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Pfister's subform theorem for reduced special groups

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Abstract

Pfister's subform theorem is an essential result in the theory of quadratic forms, but its proof uses transcendental field extensions which have no equivalent in the axiomatic theory. We present here a proof of this result for reduced special groups/spaces of orderings which avoids this difficulty.

MSC: 11E81; 03C65

1. Introduction

The algebraic theory of quadratic forms admits several axiomatizations, for example abstract Witt rings, special groups, or, for the reduced theory, abstract spaces of orderings (these axiomatizations are essentially equivalent).

They provide the usual advantages such as unified proofs, but the question of whether or not they exactly describe the theory of quadratic forms over fields (i.e. is every "model" of these axiomatizations the Witt ring or special group, or space of orderings of a field) is still open.

A natural question is then to try to prove in this new context the classical results about quadratic forms over fields. Pfister's subform theorem is such a result, which can be stated as follows:

Theorem 1 (Pfister's subform theorem). *Let K be a field of characteristic different from 2, and let ϕ, ψ be quadratic forms over K, with ψ anisotropic. Then ϕ is a*

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subform of ψ *if and only if the polynomial* $\phi(\bar{x})$ *is represented in* $K(\bar{x})$ *by the form* $\psi \otimes K(\bar{x})$ *(seen as a form with coefficients in* $K(\bar{x})$*).*

In this form this result offers few possibilities of generalization to the abstract setting, because of the lack of notion corresponding to transcendental extensions. But its equivalent conditions are also equivalent to the following (see [3, Theorem 2.1]):

For any field extension E of K, $D_E(\phi \otimes E) \subseteq D_E(\psi \otimes E)$, where $D_E q$ is the set of elements represented by the quadratic form q over the field E.

This latter formulation has an analogue in the axiomatic case (see Theorem 3), and this note presents a proof of the corresponding reformulation of Pfister's subform theorem for the (axiomatic) reduced theory, and states one immediate consequence.

The proof of this result is also interesting from another point of view, because it shows that under this form, Pfister's subform theorem cannot be used to find a difference between the reduced algebraic theory of quadratic forms and such axiomatizations as reduced special groups and abstract spaces of orderings.

2. Preliminaries

We use the terminology of special groups and abstract spaces of orderings, as described in [1,2,5] (for example), and will only give here a brief description of these objects:

If (X, G) is a space of orderings, G is a group of exponent 2 with distinguished element -1, and X is a closed subset of $\mathrm{Hom}(G, \{-1, 1\}) \subseteq \{-1, 1\}^G$ equipped with the topology induced by the product topology (of the trivial topology on $\{-1, 1\}$), which separates the points of G. X satisfies other properties, in particular $\sigma(-1) = -1$ for every $\sigma \in X$. X is then compact, Hausdorff and totally disconnected.

The sets $X(a_1, \ldots, a_n) = \{\sigma \in X \mid \sigma(a_i) = 1,\ i = 1, \ldots, n\}$ (for $n \in \mathbb{N},\ a_1, \ldots, a_n \in G$) are clopen sets which form a basis for the topology on X.

An example of space of orderings is the space of orderings of a formally real field K, where $G = \dot{K}/\sum \dot{K}^2$ and X is the set of orderings on K.

The language of special groups is $L_{\mathrm{SG}} = \{1, -1, \cdot, \equiv\}$, and a special group is an L_{SG}-structure G where G is a group of exponent 2 satisfying some axioms (expressible as first-order L_{SG}-formulas), and $\equiv$ is a binary relation on G^2.

The classical example of special group is the special group of a field K, where $G = \dot{K}/\dot{K}^2$, and $\equiv$ is interpreted as the isometry between diagonal quadratic forms of dimension 2. The special group of any real closed field contains exactly two elements 1 and -1 and is denoted by $\mathbb{Z}_2$.

A special group G is reduced if for every $a \in G$, $\langle a, a \rangle \equiv \langle 1, 1 \rangle$ implies $a = 1$.

There is an isomorphism of categories between spaces of orderings and reduced special groups (see [2, Chapter 3]), which can be partially described as follows:

- If (X, G) is a space of orderings, the binary relation on G^2 defined by $\langle a, b \rangle \equiv \langle c, d \rangle$ if and only if $\forall \sigma \in X_G\ \sigma(a) + \sigma(b) = \sigma(c) + \sigma(d)$ (sum in $\mathbb{Z}$), turns G into a reduced special group.

- If G is a reduced special group, then (X, G) is a space of orderings, with $X = \{\sigma : G \to \mathbb{Z}_2 \mid \sigma$ morphism of special groups$\}$.

Let (X, G) be a space of orderings with associated reduced special group G. A quadratic form $\langle a_1, \ldots, a_n \rangle$ of dimension n over (X, G) (or over G) is a n-tuple of elements of G.

Using as a definition the inductive description of isometry for fields, it is possible to define an isometry $\equiv_G$ between forms $\langle a_1, \ldots, a_n \rangle$, $\langle b_1, \ldots, b_n \rangle$ of dimension n over G, by a first-order L_{SG}-formula with parameters $a_1, \ldots, a_n$, $b_1, \ldots, b_n$.

This isometry coincide with the one defined as follows (Pfister's local-global principle; see [2, Proposition 3.7]):

$$\langle a_1, \ldots, a_n \rangle \equiv_X \langle b_1, \ldots, b_n \rangle \quad \text{if and only if} \quad \forall \sigma \in X_G \sum_{i=1}^{n} \sigma(a_i) = \sum_{i=1}^{n} \sigma(b_i) \in \mathbb{Z}.$$

3. Pfister's subform theorem

Lemma 2. *Let (X, G) be an abstract space of orderings and let f be a form over G, of dimension n, with $i(f) < l$ over X (where $i(f)$ denotes the Witt index of f).*

Then there exists a finite subspace X_0 of X such that $i(f) < l$ over X_0.

Proof. We write $f \equiv_X i(f)\langle -1, 1 \rangle \oplus g$ with g anisotropic in (X, G). By the isotropy theorem (see [4]) there exists a finite subspace X_0 of X over which g is anisotropic. We then have $f \equiv_{X_0} i(f)\langle -1, 1 \rangle \oplus g$. Since g is anisotropic over X_0 the Witt index of f over X is the same as the Witt index of f over X_0, so $i(f) < l$ over X_0. □

As a consequence we have:

Theorem 3 (Pfister's subform theorem). *Let G be a reduced special group, and let p, q be forms over G, such that q is anisotropic, and for every morphism of special groups $\lambda : G \to L$, $D_L \lambda(p) \subseteq D_L \lambda(q)$.*

Then p is a subform of q in G, i.e. there exists a form θ over G such that $p \oplus \theta \equiv_G q$.

Proof. By the isotropy theorem, there exists a finite subspace X_0 of X such that q is anisotropic over X_0.

Let $dim\, p = n$ and $dim\, q = m$, and suppose $i(q - p) < n$ over X. Then, by Lemma 2, there exists a finite subspace Y of X such that $i(q - p) < n$ over Y.

Replacing, if necessary, Y by the finite subspace generated by Y and X_0, we may assume $Y \supseteq X_0$, which gives that q is anisotropic over Y (and, of course, we still have $i(q - p) < n$ over the new Y).

But Y (being finite) is the space of orderings of a Pythagorean field K_Y (see [6]), and its associated special group G_Y is the special group of K_Y. Moreover, if p_Y, q_Y are the images of p, q in G_Y, p_Y, q_Y are diagonal quadratic forms over K_Y.

The map $G \to G_Y$ is a morphism of special groups, and the hypothesis gives:

$$\forall \lambda : G_Y \to L \text{ morphism of special groups, } D_L\lambda(p_Y) \subseteq D_L\lambda(q_Y). \qquad (1)$$

But if L is the special group of a field extension E of K_Y, we have a morphism of special groups λ from G_Y to L, and for any quadratic form ϕ over G_Y, $\lambda(\phi) = \phi \otimes_{K_Y} E$. Then (1) yields:

$$\text{For every field extension } E \text{ of } K_Y,\ D_E(p_Y \otimes E) \subseteq D_E(q_Y \otimes E).$$

Since q (i.e. q_Y) is anisotropic over Y, Pfister's subform theorem (for fields) applies and gives a form τ with coefficients in G such that $p_Y \oplus \tau_Y \equiv_{G_Y} q_Y$, i.e., in the language of spaces of orderings: $p \oplus \tau \equiv_Y q$.

This gives $q - p \equiv_Y \tau \oplus n\langle -1, 1\rangle$, in contradiction with $i(q - p) < n$ over Y.

Then $i(q - p) \geqslant n$ over X, i.e. there exists a form θ with coefficients in G such that $q - p \equiv_G \theta \oplus n\langle -1, 1\rangle$.

Adding p to both sides we get $q \oplus (dim\ p)\langle -1, 1\rangle \equiv_G p \oplus \theta \oplus n\langle -1, 1\rangle$, and comparing dimensions: $q \equiv_G p \oplus \theta$. □

One of the classical consequences of Pfister's subform theorem for fields is the fact that a multiplicative form is isometric to a Pfister form. We can now get it with the following definition of multiplicative form:

Definition 4. Let q be a form over a special group G. We say that q is multiplicative if for every $\lambda : G \to L$ morphism of special groups, $D_L\lambda(q)$ is a subgroup of L.

We then have, with a minor modification of the proof in the field case (see for example, [3, Theorem 4.1], (iii) ⇒ (i)):

Corollary 5. *Let ϕ be an anisotropic multiplicative form over a reduced special group G. Then there exists a Pfister form ψ over G such that $\phi \equiv_G \psi$.*

Proof. Let ψ be the Pfister form of largest dimension which is a subform of ϕ (it exists since $\langle 1\rangle$ is a subform of ϕ). Suppose that $dim\,\psi < dim\,\phi$. Then there is a nonempty form ρ over G such that $\phi = \psi \oplus \rho$. Take $\alpha \in D_G\rho$. We consider the Pfister form $\langle 1, \alpha\rangle \otimes \psi = \psi \oplus \alpha\psi$.

Let $\lambda : G \to L$ be any morphism of special groups, and let $x \in D_L\lambda(\psi \oplus \alpha\psi)$. There exists $a, b \in D_L\lambda(\psi)$ such that $x \in D_L\langle a, \lambda(\alpha)b\rangle = bD_L\langle ab, \lambda(\alpha)\rangle$. Since $ab \in D_L\lambda(\psi)$ (which is multiplicative since $\lambda(\psi)$ is a Pfister form) and $\lambda(\alpha) \in D_L\lambda(\rho)$, we have $D_L\langle ab, \lambda(\alpha)\rangle \subseteq D_L\lambda(\phi)$. Using that $D_L\lambda(\phi)$ is multiplicative, we get $x \in D_L\lambda(\phi)$.

By Theorem 3, $\langle 1, \alpha\rangle \otimes \psi$ is then a subform of ϕ, a contradiction. □

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