V such that the index of U_0 in U and the index of V_0 in V are finite, and $b : U_0 \times V_0 \to \{0\}$.

Theorem 2.2 ([4], theorem 56)**.** *Let (G, B, q) be an $\aleph_0$-stable $\aleph_0$-categorical quaternionic mapping. Then q is trivial-by-finite.*

Proposition 2.3. *Let G be the special group associated to an $\aleph_0$-stable $\aleph_0$-categorical quaternionic mapping (G, B, q). Then*

1. *G is $\aleph_0$-stable and $\aleph_0$-categorical.*
2. *The isometry relation of G is a finite union of cosets of G^4.*
3. *$\lambda(G, B, q)$ is finite.*

Proof. (1) G is $\aleph_0$-stable and $\aleph_0$-categorical since it is definable in (G, B, q).

(2) It is easy to check that the isometry relation is a finite union of cosets of subgroups of G^4 if and only if the representation relation is a finite union of cosets of subgroups of G^2. So it is enough to show the conclusion for the representation relation, i.e. for $\ker q$. By theorem 2.2 there is a subspace U of G (considered as a vector space over $\mathbb{F}_2$) of finite codimension such that

$q \restriction U \times U = 0$. Let E be a complement of U in G (E is then finite) and let $e, e' \in E$, $u, u' \in U$. Then

$$\begin{array}{rrcl} & q(e \cdot u, e' \cdot u') & = & 0 \\ \Leftrightarrow & q(e, u') + q(u, e') & = & -q(e, e') \\ \Leftrightarrow & (u, u') \in g_{e,e'}^{-1}(-q(e, e')) & & \end{array}$$

where $g_{e,e'} : G \times G \to B$, $(u, u') \mapsto q(e, u') + q(u, e')$ is a linear map. In particular $g_{e,e'}^{-1}(-q(e, e'))$ is a coset of $G \times G$ and

$$q^{-1}(0) = \bigcup_{e,e' \in E} (e, e') \cdot g_{e,e'}^{-1}(-q(e, e')),$$

is a finite union of cosets since E is finite.

(3) Consider the definable subsets

$$A_m := \{b \in B \mid \exists a_1, b_1, \ldots, a_m, b_m \in G \ \ (b = \sum_{i=1}^{m} q(a_i, b_i))\}.$$

If $\lambda(G, B, q) = \infty$ then the sequence $A_0 \subseteq A_1 \subseteq A_2 \cdots$ has a strictly increasing subsequence, giving an infinite number of types over $\emptyset$, a contradiction to the $\aleph_0$-categoricity of (G, B, q).

□

We are now interested in the converse and want to determine which special groups give rise to $\aleph_0$-stable $\aleph_0$-categorical linked quaternionic mappings.

Lemma 2.4. *Let G be an $\aleph_0$-categorical special group. Then $I(n, m, G)$ is finite for every $n, m \in \mathbb{N}$.*

Proof. This is a direct consequence of the fact that, for $a_1, \ldots, a_m \in G$, the property $I(n, \langle a_1, \ldots, a_m \rangle, G) \leq k$ can be expressed by a first-order formula with parameters $a_1, \ldots, a_m$ in the language of special groups: Suppose that $I(n, m, G)$ is infinite. Then for every $k \in \mathbb{N}$ there is $(a_1, \ldots, a_m) \in G^m$ such that $I(n, \langle a_1, \ldots, a_m \rangle, G) > k$. In particular $|S_m(\emptyset)| \geq \aleph_0$, which contradicts $\aleph_0$-categoricity. □

Lemma 2.5. *Let G be an $\aleph_0$-categorical special group of finite 2-symbol length. Then the mapping associated to G, q_G, is interpretable in G.*

Proof. Let $k := \lambda(G)$. Then

$$I(G)^2/I(G)^3 = \{\langle\langle a_1, b_1 \rangle\rangle + \cdots + \langle\langle a_k, b_k \rangle\rangle + I(G)^3 \mid a_1, b_1, \ldots, a_k, b_k \in G\}.$$

Since $q_G(a, b) = \langle\langle -a, -b \rangle\rangle + I(G)^3 \in I(G)^2/I(G)^3$, we only have to show that $I(G)^2/I(G)^3$ is interpretable in G. As a set, it is equal to G^{2k}/E, where an element $(a_1, b_1, \ldots, a_k, b_k) \in G^{2k}$ represents the quadratic form $\langle\langle a_1, b_1 \rangle\rangle + \cdots + \langle\langle a_k, b_k \rangle\rangle$ and E is the equivalence relation

$$(a_1, b_1, \ldots, a_k, b_k) \, \mathrm{E} (c_1, d_1, \ldots, c_k, d_k)$$

$$\Leftrightarrow$$

$$(*) \qquad \langle\langle a_1, b_1 \rangle\rangle + \cdots + \langle\langle a_k, b_k \rangle\rangle - \langle\langle c_1, d_1 \rangle\rangle - \cdots - \langle\langle c_k, d_k \rangle\rangle \in I(G)^3.$$

The quadratic form in $(*)$ has dimension $8k$ and by lemma 2.4 we know that $I(3, 8k, G) = l < \aleph_0$. From this we see that $(*)$ is equivalent to the first-order L_{SG}-formula

$$\exists \bar{\alpha}_1, \ldots, \bar{\alpha}_l \in G^3 \ \exists \beta_1, \ldots, \beta_l \in G$$
$$\langle\langle a_1, b_1 \rangle\rangle + \cdots + \langle\langle a_k, b_k \rangle\rangle - \langle\langle c_1, d_1 \rangle\rangle - \cdots - \langle\langle c_k, d_k \rangle\rangle =$$
$$\beta_1 \langle\langle \bar{\alpha}_1 \rangle\rangle + \cdots + \beta_l \langle\langle \bar{\alpha}_l \rangle\rangle$$

(the equality is actually in the Witt ring, and the L_{SG}-formula has to express it in terms of Witt-equivalence, but this is possible since the dimensions of the forms on both sides of the equality are fixed).
To complete the proof we have to check that sum in $I(G)^2/I(G)^3$ can also be expressed by a first-order formula. For $(a_1, b_1, \dots, a_k, b_k)$, $(c_1, d_1, \dots, c_k, d_k)$, $(e_1, f_1, \dots, e_k, f_k)$ in G^{2k} we have $\langle\langle a_1, b_1\rangle\rangle + \cdots + \langle\langle a_k, b_k\rangle\rangle + \langle\langle c_1, d_1\rangle\rangle + \cdots + \langle\langle c_k, d_k\rangle\rangle = \langle\langle e_1, f_1\rangle\rangle + \cdots + \langle\langle e_k, f_k\rangle\rangle$ in $I(G)^2/I(G)^3$ if and only if $\langle\langle a_1, b_1\rangle\rangle + \cdots + \langle\langle c_k, d_k\rangle\rangle - \langle\langle e_1, f_1\rangle\rangle - \cdots - \langle\langle e_k, f_k\rangle\rangle \in I(G)^3$, which can, as above, be expressed by a first-order L_{SG}-formula. □

Proposition 2.6. *Let G be a special group satisfying $AP(3)$ and assume that G is $\aleph_0$-stable $\aleph_0$-categorical of finite 2-symbol length. Then its associated linked quaternionic mapping, $(G, I(G)^2/I(G)^3, q_G)$, is $\aleph_0$-stable and $\aleph_0$-categorical.*

Proof. Since G satisfies $AP(3)$, we know that $(G, I(G)^2/I(G)^3, q_G)$ is a linked quaternionic mapping. By lemma 2.5 it is interpretable in G, so it is $\aleph_0$-stable and $\aleph_0$-categorical. □

We now consider the reduced case:

Corollary 2.7. *Let (G, B, q) be a reduced $\aleph_0$-stable $\aleph_0$-categorical quaternionic mapping. Then G is finite.*

Proof. We know by theorem 2.2 that (G, B, q) is trivial-by-finite. Let U be a subgroup of finite index of G, such that $q \upharpoonright U \times U = 0$. Let $u \in U$. Then $q(u, u) = 0$, i.e. $u \in D_G\langle 1, -u\rangle$, i.e. $u \in D_G\langle 1, 1\rangle = \{1\}$. So $U = \{1\}$ and G is finite. □

Remark 2.8. Recalling that the $\aleph_0$-stable reduced special groups are the reduced special groups of finite type, and that they are $\aleph_0$-categorical (see [3] corollary 4.5 and the paragraph following it, together with proposition 6.2), shows that if G is a finite reduced special group and H is an infinite group of exponent 2, then $\lambda(G[H]) = \infty$ (otherwise $(G[H], I(G[H])^2/I(G[H])^3, q_{G[H]})$ would be interpretable in $G[H]$ which is $\aleph_0$-stable and $\aleph_0$-categorical, so would be itself $\aleph_0$-stable and $\aleph_0$-categorical, implying that $G[H]$ should be finite).
The condition that a linked quaternionic mapping is $\aleph_0$-stable and $\aleph_0$-categorical is then quite restrictive. However, the condition appearing in proposition 2.3 that the isometry relation is a finite union of cosets is weaker (see remark 3.2) but still produces very manageable special groups, which will be our main object of study for the rest of the paper.

3. Special groups whose isometry relation is a finite union of cosets

Definition 3.1. We denote by SG_{FC} the class of $\aleph_0$-categorical special groups whose representation relation is a finite union of cosets.

We check easily that for a special group G the representation relation is a union of m cosets if and only if the isometry relation is a union of m cosets.
We consider the representation relation of a special group G as a subset R^G of G^2: $R^G := \{(a, b) \in G^2 \mid a \in D_G\langle 1, b\rangle\}$.

The simplest examples of special groups in SG_{FC} are the special groups of finite type: If G is such a special group, we can see by induction on the construction of G that its representation relation is a finite union of cosets. The only non obvious step is the extension, and for this it is enough to see that the standard presentation of the representation relation of $G[H]$ (see [8] p. 10) can be reformulated as follows:

$$\begin{aligned} R^{G[H]} = &\{(g'h', gh) \in (G \times H)^2 \mid h = 1 \wedge h' = 1 \wedge (g', g) \in R^G\} \\ &\cup \{(g'h', gh) \in (G \times H)^2 \mid h = 1 \wedge g = -1\} \\ &\cup \{(g'h', gh) \in (G \times H)^2 \mid g'h' \in \{1, gh\}\}. \end{aligned}$$

Note that all special groups of finite type are $\aleph_0$-categorical (and hence in SG_{FC}) since they are obtained from $\aleph_0$-categorical structures by a finite number of products and extensions, which are generalized products in the sense of [10] (see for instance [3, lemma 2.6]).

Remark 3.2. It follows that there are special groups (even reduced special groups) G whose isometry relation is a finite union of cosets but whose associated quaternionic mapping $(G, I(G)^2/I(G)^3, q_G)$ is not $\aleph_0$-stable and $\aleph_0$-categorical: just take $G = G_0[H]$ where G_0 is any finite reduced special group and H is an infinite group of exponent 2. Its isometry relation is a finite union of cosets, but $\lambda(G_0[H]) = \infty$ by remark 2.8.

Lemma 3.3. *Let $G \in SG_{FC}$. To G we associate two languages*

$$\begin{aligned} &L_m = \{1, -1, \cdot, \bar{a}_1, \ldots, \bar{a}_m, \Delta_1, \ldots, \Delta_m\}, \text{ and} \\ &L_m^- = \{1, \cdot, \Delta_1, \ldots, \Delta_m\}, \end{aligned}$$

and there is an interpretation in G of these symbols that turns G into an L_m-structure with the following properties

1. *$R^G = \bigcup_{i=1}^m \bar{a}_i^G \Delta_i^G$, where $\bar{a}_i^G \in G^2$ and Δ_i^G is a subgroup of G^2 (the notation is a little abusive: each $\bar{a}_i$ denotes in fact two constant symbols since $\bar{a}_i^G \in G^2$);*
2. *each Δ_i^G is L_{SG}-definable in G;*
3. *G is $\aleph_0$-categorical and $\aleph_0$-stable as L_m- and L_{SG}-structure.*

Any future reference to G as an L_m-structure will assume an interpretation of the symbols possessing the three properties listed above.

Proof. Since the relation R^G is a finite union of cosets of subgroups of G^2, we know that G is one-based (see [14]). In particular R^G is a finite boolean combination of cosets of L_{SG}-definable subgroups of G^2 ([14, theorem 4.1]). We now use the same argument as in Theorem 4.3 (in this paper) $2 \Rightarrow 1$: Since R^G is a finite union of cosets, it is closed for the topology introduced there, and the same proof shows that the finite boolean combination of L_{SG}-definable cosets that describes R^G can be chosen to be a finite union of L_{SG}-definable cosets. The L_m-structure G is then $\aleph_0$-categorical (since G is $\aleph_0$-categorical in L_{SG}), and the $\aleph_0$-stability (as L_m-structure and then as L_{SG}-structure by interpretation) follows by Proposition 1.2. □

Proposition 3.4. *SG_{FC} is closed under elementary equivalence in the language L_{SG} of special groups.*

Proof. Let $G \in SG_{FC}$ and let K be a special group such that $G \equiv K$ in L_{SG}. Let L_m be a language associated to G. By the Keisler-Shelah theorem there is a set I and an ultrafilter $\mathcal{U}$ on I such that $G^I/\mathcal{U} \cong K^I/\mathcal{U}$ in L_{SG}. Let $G' = G^I/\mathcal{U}$ and $K' = K^I/\mathcal{U}$. Since G is an L_m-structure, G' is naturally an L_m-structure such that $G \prec G'$ in L_m, and $R_G = \bigcup_{i=1}^m \bar{a}_i^G \Delta_i^G$ implies $R_{G'} = \bigcup_{i=1}^m \bar{a}_i^{G'} \Delta_i^{G'}$. Let f be the isomorphism of special groups between G' and K'. Then $R_{K'} = \bigcup_{i=1}^m f(\bar{a}_i^{G'}) f(\Delta_i^{G'})$, which shows that the isometry relation of K' is a finite union of cosets. Since $R^K = R^{K'} \cap K^2$, the isometry relation of K is also a finite union of cosets (and the number of cosets can be bounded by m). □

3.1. **Using the model theory of modules.**

Proposition 3.5. *Let $G \in SG_{FC}$. Then G is algebraically compact in the languages L_m and L_{SG}. In particular, if $L = L_m$ or L_{SG} then every pure L-morphism $f : G \to H$ (where H is an L-structure) has a retract which is an L-morphism.*

Proof. We first prove algebraic compactness. Using the equivalence between 1. and 2. in theorem 1.3, it is enough to show that for every $n \in \omega$ the subsets of G^n defined by a pp-formula satisfy the descending chain condition (DCC; i.e. there is no infinite strictly descending chain of pp-definable subsets of G^n) for the languages L_m and L_{SG}.
We start with L_m: The subsets of G^n defined by pp-formulas are cosets of definable subgroups of G^n, and satisfy the DCC since G is an $\aleph_0$-stable group.
For the language L_{SG}: Since R^G is a finite union of cosets it is easy to see that the subsets defined by L_{SG}-pp-formulas are finite unions of cosets which are themselves defined by L_m-pp-formulas. We conclude using [25, lemma 2.7], whose terminology we follow:
Let Γ be the set of L_m-pp-formulas. Then $cl_1(\Gamma)$, the closure of Γ under substitution of variables and conjunctions, is equal to Γ and has the DCC. By [25, lemma 2.7], $cl_2(\Gamma)$, the closure of Γ under substitution of variables and positive boolean combinations, also has the DCC. In particular, since every L_{SG}-pp-formula is equivalent to some formula in $cl_2(\Gamma)$, the subsets of G^n defined by L_{SG}-pp-formulas have the DCC.

The second part of the proposition follows at once from theorem 1.3, $1 \Leftrightarrow 4$, if we take for $\bar{b}$ an enumeration of G and for $\bar{a}$ the tuple $f(\bar{b})$. □

The results in this section are inspired by the fact that if $G \in SG_{FC}$, then G, considered as an L_m^--structure (with $R^G = \bigcup_{i=1}^m \bar{a}_i^G \Delta_i^G$), is an abelian structure in the sense of [11]. Since the properties of abelian structures are close to those of modules, some results from the model theory of modules apply to G. The main one comes from [12]:

Theorem 3.6 ([12], theorem 7)**.** *If A is an R-module then $S_2(\mathrm{Th}(A))$ is finite if and only if there are finite R-modules $A_1, \dots, A_n$ and cardinals $\alpha_1, \dots, \alpha_n$ such that $A = A_1^{(\alpha_1)} \oplus \dots \oplus A_n^{(\alpha_n)}$.*

This result remains valid for a special group in SG_{FC}, in the language L_m^-:

Proposition 3.7. *Let $G \in SG_{FC}$ and let L_m be a language associated to G. Then there exist $n \in \omega$, finite L_m^--substructures $G_1, \dots, G_n$ of G and cardinals $\alpha_1, \dots, \alpha_n$ such that $G = G_1^{(\alpha_1)} \times \dots \times G_n^{(\alpha_n)}$ as a L_m^--structure.*

Proof. We work in the language L_m^-. $S_2(\mathrm{Th}(G))$ is finite since $\mathrm{Th}(G)$ is $\aleph_0$-categorical (see lemma 3.3). The proof consists in checking that all the results used in the proof of theorem 7 in Garavaglia's paper [12] remain valid for special groups in SG_{FC} in the language L_m^-. The key property that makes everything work is that every pure L_m^--substructure of G is a direct factor of G (in the language L_m^-), as recalled in theorem 1.3 5. □

We use proposition 3.7 to built retracts of the inclusion of some substructures of G. A special case of [8, lemma 5.17] will be useful:

Lemma 3.8. *Let G_0 be a L_{SG}-substructure of a special group G and let $\sigma : G \to G_0$ be an L_{SG}-morphism which is a retract of the inclusion of G_0 in G. Then $G = G_0 \times \ker \sigma$ as groups and*

(1) *G_0 is a special group;*

(2) *The L_{SG}-structure induced on $G/\ker\sigma$ (by $R^{G/\ker\sigma} := \sigma(R^G)$) coincides with the L_{SG}-structure of G_0.*

Proof. It suffices to observe that, in the notation of [8, lemma 5.17], G (our G_0) needs only be an L_{SG}-structure. The fact that it is then necessarily a special group follows from [8, lemma 5.17 i)]. □

The following is an easy consequence of proposition 3.7:

Lemma 3.9. *Let $G \in SG_{FC}$ and let L_m be a language associated to G, $L_m = \{1, -1, \cdot, \bar{a}_1, \ldots, \bar{a}_m, \Delta_1, \ldots, \Delta_m\}$. Write*

$$G = G_1^{(\alpha_1)} \times \cdots \times G_n^{(\alpha_n)} \tag{3}$$

as L_m^--structure, as in proposition 3.7. Let $J_k \subseteq \alpha_k$ for $k = 1, \ldots, n$, and assume $-1, \bar{a}_1, \ldots, \bar{a}_m \in H := G_1^{(J_1)} \times \ldots \times G_n^{(J_n)}$ (where $G_i^{(J_i)}$ denotes the subspace of $G_i^{(\alpha_i)}$ consisting of elements whose coordinates are 1 outside of J_i). Consider now the L_{SG}-structure induced on H by its inclusion in G. Then the canonical projection from G onto H induced by (3) in a L_{SG}-morphism, a retract of the inclusion of H in G, and H is a special group.

Proof. Let σ be the canonical projection from G onto H induced by (3). σ is clearly a retract of the inclusion of H in G. We only have to check that σ is an L_{SG}-morphism, since lemma 3.8 will then give that H is a special group:

$$\begin{array}{rcl} \sigma(R^G) & = & \sigma(\bigcup_{i=1}^m \bar{a}_i\Delta_i) \\ & = & \bigcup_{i=1}^m \sigma(\bar{a}_i\Delta_i) \\ & = & \bigcup_{i=1}^m \bar{a}_i\ \sigma(\Delta_i) \text{ since } \sigma(\bar{a}_i) = \bar{a}_i \\ & \subseteq & \bigcup_{i=1}^m \bar{a}_i\Delta_i \text{ since } \sigma \text{ is an } L_m^-\text{-morphism} \\ & \subseteq & R^G. \end{array}$$

□

Proposition 3.10. *Let $G \in SG_{FC}$. For every finite subset A of G there is a finite special subgroup G_A of G, G_A containing A, and a morphism of special groups $\sigma : G \to G_A$ which is a retract of the inclusion of G_A in G.*

Proof. Since A is finite there are a finite number of factors from the decomposition of G given in 3.9 (3), say $H_1, \ldots, H_k$, such that

$$A \cup \{-1, \bar{a}_1, \ldots, \bar{a}_m\} \subseteq H_1 \times \cdots \times H_k \subseteq G,$$

Let $G_A := H_1 \times \cdots \times H_k$ and let σ be the projection from G onto G_A induced by the decomposition of G. Lemma 3.9 then yields the conclusion. □

Definition 3.11. For a special group G and a special subgroup K of G, we denote by $X_{G,K}$ the set of special group morphisms from G to K which are retracts of the inclusion of K in G. If $\sigma \in X_{G,K}$ we define $G_\sigma := K$.
X_G^+ denotes the union of all $X_{G,K}$ for K finite special subgroup of G.

The following two results are straightforward consequences of proposition 3.10:

Lemma 3.12. *Let $G \in SG_{FC}$. For every existential sentence $\phi(\bar{g})$ with parameters $\bar{g}$ in G, there is a finite special subgroup G_0 of G containing $\bar{g}$ such that*

$$G \models \phi(\bar{g}) \Leftrightarrow G_0 \models \phi(\bar{g}).$$

Proposition 3.13. *Let $G \in SG_{FC}$. For every positive existential sentence $\phi(\bar{g})$ with parameters $\bar{g}$ in G we have*

$$G \models \phi(\bar{g}) \Leftrightarrow \forall\sigma \in X_G^+ \quad G_\sigma \models \phi(\sigma(\bar{g})).$$

Proposition 3.14. *Let $G \in SG_{FC}$, $G^* = \prod\{G_\sigma \mid \sigma \in X_G^+\}$ and $\nu : G \to G^*$, $g \mapsto (\sigma(g))_{\sigma \in X_G^+}$. Then ν is a monomorphism of special groups and, identifying ν with an inclusion, we have $G \prec_{\exists^+} G^*$, i.e. for every positive existential formula $\phi(\bar{g})$ with parameters in G, $G \models \phi(\bar{g})$ if and only if $G^* \models \phi(\nu(\bar{g}))$.*
In particular ν is a complete morphism of special groups and ν reflects the isotropy of quadratic forms.

Proof. Only the implication from right to left requires a justification. Let $\phi(\bar{g}) = \exists \bar{z} \ \bigvee_{i=1}^{m}(\bigwedge_{j=1}^{n_i} \theta_{i,j}(\bar{z}, \bar{g}))$ be a positive existential formula with parameters $\bar{g} \in G$, where the $\theta_{i,j}$ are atomic L_{SG}-formulas. Assume $G^* \models \phi(\nu(\bar{g}))$. Then there exists $i_0 \in \{1, \dots, m\}$ such that $G^* = \prod G_\sigma \models \exists \bar{z} \ \bigwedge_{j=1}^{n_{i_0}} \theta_{i_0,j}(\bar{z}, \nu(\bar{g}))$. This is a pp-formula and the satisfaction of such formulas is preserved under projections, so

$$\forall \sigma \in X_G^+, \quad G_\sigma \models \exists \bar{z} \bigwedge_{j=1}^{n_{i_0}} \theta_{i_0,j}(\bar{z}, \sigma(\bar{g})).$$

Let $\sigma_0 \in X_G^+$ be such that $\bar{g} \in G_{\sigma_0}$ (such a σ exists by proposition 3.10). We then have $\sigma_0(\bar{g}) = \bar{g}$ and $G_{\sigma_0} \models \exists \bar{z} \ \bigwedge_{j=1}^{n_{i_0}} \theta_{i_0,j}(\bar{z}, \bar{g})$, which implies $G \models \exists \bar{z} \ \bigwedge_{j=1}^{n_{i_0}} \theta_{i_0,j}(\bar{z}, \bar{g})$ and $G \models \phi(\bar{g})$. □

G^*, equipped with the product topology of the discrete topologies on each G_σ, is compact, Hausdorff, totally disconnected, and the isometry relation is a closed subset of G^{*2}. G^* is in fact a profinite special group (which have been considered in the reduced case in [17], chapitre 1, section 9, and, in the non-reduced case, in [9], §3.4, pp. 233-237) and it is possible to get a little bit more concerning them:

Definition 3.15. A special group is profinite if it is the projective limit of finite special groups (such a projective limit is always a special group; see [9, Theorem 3.24]).

We recall the following simple case of more general result appearing in [18]:

Theorem 3.16 ([18], Theorem 2.3)**.** *Let $\varprojlim G_i$ be the projective limit of an inverse system of finite special groups $\{G_i\}_{i \in I}$. Then there is an ultrafilter $\mathcal{U}$ on I such that the map*

$$\varprojlim G_i \to \prod_{i \in I} G_i/\mathcal{U}, \ (g_i)_{i \in I} \mapsto (g_i)_{i \in I} \bmod \mathcal{U}$$

is pure.

Proposition 3.17. *Let $m \in \mathbb{N}$ and let $G \in SG_{FC}$. Then G is an elementary substructure of a projective limit of finite special subgroups of G, all belonging to SG_{FC}.*

Proof. Proposition 3.7 gives the following decomposition of G as a L_m^--structure:

$$G = G_1^{(\alpha_1)} \times \cdots \times G_n^{(\alpha_n)}. \tag{4}$$

There is then a finite number of factors appearing in this decomposition, say $L_1, \dots, L_k$, such that

$$-1, \bar{a}_1, \dots, \bar{a}_m \in L_1 \times \cdots \times L_k.$$

Let $G_0 := L_1 \times \cdots \times L_k$. By lemma 3.9, the projection from G onto G_0 induced by the above decomposition of G is an L_{SG}-morphism, and G_0 is a special group.

Changing the notation in (4) we write

$$G = G_0 \times H_1^{(\alpha_1)} \times \cdots \times H_n^{(\alpha_n)} \text{ as } L_m^-\text{structures}, \tag{5}$$

where the H_i are finite L_m^--substructures of G (the α_i's in (5) may be different from those in (4)).

We record the following fact which is a direct application of lemma 3.9, since $-1, \bar{a}_1, \dots, \bar{a}_m \in G_0$:
Fact 1: If $\{L_i\}_{i\in I}$ is a (possibly infinite) set of factors appearing in the product $H_1^{(\alpha_1)} \times \cdots \times H_n^{(\alpha_n)}$ then the natural retract of the natural inclusion of $G_0 \times \oplus_{i\in I} L_i$ into G is a morphism of special groups and $G_0 \times \oplus_{i\in I} L_i$ is a special subgroup of G.

But it is possible to show, for L_m^--structures over a group of exponent 2 the same result about elimination of quantifiers modulo positive-primitive formulas as the one for modules, and this with the same proof (see for instance [15] theorem 6.14 p. 99, [13], or [26] corollary 2.16 p. 37). This result has the same consequences as for modules and in particular

$$H_1^{(\alpha_1)} \times \cdots \times H_n^{(\alpha_n)} \equiv H_1^{\alpha_1} \times \cdots \times H_n^{\alpha_n} \text{ in } L_m^-$$

(a proof of this for modules can be found in [15], theorem 6.19 p. 106).
Moreover, the inclusion of $H_1^{(\alpha_1)} \times \cdots \times H_n^{(\alpha_n)}$ in $H_1^{\alpha_1} \times \cdots \times H_n^{\alpha_n}$ is pure in L_m^-, and since these two L_m^--structures are elementarily equivalent, we have

$$H_1^{(\alpha_1)} \times \cdots \times H_n^{(\alpha_n)} \prec H_1^{\alpha_1} \times \cdots \times H_n^{\alpha_n} \text{ in } L_m^-$$

(a proof of this result for modules can be found in [26], corollary 2.26 p. 40).
Using that products of structures preserve elementary equivalence and since the constants in L_m are in G_0 we get

$$G = G_0 \times H_1^{(\alpha_1)} \times \cdots \times H_n^{(\alpha_n)} \prec G_0 \times H_1^{\alpha_1} \times \cdots \times H_n^{\alpha_n} =: G' \text{ in } L_m.$$

This elementary inclusion also holds for L_{SG} (which is definable in L_m). In particular the right hand side structure is a special group.

To prove the result we just have to check that $G_0 \times H_1^{\alpha_1} \times \cdots \times H_n^{\alpha_n}$ is a profinite special group. We change the notation for easier reading and write $G = G_0 \times \bigoplus_{i\in I} H_i$, $G' = G_0 \times \prod_{i\in I} H_i$ as L_m^--structures, and we define $G_J := G_0 \times \prod_{i\in J} H_i \times \prod_{i\in I\setminus J}\{1\}$ for every finite subset J of I. The G_J are finite special subgroups of G and belong to SG_{FC} (see fact 1).
We also denote by $p_J : G' \to G_J$ the canonical projection (which will also be seen as a map from G' to G'), and if J_1, J_2 are finite subsets of I such that $J_1 \subseteq J_2$, then $p_{J_2J_1}$ will denote the canonical projection from G_{J_2} onto G_{J_1}. The maps p_J and $p_{J_2J_1}$ are L_{SG}-morphisms (apply the obvious modification of fact 1 to G'). It is clear that the G_J, together with the morphisms $p_{J_2J_1}$, form a projective system of finite special groups. We equip each G_J with the discrete topology. The proof that $G' \cong \varprojlim\,(G_J, p_{J_2J_1})$ as topological groups is standard. The fact that they are isomorphic as L_{SG}-structures has been done in [17], proposition 1.9.11, 1) $\Rightarrow$ 2); we reproduce it here since it is unpublished:
Let $f : G' \to \varprojlim\,(G_J, p_{J_2J_1})$, $f(a) = (p_J(a))_J$ be the isomorphism of groups between G' and $\varprojlim\,(G_J, p_{J_2J_1})$. We check that it is an L_{SG}-monomorphism. Let $a, b \in G'$. If $a \in D_{G'}\langle 1, b\rangle$ then $f(a) \in D\langle 1, f(b)\rangle$ because the p_J are morphisms of special groups. Assume now $f(a) \in D\langle 1, f(b)\rangle$, i.e. $p_J(a) \in D_{G_J}\langle 1, p_J(b)\rangle$ for every finite subset J of I. So $(p_J(a), p_J(b)) \in R^{G_J} \subseteq R^{G'}$ for every finite subset J of I. But (a, b) is in the closure (for the product topology) of $\{(p_J(a), p_J(b)) \mid J \text{ finite subset of } I\}$ and we just saw that this subset is included in $R^{G'}$, which is closed. So $(a, b) \in R^{G'}$, i.e. $a \in D_{G'}\langle 1, b\rangle$. □

Corollary 3.18. *Let $m \in \mathbb{N}$ and let $G \in SG_{FC}$. Then G is a pure substructure of an ultraproduct of finite special groups belonging to SG_{FC}.*

Proof. By proposition 3.17 and theorem 3.16. □

Remark 3.19. We also know that a special group G in SG_{FC} is an inductive limit of finite special groups belonging to SG_{FC}: We can use proposition 3.10 or just the fact that G is $\aleph_0$-stable and $\aleph_0$-categorical, because then corollary 7.4 from [6] gives that for every finite subset A of G there is a finite special subgroup G_A of G, containing A. If A varies among all finite subsets of G we get an inductive system of finite special groups whose limit is G.

3.2. **Retracts and local-global principles.** We now use the decomposition of special groups in SG_{FC} given in the preceding section and profit of the ease with which it allows us to build retracts to present some results about preservation and reflection of some kinds of formulas.

Proposition 3.20. *Let $G \in SG_{FC}$. Then for every $N \in \mathbb{N}$ there is a finite special subgroup G_N of G such that*

(1) $X_{G,G_N} \neq \emptyset$;
(2) *For every pp-formula $\phi(\bar{g})$ in L_{SG} with at most N conjunctions and with parameters $\bar{g}$ in G we have*

$$G \models \phi(\bar{g}) \text{ if and only if } \forall \sigma \in X_{G,G_N}\ G_N \models \phi(\sigma(\bar{g})).$$

The following lemmas will be needed in the proof:

Lemma 3.21. *Let $l, d \in \mathbb{N}$, $p = l^{d-1} + 1$ and let $\bar{\delta}_1, \ldots, \bar{\delta}_p$ be pairwise distinct elements of $\{1, \ldots, l\}^d$. Then for every $k \in \{1, \ldots, d\}$ there exist two elements $\bar{\delta}_i, \bar{\delta}_j$ which only differ by their k-th coordinates.*

Proof. We can take $k = 1$. Assume that for every $1 \leq i \neq j \leq p$, $\bar{\delta}_i$ and $\bar{\delta}_j$ differ on a coordinate different from the first one. If we now remove the first coordinate from every $\bar{\delta}_i$, we get $l^{d-1}+1$ different elements in $\{1, \ldots, l\}^{d-1}$, which is impossible. □

Lemma 3.22. *Let $M \in \mathbb{N}$ and let $A_1, \ldots, A_M$ be sets. Assume $u_1, \ldots, u_{(M+1)^k}$ belong to the union $\bigcup_{j=1}^{M} A_j$. Then there are $(M+1)^{k-1}+1$ elements of $\{u_1, \ldots, u_{(M+1)^k}\}$ belonging to the same set A_i.*

Proof. Since $M(M+1)^{k-1} < (M+1)^k$, one of the A_i must contain more than $(M+1)^{k-1}$ elements. □

Proof of proposition 3.20. As in the beginning of the proof of proposition 3.17, we can write

$$G = G_0 \times H_1^{(\alpha_1)} \times \cdots \times H_n^{(\alpha_n)} \text{ in } L_m^-, \tag{6}$$

where G_0 and the H_i are finite L_m^--substructures of G. We can also assume that $\alpha_1, \ldots, \alpha_n$ are all infinite (taking a larger G_0 if necessary), and that G_0 was chosen such that the constants of L_m are elements of G_0.
In particular a subgroup Δ_i^G has the form $\Delta_i^{G_0} \times (\Delta_i^{H_1})^{(\alpha_1)} \times \cdots \times (\Delta_i^{H_n})^{(\alpha_n)}$ because the decomposition (6) is in the language L_m^-, and the corresponding coset is $\bar{a}_i \Delta_i^{G_0} \times (\Delta_i^{H_1})^{(\alpha_1)} \times \cdots \times (\Delta_i^{H_n})^{(\alpha_n)}$ (since $\bar{a}_i \in G_0$).

To simplify notation we start with the case

$$G = G_0 \times H_1^{(\alpha)}. \tag{7}$$

The proof is split into three steps, and we will indicate at the end how to proceed in the general case.

1) For our first step we consider the general form of such a formula $\phi(\bar{g})$ and define G_N. The formula $\phi(\bar{g})$ is of the form

$$\exists \bar{x} \bigwedge_{i=1}^{N_1} T_i(\bar{x}, \bar{g}) \in R \wedge \bigwedge_{i=1}^{N_2} t'_i(\bar{x}, \bar{g}) = 1,$$

where the t'_i are L_{SG}-terms, the T_i are pairs $(t_{i,1}, t_{i,2})$ of L_{SG}-terms and R is the representation relation seen as a binary relation. By considering (if necessary) -1 as one of the parameters $\bar{g}$ we can assume that the terms $t'_i, t_{i,j}$ are actually terms in the language $\{\cdot, 1\}$. Such a term $t(\bar{z})$ in the language $\{\cdot, 1\}$ has the property $t(\bar{a})t(\bar{b}) = t(\bar{a}\bar{b})$ for every $\bar{a}, \bar{b} \in G$. So $\phi(\bar{g})$ can be seen as

$$\exists \bar{x}\ \bar{T}(\bar{x}, \bar{g}) \in R^{N_1} \times \{1\}^{N_2}, \tag{8}$$

where $\bar{T} = T_1 \times \cdots \times T_{N_1} \times t'_1 \times \cdots \times t'_{N_2}$ has the property $\bar{T}(\bar{a})\bar{T}(\bar{b}) = \bar{T}(\bar{a}\bar{b})$ for every $\bar{a}, \bar{b} \in G^{\ell(\bar{x})+\ell(\bar{g})}$ (where $\ell(\bar{z})$ is the length of the tuple $\bar{z}$), and

$$R^{N_1} \times \{1\}^{N_2} = \bigcup_{i=1}^{M} \omega_i \Omega_i \tag{9}$$

is a finite union of cosets in $G^{2N_1+N_2}$. Note that $\omega_i \in G_0^{2N_1+N_2}$ since the constants of L_m are in G_0, and that Ω_i is a product of some Δ_j^G and $\{1\}$. Since the decomposition (7) is in L_m^-, we get $\omega_i \Omega_i = \omega_i \Omega_i^{G_0} \times (\Omega_i^{H_1})^{(\alpha)}$, for $i = 1, \ldots, M$.
Clearly $M \leq m^{N_1} \leq m^N$, so we can assume $M = m^N$. We take $G_N := G_0 \times H_1^M$ (it only depends on N since G is fixed), which we consider as the subgroup of $G_0 \times H_1^{(\alpha)}$ consisting of the first $M + 1$ coordinates. Note that

$$R^{G_N} = \bigcup_{i=1}^{M} \omega_i \Omega'_i, \text{ with } \Omega'_i := \Omega_i^{G_0} \times (\Omega_i^{H_i})^M.$$

Using (7) and since $\bar{g}$ is a finite tuple, there is $k \in \mathbb{N} \cup \{0\}$ such that $\bar{g} \in G_0 \times H_1^{M+k}$.

2) Our second step consists in defining well-chosen retracts of the inclusion of G_N in G, which we will then exclusively use:

Let σ_0 be the projection on the $M + 1$ first coordinates from $G = G_0 \times H_1^{(\alpha)}$ onto $G_0 \times H_1^M$. It is an L_m^--morphism and a retract of the inclusion of G_N in G, and thus a morphism of special groups (see lemma 3.9).
We denote by $(a_i)_{i<\alpha}$ an element of $G_0 \times H_1^{(\alpha)}$, with $a_0 \in G_0$ and $a_i \in H_1$ for $1 \leq i < \alpha$, and we define for $k_1 \geq 1$ and $1 \leq i_1 \leq M$:

$$\begin{array}{rccl} p_{M+k_1\ i_1} : & G_0 \times H_1^{(\alpha)} & \longrightarrow & G_N := G_0 \times H_1^M \\ & (a_i)_{i<\alpha} & \longmapsto & (1, \ldots, 1, a_{M+k_1}, 1, \ldots, 1) \end{array}$$

(a_{M+k_1} is put in the i_1-th coordinate). $p_{M+k_1\ i_1}$ is clearly an L_m^--morphism.

So for every $1 \leq k_1, \ldots, k_l \leq k$ and every $1 \leq i_1, \ldots, i_l \leq M$, we have a projection $\sigma_0 p_{M+k_1\ i_1} \cdots p_{M+k_l\ i_l}$ in the language L_m^- from $G = G_0 \times H_1^{(\alpha)}$ onto $G_0 \times H_1^M$. It is then, as for σ_0, a morphism of special groups. We will only consider retracts of this form, which we call "useful retracts." If we define $p_{M+k\ 0} := 1$, then every useful retract is of the form $\sigma_0 p_{M+1\ i_1} \cdots p_{M+k\ i_k}$ and is uniquely determined by the k-tuple $(i_1, \ldots, i_k) \in \{0, \ldots, M\}^k$.

3) Our third step is the actual proof, for which we now have all the necessary tools. We clearly have $X_{G,G_N} \neq \emptyset$ (it contains every useful retract). For the second statement of the proposition, the implication from left to right is obvious. Assume now that for every $\sigma \in X_{G, G_0 \times H_1^M}$, $G_0 \times H_1^M \models \phi(\sigma(\bar{g}))$. We will find a realisation in G of the existential quantifier in $\phi(\bar{g})$ by induction on k (which was defined at the end of step 1):

- $k = 0$. The result is clear, since we then have $\bar{g} \in G_N$, $\sigma_0(\bar{g}) = \bar{g}$ and $G_N \models \phi(\sigma_0(\bar{g}))$ if and only if $G \models \phi(\bar{g})$.
- $k \geq 1$. We are looking for a realization of the tuple $\bar{x}$ from $\exists \bar{x}\ \bar{T}(\bar{x}, \bar{g}) \in \bigcup_{i=1}^{M} \omega_i \Omega_i$.

There are $(M+1)^k$ useful retracts, say σ_i for $i = 1, \dots, (M+1)^k$, and we have, by hypothesis, for $i = 1, \dots, (M+1)^k$:

$$G_0 \times H_1^M \models \exists \bar{x}\ \bar{T}(\bar{x}, \sigma_i(\bar{g})) \in \bigcup_{j=1}^{M} \omega_j \Omega'_j.$$

By lemma 3.22 there are one coset and at least $(M+1)^{k-1} + 1 =: p$ useful retracts (we can assume they are $\omega_1 \Omega'_1$ and $\sigma_1, \dots, \sigma_p$) such that

$$G_0 \times H_1^M \models \exists \bar{x}\ \bar{T}(\bar{x}, \sigma_i(\bar{g})) \in \omega_1 \Omega'_1, \text{ for } i = 1, \dots, p.$$

As mentioned above, each of these retracts is determined by an element of $\{0, \dots, M\}^k$. We are then considering $p = (M+1)^{k-1} + 1$ elements of $\{0, \dots, M\}^k$. By lemma 3.21, there are two of them which only differ by one coordinate, say the first. Let $\bar{\delta} := (\delta_1, \delta_2, \dots, \delta_k)$ and $\bar{\delta}' := (\delta'_1, \delta_2, \dots, \delta_k)$ be these two elements, representing the retracts γ and τ, with $\delta_1 \neq \delta'_1$. One of δ_1, δ'_1 is different from 0 so we assume $\delta_1 \neq 0$.

We then have

$$G_0 \times H_1^M \models \exists \bar{x}\ \bar{T}(\bar{x}, \gamma(\bar{g})) \in \omega_1 \Omega_1^{G_0} \times (\Omega_1^{H_1})^M,$$

$$G_0 \times H_1^M \models \exists \bar{x}\ \bar{T}(\bar{x}, \tau(\bar{g})) \in \omega_1 \Omega_1^{G_0} \times (\Omega_1^{H_1})^M,$$

i.e. there exist $\bar{a}, \bar{b} \in G_0 \times H_1^M$ such that:

$$G_0 \times H_1^M \models \bar{T}(\bar{a}, \gamma(\bar{g})) \in \omega_1 \Omega_1^{G_0} \times (\Omega_1^{H_1})^M,$$

$$G_0 \times H_1^M \models \bar{T}(\bar{b}, \tau(\bar{g})) \in \omega_1 \Omega_1^{G_0} \times (\Omega_1^{H_1})^M.$$

In particular, looking at the δ_1-th coordinate:

$$\bar{T}(\bar{a}_{\delta_1}, \bar{g}_{\delta_1} \bar{g}_{M+1} \bar{h}) \in \Omega_1^{H_1} \text{ and } \bar{T}(\bar{b}_{\delta_1}, \bar{g}_{\delta_1} \bar{h}) \in \Omega_1^{H_1},$$

where $\bar{h}$ is a product of elements of $\{g_{M+2}, \dots, g_{M+k}\}$ (since $\bar{\delta}$ and $\bar{\delta}'$ only differ by their first coordinates). Taking the product of these last two terms we get

$$\bar{T}(\bar{a}_{\delta_1} \bar{b}_{\delta_1}, \bar{g}_{M+1}) \in \Omega_1^{H_1},$$

and we choose $\bar{x}_{M+1} := \bar{a}_{\delta_1} \bar{b}_{\delta_1}$.

The idea is now to "forget" the $M+1$-st coordinate. It leaves us with at least $(M+1)^{k-2}+1$ useful retracts belonging to the same coset. We are then back to the case $k-1$ where a realisation exists by induction hypothesis.

This completes the proof in the case $G = G_0 \times H_1^{(\alpha)}$. If $G = G_0 \times H_1^{(\alpha_1)} \times \cdots \times H_n^{(\alpha_n)}$ a similar proof will work if we consider the retracts of the inclusion of $G = G_0 \times H_1^M \times \cdots \times H_n^M$ in G. □

The result of proposition 3.20 can be improved in two steps. The first one consists in using positive existential formulas instead of pp-formulas: A positive existential L_{SG}-formula is of the form

$$\exists \bar{x} \quad (\bigwedge_{i=1}^{l_1} T_{1,i}(\bar{x}, \bar{g}) \in R \wedge \bigwedge_{i=1}^{l'_1} t'_{1,i}(\bar{x}, \bar{g}) = 1) \vee \cdots$$
$$\vee (\bigwedge_{i=1}^{l_n} T_{n,i}(\bar{x}, \bar{g}) \in R \wedge \bigwedge_{i=1}^{l'_n} t'_{n,i}(\bar{x}, \bar{g}) = 1),$$

where the $t'_{i,j}(\bar{x}, \bar{g})$ are L_{SG}-terms and the $T_{i,j}$ are pairs of L_{SG}-terms. The formula can then be written

$$\exists \bar{x}\ f_1(\bar{x}, \bar{g}) \in B_1 \vee \cdots \vee f_n(\bar{x}, \bar{g}) \in B_n,$$

for $f_k = T_{k,1} \times \cdots \times T_{k,l_k} \times t'_{k,1} \times \cdots \times t'_{k,l'_k}$ and $B_k = R^{l_k} \times \{1\}^{l'_k}$ (note that B_k is a subset of G^{p_k} where $p_k = 2l_k + l'_k$ and is a finite union of cosets). The positive existential formula can then be written as follows:

$$\begin{array}{rl} \exists \bar{x}\ (f_1(\bar{x},\bar{g}),\ldots,f_n(\bar{x},\bar{g})) \in & B_1 \times G^{p_2} \times \cdots \times G^{p_n} \cup \\ & G^{p_1} \times B_2 \times G^{p_3} \times \cdots \times G^{p_n} \cup \\ & \cdots \cup G^{p_1} \times \cdots \times G^{p_{n-1}} \times B_n. \end{array}$$

But this describes the same situation as in lines (8) and (9) in the proof of proposition 3.20. We then just have to follow the proof after that point.

The second step consists in observing that since the parameters remain unchanged along the proof of proposition 3.20 and can be freely chosen, we can universally quantify our formula. This gives the following version of proposition 3.20:

Theorem 3.23. *Let $G \in SG_{FC}$. Then for every $N \in \mathbb{N}$ there is a finite special subgroup G_N of G such that*

(1) $X_{G,G_N} \neq \emptyset$;
(2) *For every $\forall\exists^+$-formula $\phi(\bar{g})$ with at most N atomic subformulas and with parameters in G:*

$$G \models \phi(\bar{g}) \text{ if and only if } \forall \sigma \in X_{G,G_N}\ G_N \models \phi(\sigma(\bar{g})).$$

We immediately deduce

Corollary 3.24. *Let $G \in SG_{FC}$. Then*

- *For every $N \in \mathbb{N}$ there is a finite special subgroup K_N of G such that, for every pair of quadratic forms f, g over G of dimension at most N:*

$$f \equiv_G g \text{ if and only if } \forall \sigma \in X_{G,K_N}\ \sigma(f) \equiv_{K_N} \sigma(g).$$

- *For every $N \in \mathbb{N}$ there is a finite special subgroup K'_N of G such that for every quadratic form f over G of dimension at most N:*

$$f \text{ is anisotropic in } G \text{ if and only if } \forall \sigma \in X_{G,K'_N}\ \sigma(f) \text{ is anisotropic in } K'_N.$$

Proof. Both statements follow from theorem 3.23, as isometry and isotropy are expressed by positive-existential formulas whose number of atomic subformulas is bounded by a function of the dimension of the forms. □

The second assertion in the corollary is similar to the isotropy theorem (see [20]), but the special group K'_N does not depend on the coefficients of the quadratic form. In case G is reduced and hence of finite chain length (reduced SG_{FC}-groups are $\aleph_0$-stable, hence of finite chain length), we obtain the following dual formulation of Corollary 3.24 in terms of spaces of orderings:

Corollary 3.25. *Let (X, G) be a space of orderings of finite chain length. Then for every $N \in \mathbb{N}$ there is a finite subspace Y of X such that for every quadratic form f with coefficients in G and dimension at most N:*

$$f \text{ is isotropic over } X \text{ if and only if } f \text{ is isotropic over } Y.$$

4. Weakly normal special groups

Weakly normal groups have been studied in [14], where the following characterisation is proved (theorem 4.1):
A group G (in a language containing the language of groups) is weakly normal if every definable subset of G^n is a finite boolean combination of cosets of definable subgroups of G^n, for every $n \in \mathbb{N}$.

This suggests exploring possible links between weakly normal special groups and SG_{FC}. We start with a simple observation:

Lemma 4.1. *Let G be a special group. Then*

1. *G is weakly normal if and only if R^G is a finite boolean combination of cosets.*
2. *If G is weakly normal and $\aleph_0$-categorical then G is $\aleph_0$-stable.*

Proof. (1) The left to right implication is clear by the above mentioned characterization. Suppose now that R^G is a boolean combination of cosets of the subgroups $A_1, \ldots, A_n$ of G^2. Let L_1 be the language of groups with predicates for $A_1, \ldots, A_n$. Then the L_{SG}-structure G is interpretable in the L_1-structure G, which is weakly normal (every definable set is a boolean combination of definable cosets). In particular, G does not type-interpret a pseudoplane in L_1 ([14, proposition 1.1]), and also does not type-interpret a pseudoplane in L_{SG} since L_{SG} is definable in L_1. Proposition 1.1 of [14] shows, then, that the special group G is weakly normal.

(2) By [14, Theorem 4.1] there are L_{SG}-definable subgroups $A_1, \ldots, A_n$ of G^2 such that R^G is a finite boolean combination of cosets of $A_1, \ldots, A_n$. The special group G is then interpretable in the group G equipped with predicates for $A_1, \ldots, A_n$ (which is $\aleph_0$-categorical by interpretation). Since the underlying group is of exponent 2 —hence a $\mathbb{F}_2$-vector space— this latter structure is also $\aleph_0$-stable by Proposition 1.2.

□

Definition 4.2. A special group G is called residually finite if for every L_{SG}-atomic formula $\theta(\bar{a})$ with parameters $\bar{a} \in G$ such that $G \not\models \theta(\bar{a})$ there is a finite special group H and a morphism of special groups $f : G \to H$ such that $H \not\models \theta(f(\bar{a}))$.

We conclude the paper by summing up some of the equivalences proved so far and linking them to weakly normal special groups:

Theorem 4.3. *Let G be a special group. Then the following statements are equivalent:*

1. *The isometry relation of G is a finite union of cosets;*
2. *G is weakly normal and residually finite;*

If furthermore G satisfies $AP(3)$ and is $\aleph_0$-categorical then the following statements are equivalent:

5. *The isometry relation of G is a finite union of cosets of subgroups of G^4 and G has finite 2-symbol length;*
6. *G is weakly normal of finite 2-symbol length;*
7. *G is $\aleph_0$-stable, $\aleph_0$-categorical and of finite 2-symbol length;*
8. *$(G, I(G)^2/I(G)^3, q_G)$ is an $\aleph_0$-stable $\aleph_0$-categorical quaternionic mapping.*
9. *G is the special group associated to an $\aleph_0$-stable $\aleph_0$-categorical quaternionic mapping.*

Proof. For the first part of the theorem: 1 implies 2 by lemma 4.1 and proposition 3.13, so we only need to check that 2 implies 1. We assume 2, and we consider the topology on G whose basis of open sets is given by the cosets of finite index. Since G is residually finite, R^G is closed for this topology, so $\overline{R^G} = R^G$. Since R^G is a boolean combination of cosets we have

$$\begin{aligned} R^G &= \bigcup_{i=1}^{n} \left(\bigcap_{r=1}^{k_i} \Omega_{i,r} \cap \bigcap_{s=1}^{l_i} (G^2 \setminus \Gamma_{i,s}) \right) \\ &= \bigcup_{i=1}^{n} \left(\Omega_i \cap \bigcap_{s=1}^{l_i} (G^2 \setminus \Gamma_{i,s}) \right) \end{aligned}$$

where the $\Omega_{i,r}, \Omega_i, \Gamma_{i,s}$ are cosets in G^2. We can also assume (up to some rewriting of the expression of R^G as boolean combination of cosets) that for every $i = 1, \dots, n$, $s = 1, \dots, l_i$, the cosets $\Gamma_{i,s}$ have infinite index in Ω_i. Taking now the topological closure on both sides of this equality, we get

$$\begin{aligned} R^G &= \bigcup_{i=1}^{n} \overline{\left(\Omega_i \cap \bigcap_{s=1}^{l_i} (G^2 \setminus \Gamma_{i,s})\right)} \\ &= \bigcup_{i=1}^{n} \overline{\left(\Omega_i \setminus \bigcup_{s=1}^{l_i} \Gamma_{i,s}\right)} \\ &= \bigcup_{i=1}^{n} \Omega_i \quad \text{(see fact below)}, \end{aligned}$$

so R^G is a finite union of cosets.
Fact: If $n \in \mathbb{N}$ and $\Omega, \Gamma_1, \dots, \Gamma_n$ are cosets of G^2 such that Γ_i has infinite index in Ω for $i = 1, \dots, n$, then $\overline{\Omega \setminus \bigcup_{i=1}^{n} \Gamma_i} = \Omega$.
Proof of the fact: Ω is the intersection of all cosets of finite index containing Ω, so is closed. Consider now $a \in \Omega$, and let H be a coset of finite index in G^2 containing a. If we show that $H \cap (\Omega \setminus \bigcup_{i=1}^{n} \Gamma_i) \neq \emptyset$, then $a \in \overline{\Omega \setminus \bigcup_{i=1}^{n} \Gamma_i}$ and the fact is proved. Assume $H \cap (\Omega \setminus \bigcup_{i=1}^{n} \Gamma_i) = \emptyset$. Then $\Omega \cap H \subseteq \bigcup_{i=1}^{n} \Gamma_i$. But $\Omega \cap H$ has finite index in Ω (since H has finite index in G^2 and $\Omega \cap H$ is nonempty), while $\bigcup_{i=1}^{n} \Gamma_i$ has infinite index in Ω (by Neumann's lemma, see for instance [26, Theorem 2.12]), a contradiction which proves the fact.

We now consider the second part of the theorem, so we assume that G satisfies $AP(3)$. Then $(G, I(G)^2/I(G)^3, q_G)$ is a linked quaternionic mapping. We have 5 implies 6 by definition of weak normality, 6 implies 7 by lemma 4.1, 7 implies 8 by proposition 2.6. 8 implies 9 is clear and 9 implies 5 by proposition 2.3. □

Remark 4.4. In the case of reduced special groups:

- The statements 1 and 2 are equivalent to G being stable, i.e. of finite chain length, i.e. built from $\mathbb{Z}_2$ by using a finite number of times the operations of product and extension (see [3], corollary 4.5 and the subsequent paragraph).
- The statements 5, 6, 7, 8 and 9 are all equivalent to G being finite (see corollary 2.7).

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