Generic splitting for special groups

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Abstract

Using model-theoretic tools, we present a generalisation of the theory of generic splitting of quadratic forms to special groups, an axiomatic version of the algebraic theory of quadratic forms.

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1 Introduction

The algebraic theory of quadratic forms over fields (of characteristic different from 2) has been introduced by Witt in the 1930's, and has been extensively studied since. It has also been the object of several axiomatisations, and this paper concentrates on one of them, the theory of special groups, developed by Dickmann and Miraglia (see [4]). This theory is equivalent to the theory of abstract Witt rings (see [11]), whose language we will occasionally use, in the sense that they define isomorphic categories.

One of the main aims of any axiomatisation of the algebraic theory of quadratic forms, so in particular of special groups or abstract Witt rings, is to identify a core set of properties of quadratic forms over fields from which the whole theory can be recovered. In this optic, it seems worthwhile to investigate how it could be possible to recover, from the axioms of special groups, the classical theory of generic splitting of quadratic forms over fields.

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This theory, initiated by Knebusch in his papers [6,7], involves careful consideration of the behaviour of quadratic forms under some field extensions and places, and has become an essential part of quadratic form theory.

We present here a possible generalisation to special groups of the notion of generic splitting of a quadratic form, using some techniques from model theory. We assume familiarity with the algebraic theory of quadratic forms and special groups, as well as with some basic notions from model theory, which can be found in any introductory textbook, such as [3] or [9]. We use the rest of this section to introduce some notation concerning generic splitting on fields, model theory, and special groups. In section 2 we present our generalisation and investigate its first properties, section 3 is devoted to the special case of reduced special groups, and we return in section 4 to the general case.

1.1 Generic splitting over fields

We recall here the definition and the very first result concerning generic splitting of quadratic forms over fields, in order to provide some background to the analogue we propose. All this can be found in [6] (definition 3.2 and theorem 3.3) or [8] (definition on page 24 and theorem 9.4 together with remark 9.5). In this paper, all fields have characteristic different from 2 and all quadratic forms are non-degenerate.

Definition 1 *Let k be a field and let φ be a quadratic form over k. A field extension K of k is called a generic zero field, or generic partial splitting field, of φ if*

(1) $\varphi \otimes K$ is isotropic,
(2) for every field extension L of k with $\varphi \otimes L$ isotropic, there exists a place $\mu : K \to L \cup \{\infty\}$ over k, i.e. such that the following diagram commutes

$$\begin{array}{ccc} k & \longrightarrow & K \\ \downarrow & \swarrow_{\mu} & \\ L \cup \{\infty\} & & \end{array} \qquad (1)$$

Theorem 2 *Let k be a field and let $\varphi(x_1, \ldots, x_n)$ be a form of dimension at least 2 over k, not isometric to the hyperbolic plane. We define $k(\varphi)$ as the quotient field of $k[x_1, \ldots, x_n]/(\varphi(x_1, \ldots, x_n))$. Then $k(\varphi)$ is a generic zero field of φ.*

We will not distinguish between a structure and its underlying set. For a first-order language L and an L-structure M:

- if c is a constant symbol in L, we write c^M for the interpretation of c in M;
- if $A \subseteq M$, $L(A)$ denotes the language L expanded by new constants for the elements of A. These constants will be interpreted in M by the corresponding elements of A. If we wish to distinguish between $a \in A$ and the new constant associated to it, we will write c_a for this constant. If $\bar{a} = (a_1, \ldots, a_n) \in A^n$ we also write $c_{\bar{a}}$ for the tuple $(c_{a_1}, \ldots, c_{a_n})$.

Definition 3 *Let L be a first-order language.*

(1) A positive-primitive L-formula (pp-formula for short) $\varphi(\bar{y})$ is a formula of the form $\exists \bar{x}\ \theta(\bar{x}, \bar{y})$, where $\theta(\bar{x}, \bar{y})$ is a conjunction of atomic L-formulas.

(2) Let $f : M \to N$ be an L-morphism between two L-structures M and N, and let $A \subseteq M$. f is pure over A if for every pp-formula $\varphi(\bar{a})$ with parameters in A we have

$$M \models \varphi(\bar{a}) \Leftrightarrow N \models \varphi(f(\bar{a})).$$

In the case $A = M$ we simply say that f is pure, and we denote this by $f : M \overset{\prec_{pp}}{\to} N$, or $M \prec_{pp} N$ if f is an inclusion.

(3) Let M be an L-structure, and let A be a subset of M. We define the pp-type of M over A as

$$\mathrm{pp}^M(A) := \{\varphi(\bar{a}) \mid \varphi \textit{ pp-formula with parameters in } A, \textit{ such that } M \models \varphi(\bar{a})\}.$$

Similarly we define

$$\text{neg-pp}^M(A) := \{\neg\varphi(\bar{a}) \mid \varphi \textit{ pp-formula with parameters in } A, \textit{ such that } M \models \neg\varphi(\bar{a})\}.$$

Note that $\mathrm{pp}^M(A)$ and $\text{neg-pp}^M(A)$ are $L(A)$-theories.

If M and N are two L-structures and $\bar{a} \in M$, $\bar{b} \in N$ are tuples of the same length, the notation $(M, \bar{a}) \equiv_{pp} (N, \bar{b})$ means

$$\text{for every pp-formula } \varphi(\bar{x}),\ M \models \varphi(\bar{a}) \text{ if and only if } N \models \varphi(\bar{b}).$$

The central role played by pp-types of special groups in this paper is explained by the following well-known observation:

If M is an L-structure, then L-morphisms defined on M and with values in some L-structure correspond to models of $\mathrm{pp}^M(M)$.
Indeed, assume we have an L-morphism $f : M \to N$. If, for every $m \in M$, we define $c_m^N := f(m)$, then N becomes a model of $\mathrm{pp}^M(M)$. Conversely, if we have an $L(M)$-structure N such that $N \models \mathrm{pp}^M(M)$, then the map $M \to N$, $m \mapsto c_m^N$ is an L-morphism.

In a similar vein, for an L-structure M, we denote by $\Delta(M)$ the diagram of M, that is, the set of all $L(M)$-formulas that are true in M. Then a model of $\Delta(M)$ corresponds to an elementary L-morphism from M into some L-structure N, i.e. an L-morphism f such that, for every L-formula $\varphi(\bar{m})$ with parameters in M,

$$M \models \varphi(\bar{m}) \text{ if and only if } N \models \varphi(f(\bar{m})).$$

We denote as usual such a morphism by $f : M \overset{\prec}{\to} N$, or simply $M \prec N$ if f is an inclusion.

1.3 Concerning special groups and spaces of orderings

As briefly mentioned in the introduction, our reference for special groups is [4]. We fix some general notation:

If K is a field, $SG(K)$ is the special group associated to K.

We denote by L_{SG} the language of special groups $\{1, -1, \cdot, \equiv\}$. Consequently, we denote by $\equiv$ the isometry between quadratic forms of same dimension.

The equality modulo Witt equivalence will be denoted by $\sim$. In particular, for a quadratic form q, $q \sim 0$ means that the form q is hyperbolic. If q is a form over a special group G, we denote by Dq (or $D_G q$ if we want to be more precise) the set of elements of G represented by q.

If q is a quadratic form over a special group G, we denote by q_{an} its anisotropic part, and the Witt index of q is the unique integer $i(q)$ such that $q \equiv i(q)\langle -1, 1\rangle \oplus q_{an}$.

The sentence "$f : G \to H$ is a morphism of special groups" means, not only that f is an L_{SG}-morphism, but also that both G and H are special groups. If f is an L_{SG}-morphism defined on the special group G and if $\langle a_1, \ldots, a_n\rangle$ is a form over G, the form $f(\langle a_1, \ldots, a_n\rangle)$ is $\langle f(a_1), \ldots, f(a_n)\rangle$.

A special group G is reduced if $-1 \neq 1$ and $D\langle 1, 1\rangle = \{1\}$. The category of reduced special groups is isomorphic (via contravariant functors) to that of abstract spaces of orderings. This result can be found in [4, chapter 3] (where spaces of orderings are called abstract order spaces), and our reference for spaces of orderings is [13].

If G is a reduced special group, we denote by X_G (more precisely (X_G, G)) its associated space of orderings. The subspaces of X_G are in bijective correspondence with the Pfister quotients G/Δ of G. If Y is a subspace of X_G we denote by Δ_Y the subgroup $\cap_{\sigma \in Y} \ker \sigma$ of G, by $G_Y := G/\Delta_Y$ the Pfister quotient associated to Y, and by $\pi_Y : G \to G_Y$ the canonical projection (which is a morphism of special groups). In this case, as is usual, we say that a property P described by the L_{SG}-formula $\varphi(\bar{g})$ with parameters in G holds over Y if and only if the formula $\varphi(\pi_Y(\bar{g}))$ holds in the special group G_Y.

2 Definition and first properties

We wish to generalise definition 1 to special groups. Consider a special group G, and let q be an anisotropic form over G. Intuitively a "generic zero" special group of q over G should consist in a morphism of special groups $\lambda : G \to G_q$ such that $\lambda(q)$ is isotropic, with some additional properties distinguishing it among the morphisms of special groups f such that $f(q)$ is isotropic.

In the field case, if q is an anisotropic form over k and K is a generic splitting field of q, we have a place μ from K to $L \cup \{\infty\}$, whenever L is a field extension of k on which q is isotropic. This place μ does not in general induce a morphism of special groups between $SG(K)$ and $SG(L)$, but does preserve some properties involving forms having good reduction with respect to μ, such as isometry (see for instance [14, chapter 4 definition 6.4] for Knebusch's notion of good reduction). It is easy to check, using the commutativity of diagram (1) in definition 1, that forms with coefficients in k always have good reduction with respect to μ. Furthermore, isometry between two forms of same dimension is expressed in the language of special groups by a positive-primitive formula. Using this as inspiration, we then wish that whenever $f : G \to L$ is a morphism of special groups such that $f(q)$ is isotropic, then, for every L_{SG}-pp-formula $\varphi(\bar{g})$ with parameters $\bar{g} \in G$:

$$G_q \models \varphi(\lambda(\bar{g})) \text{ implies } L \models \varphi(f(\bar{g})).$$

This motivates the following definition:

Definition 4 *Let G be a special group and let q be a quadratic form over G of dimension at least 2. We define the $L_{SG}(G)$-theories*

$$T^0_{q,G} := SG \cup \mathrm{pp}^G(G) \cup \text{“}q \textit{ isotropic}\text{”}$$

and

$$T_{q,G} := T^0_{q,G} \cup \{\neg\varphi(\bar{g}) \mid \varphi(\bar{g}) \text{ pp-formula over } G \text{ such that } T^0_{q,G} \not\vdash \varphi(\bar{g})\},$$

and we say that a special group G_q is a generic zero special group (GZSG for short) of q over G if and only if $G_q \models T_{q,G}$. In this case we refer to the morphism of special groups $\lambda : G \to G_q$, $g \mapsto c_g^{G_q}$ as a generic partial splitting (GPS for short) of q over G.

Remark 5 *(1) By the completeness theorem ([3, theorem 1.3.20] or [9, theorem 2.1.2]), the condition $T^0_{q,G} \not\vdash \varphi(\bar{g})$ simply means that there is a model L of $T^0_{q,G}$ in which $\varphi(\bar{g})$ is false. According to the observation made in section 1.2, it means that there is a morphism of special groups $f : G \to L$ such that $L \models$ "$f(q)$ isotropic" and $L \not\models \varphi(f(\bar{g}))$, which agrees with the intuitive motivation of the definition.*

(2) $T_{q,G} = SG \cup \mathrm{pp}^G(G) \cup \text{neg-pp}^G(G)$ if and only if q is isotropic in G.

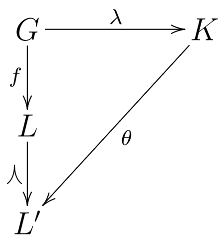
Proposition 6 *Let G be a special group and let q be a quadratic form over G, of dimension at least 2. Then $T_{q,G}$ has a model.*

Proof: We distinguish two cases. The first one is $T_{q,G} = T^0_{q,G}$. Then a model of $T_{q,G}$ will be given by any morphism of special groups $f : G \to H$ such that $f(q)$ is isotropic. The canonical morphism from G onto the one-element special group is such a morphism.

The second case is $T_{q,G} \neq T^0_{q,G}$. Let $\{\neg\varphi_1(\bar{g}), \ldots, \neg\varphi_k(\bar{g})\}$ be a finite subset of $T_{q,G} \setminus T^0_{q,G}$. By remark 5 1., for each $1 \leq i \leq k$ there is a morphism of special groups $f_i : G \to L_i$ such that $f_i(q)$ is isotropic and $L_i \models \neg\varphi(f_i(\bar{g}))$. Define $f : G \to \prod_{1\leq i\leq k} L_i$, $x \mapsto (f_1(x), \ldots, f_k(x))$. Then f is a morphism of special groups, $f(q)$ is isotropic, and $\prod_{1\leq i\leq k} L_i \models \neg\varphi_1(f(\bar{g})) \wedge \ldots \wedge \neg\varphi_k(f(\bar{g}))$, so $\prod_{1\leq i\leq k} L_i$, together with the interpretation of the elements of G given by f, is a model of $T^0_{q,G} \cup \{\neg\varphi_1(\bar{g}), \ldots, \neg\varphi_k(\bar{g})\}$. $T_{q,G}$ has then a model by the compactness theorem ([3, theorem 1.3.22] or [9, theorem 2.1.4]). □

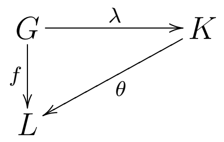
We present two direct reformulations of the definition in terms of diagrams:

Proposition 7 *Let G be a special group and let q be a form of dimension at least 2 over G. A morphism of special groups $\lambda : G \to K$ such that $\lambda(q)$ is isotropic is a GPS of q over G if and only if for every $f : G \to L$ morphism of special groups with $f(q)$ isotropic, there is an elementary extension L' of L and a morphism of special groups $\theta : K \to L'$ such that the following diagram* *is commutative*



PROOF: "$\Leftarrow$" That $K \models T_{q,G}$, where the interpretation of the elements of G is given by λ, follows from the hypothesis using remark 5 1.
"$\Rightarrow$" It suffices to take for L' a model of $T = \Delta(L) \cup \mathrm{pp}^K(K) \cup \{c_{\lambda(g)} = c_{f(g)} \mid g \in G\}$. We check that T has a model: consider of finite subset of T. It is included in $A = \Delta(L) \cup \{\varphi(c_{\lambda(\bar{g})}, c_{\bar{k}})\} \cup \{c_{\lambda(\bar{g})} = c_{f(\bar{g})}\}$ where $\varphi(c_{\lambda(\bar{g})}, c_{\bar{k}}) \in \mathrm{pp}^K(K)$, $\bar{g} \in G$, and $\bar{k} \in K \setminus \lambda(G)$. Since $K \models \varphi(\lambda(\bar{g}), \bar{k})$ we have $K \models \exists\bar{x}\ \varphi(\lambda(\bar{g}), \bar{x})$ and $T^0_{q,G} \vdash \exists\bar{x}\ \varphi(c_{\bar{g}}, \bar{x})$. But $L \models T^0_{q,G}$ if the interpretation of the elements of G is given by f. So $L \models \exists\bar{x}\ \varphi(f(\bar{g}), \bar{x})$, and L is a model of A where, for every $g \in G$, $c_{\lambda(g)}$ and $c_{f(g)}$ are interpreted by $f(g)$ (the interpretation $c^L_{\lambda(g)} := f(g)$ is well defined: Assume $\lambda(g_1) = \lambda(g_2)$ for some g_1, $g_2 \in G$. By definition of λ we have $T^0_{q,G} \vdash c_{g_1} = c_{g_2}$. But L is a model of $T^0_{q,G}$ if the interpretation of the elements of G is given by f. So $f(g_1) = f(g_2)$). $\square$

Proposition 8 *Let G be a special group and let q be a form of dimension at least 2 over G. A morphism of special groups $\lambda : G \to K$ such that $\lambda(q)$ is isotropic is a GPS of q over G if and only if for every $f : G \to L$ morphism of special groups with $f(q)$ isotropic and L $|K|^+$-saturated, there is a morphism of special groups $\theta : K \to L$ such that the following diagram is commutative*



PROOF: It is the same proof as for proposition 7, except that, since L is $|K|^+$-saturated, we can satisfy the theory T in L, for a proper interpretation of the constants c_a, $a \in K$. $\square$

The following observation provides another analogy with the field case, where a place between fields gives rise to a morphism of groups between their respective Witt rings:

Remark 9 *Let K and L be special groups, and let K_0 be a (possibly empty) subset of K. Let $\lambda_0 : K_0 \to L$ be a map, and let $\lambda : K \to L'$ be a morphism of special groups from K to an elementary extension L' of L, such that the* *following diagram commutes*

$$\begin{array}{ccc} K_0 & \hookrightarrow & K \\ {\scriptstyle \lambda_0}\downarrow & & \downarrow{\scriptstyle \lambda} \\ L & \underset{\prec}{\longrightarrow} & L' \end{array}$$

Then the map

$$\lambda_1 : W(K) \to W(L)$$
$$\langle a \rangle \mapsto \begin{cases} \langle \lambda(a) \rangle & \text{if } \lambda(a) \in L \\ 0 & \text{otherwise} \end{cases}$$

induces a morphism of groups and, for every $a \in K_0$*,* $\lambda_1(\langle a \rangle) = \langle \lambda_0(a) \rangle$*.*

We now turn our attention to the satisfaction, in models of $T_{q,G}$, of pp-formulas with parameters in G. We first need a new notion to deal with the following inconvenience: It is clear that if $a \in G \setminus \{1\}$, then the generic partial splittings of q over G and of aq over G are the same. But the theories $T_{q,G}$ and $T_{aq,G}$ are different since the pp-formula "q isotropic" belongs to $T_{q,G}$ but not to $T_{aq,G}$ (if q is anisotropic over G, of course). The following definition will solve this problem:

Definition 10 *Let* L *be a first-order language and let* T *be an* L*-theory. We define*

$$\overline{T}^{pp} = T \cup \{\varphi \mid \varphi \text{ pp-}L\text{-formula},\ T \vdash \varphi\}.$$

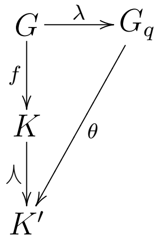
It is clear that $T_{q,G}$ and $\overline{T_{q,G}}^{pp}$ have the same models. Applying this definition to $T_{q,G}$ and $T_{aq,G}$, and using the completeness theorem, we immediately see that $\overline{T_{q,G}}^{pp} = \overline{T_{aq,G}}^{pp}$. (The comparison of the theories $\overline{T_{q,G}}^{pp}$ for q Pfister form over G is completed for G reduced in corollary 20.) More generally:

Lemma 11 *Let* G *be a special group and let* p *and* q *be forms over* G*, of dimension at least* 2*. The following are equivalent:*

(1) $\overline{T_{q,G}}^{pp} = \overline{T_{p,G}}^{pp}$*;*
(2) The theories $T_{q,G}$ *and* $T_{p,G}$ *have the same models;*
(3) For every morphism of special groups $f : G \to K$*,* $f(q)$ *is isotropic if and only if* $f(p)$ *is isotropic.*

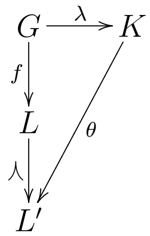
Proof: 1.⇔2. is a direct consequence of the completeness theorem, and we only check the equivalence of 2. and 3.

Assume 2., and let $f : G \to K$ be a morphism of special groups such that $f(q)$ is isotropic. Let $\lambda : G \to G_q$ be a GPS of q over G, i.e. a model of $T_{q,G}$. By proposition 7 we then have the following commutative diagram, for some elementary extension K' of K and some morphism of special groups θ



By hypothesis, λ is also a GPS of p over G, from which follows that $\lambda(p)$ and then $f(p)$ is isotropic.

Assume 3., and let $\lambda : G \to K$ be a GPS of q over G. We use proposition 7 to show that it is a GPS of p over G. Let $f : G \to L$ be a morphism of special groups such that $f(p)$ is isotropic. By hypothesis $f(q)$ is isotropic and we have an elementary extension L' of L and a morphism of special groups θ such that the following diagram commutes, completing the proof:



□

We also observe that the theory $\overline{T_{q,G}}^{pp}$ is complete with respect to pp-formulas with parameters in G, i.e. if $\varphi(\bar{g})$ is such a pp-formula, then either $\varphi(\bar{g}) \in \overline{T_{q,G}}^{pp}$ or $\neg\varphi(\bar{g}) \in \overline{T_{q,G}}^{pp}$. Written a bit more generally, it gives:

Proposition 12 *Let G_1 and G_2 be special groups, let $n \geq 2$, and let, for $i = 1, 2$, $q_i = \langle a_{i,1}, \ldots, a_{i,n} \rangle$ be a form over G_i, and $\lambda_i : G_i \to G_{iq_i}$ be a GPS of q_i over G_i. Assume that $(G_1, a_{1,1}, \ldots, a_{1,n}) \equiv_{pp} (G_2, a_{2,1}, \ldots, a_{2,n})$.*
Then $(G_{1q_1}, \lambda_1(a_{1,1}), \ldots, \lambda_1(a_{1,n})) \equiv_{pp} (G_{2q_2}, \lambda_2(a_{2,1}), \ldots, \lambda_2(a_{2,n}))$.

Proof: Let $\bar{a}_1 = (a_{1,1}, \ldots, a_{1,n})$ and $\bar{a}_2 = (a_{2,1}, \ldots, a_{2,n})$. Let $\varphi(\bar{x})$ be a pp-formula such that $G_{1q_1} \models \varphi(\lambda_1(\bar{a}_1))$. Then $T^0_{q_1,G_1} \vdash \varphi(\bar{a}_1)$, and by compactness we find formulas $\psi_1(\bar{g}_1, \bar{a}_1), \ldots, \psi_k(\bar{g}_1, \bar{a}_1)$ in $\mathrm{pp}^{G_1}(G_1)$ such that

$$SG \cup \text{“}q_1 \text{ isotropic”} \cup \{\psi_1(\bar{g}_1, \bar{a}_1), \ldots, \psi_k(\bar{g}_1, \bar{a}_1)\} \vdash \varphi(\bar{a}_1). \tag{2}$$

We can of course assume $\bar{g}_1 \cap \bar{a}_1 = \emptyset$. Thus $G_1 \models \exists \bar{g} \ \bigwedge_{i=1}^{k} \psi_i(\bar{g}, \bar{a}_1)$ and by hypothesis $G_2 \models \exists \bar{g} \ \bigwedge_{i=1}^{k} \psi_i(\bar{g}, \bar{a}_2)$. Let $\bar{g}_2 \in G_2$ be such that $G_2 \models \bigwedge_{i=1}^{k} \psi_i(\bar{g}_2, \bar{a}_2)$. Then $\psi_1(\bar{g}_2, \bar{a}_2), \ldots, \psi_k(\bar{g}_2, \bar{a}_2) \in \mathrm{pp}^{G_2}(G_2)$, which, together with (2), implies $T^0_{q_2,G_2} \vdash \varphi(\bar{a}_2)$ and $G_{2q_2} \models \varphi(\lambda_2(\bar{a}_2))$. □

We can now define the notion of generic splitting tower of a quadratic form (see [6, chapter 5] or [14, chapter 4 definition 6.9]):

Definition 13 (Generic splitting tower of q over G) *Let G be a special group and let q be any form aver G. We define a tower of generic partial splittings*

$$G = G_0 \stackrel{\lambda_0}{\to} G_1 \stackrel{\lambda_1}{\to} \cdots \stackrel{\lambda_{h-1}}{\to} G_h,$$

together with quadratic forms q_k over G_k and integers i_k ($1 \leq k \leq h$), as follows:

(1) $G_0 = G$, $q_0 = q_{an}$, and $i_0 = i(q)$ (so $q \equiv q_{an} \oplus i_0\langle -1, 1\rangle$);

(2) Assume G_k, q_k and i_k are defined. Then

- *If $\dim q_k \leq 1$, then $h = k$;*
- *If $\dim q_k \geq 2$, we take for $\lambda_k : G_k \to G_{k+1}$ any GPS of q_k over G_k and define $q_{k+1} = (\lambda_k(q_k))_{an}$ and $i_{k+1} = i(\lambda_{q_k}(q_k))$ (in other words $\lambda_k(q_k) \equiv \lambda_k(q_k)_{an} \oplus i_{k+1}\langle -1, 1\rangle \equiv q_{k+1} \oplus i_{k+1}\langle -1, 1\rangle$).*

The length of this generic splitting tower of q over G is the integer h, and the i_k are its indices, also called higher Witt indices of q.

The uniqueness implicitly assumed in this last sentence is justified by the following result:

Corollary 14 *Let G, H be special groups, with $\bar{a} = (a_1, \ldots, a_n) \in G^n$ and $\bar{b} = (b_1, \ldots, b_n) \in H^n$. Assume that $(G, \bar{a}) \equiv_{pp} (H, \bar{b})$. Then any two generic splitting towers of $q = \langle a_1, \ldots, a_n\rangle$ over G and $p = \langle b_1, \ldots, b_n\rangle$ over H have the same length and the same indices.*

Proof: Let $G_0 \stackrel{\lambda_0}{\to} G_1 \stackrel{\lambda_1}{\to} \cdots \stackrel{\lambda_{h-1}}{\to} G_h$, and $H_0 \stackrel{\mu_0}{\to} H_1 \stackrel{\mu_1}{\to} \cdots \stackrel{\mu_{k-1}}{\to} H_k$ be two generic splitting towers of q over G and of p over H. We can assume $h \leq k$. Then by induction on i, applying proposition 12, we get, for $1 \leq i \leq h$, $(G_i, \lambda_{i-1} \circ \cdots \circ \lambda_0(\bar{a})) \equiv_{pp} (H_i, \mu_{i-1} \circ \cdots \circ \mu_0(\bar{b}))$, from which the result follows since the properties under consideration are expressed by pp-formulas with parameters $\lambda_{i-1} \circ \cdots \circ \lambda_0(\bar{a})$ and $\mu_{i-1} \circ \cdots \circ \mu_0(\bar{b})$. □

The next theorem corresponds to [8, theorem 10.1] or [14, chapter 4 corollary 6.10]

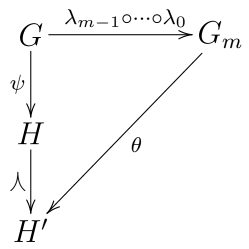
Theorem 15 *Let G be a special group, let q be a form aver G, and let*

$$G = G_0 \stackrel{\lambda_0}{\to} G_1 \stackrel{\lambda_1}{\to} \cdots \stackrel{\lambda_{h-1}}{\to} G_h,$$

be a generic splitting tower of q over G.
Let $\psi : G \to H$ be a morphism of special groups, and write $\psi(q) \equiv \psi(q)_{an} \oplus j\langle -1, 1\rangle$. Then there is $0 \leq m \leq h$ such that

(1) $j = i_0 + \cdots + i_m$;

(2) there is an elementary extension H' of H and a morphism of special groups $\theta : G_m \to H'$, such that the following diagram is commutative



(3)

(3) if $m < h$, statement 2. does not hold if m is replaced by $m+1$.

Proof: We use the following notation for the duration of this proof: if $1 \leq i \leq h$, we say that $\mathrm{pp}^{G_i}(G) \subseteq \mathrm{pp}^{H}(G)$ if for every pp-formula $\varphi(\bar{x})$ and every $\bar{g} \in G$, we have $G_i \models \varphi(\lambda_{i-1} \circ \cdots \circ \lambda_0(\bar{g}))$ implies $H \models \varphi(\psi(\bar{g}))$.
Note that $\mathrm{pp}^{G_i}(G) \subseteq \mathrm{pp}^{H}(G)$ if and only if there is an elementary extension H' of H and a morphism of special groups $\theta : G_i \to H'$ such that diagram (3) above (with $m = i$) is commutative. One implication is clear. For the other, it suffices to take for H' a model of $T = \Delta(H) \cup \mathrm{pp}^{G_i}(G_i) \cup \{c_{\lambda_{i-1}\circ\cdots\circ\lambda_0(g)} = c_{\psi(g)} \mid g \in G\}$. The first part of this theory ($\Delta(H)$) ensures that H' will be an elementary extension of H, the second part ($\mathrm{pp}^{G_i}(G_i)$) gives the morphism θ, and the third part assures the commutativity of the diagram. Finally T has a model by compactness since we easily see that H is a model of every finite subset of T, for a well-chosen interpretation of the constants.

To complete the proof we simply take m maximal in $\{0, \ldots, h\}$ such that $\mathrm{pp}^{G_m}(G) \subseteq \mathrm{pp}^{H}(G)$. Such an m exists since $\mathrm{pp}^{G_0}(G) = \mathrm{pp}^{G}(G) \subseteq \mathrm{pp}^{H}(G)$. As observed above we then get conclusion 2. of the theorem, and conclusion 3. expresses the maximality of m. We just need to check that $j = i_0 + \cdots + i_m$.

In H' we have $\psi(q) \equiv \psi(q)_{an} \oplus j\langle -1, 1\rangle \equiv \theta(\lambda_{m-1} \circ \cdots \circ \lambda_0(q)) \equiv \theta(q_m) \oplus (i_0 + \cdots + i_m)\langle -1, 1\rangle$. This yields $j \geq i_0 + \cdots + i_m$. Assume now that $j > i_0 + \cdots + i_m$. Then $\theta(q_m)$ is isotropic, so $\dim q_m \geq 2$. Consequently $m < h$ and the morphism λ_m is defined. Since λ_m is a GPS of q_m over G_m, we get an elementary extension $i : H' \to H''$ and a morphism of special groups $\tau : G_{m+1} \to H''$ such that $i \circ \theta = \tau \circ \lambda_m$, which yields $\mathrm{pp}^{G_{m+1}}(G) \subseteq \mathrm{pp}^{H}(G)$, a contradiction. □

3 For reduced special groups

The next lemma collects without proof two consequences of Marshall's isotropy theorem ([12, theorem 1.4]). The first one is [1, lemma 2] and the second one follows directly from the proof of [1, theorem 3].

Lemma 16 *Let G be a reduced special group and let q be a form over G.*

(1) Assume $i(q) < n$ for some integer n. Then there is a finite subspace Y of X_G such that $i(q) < n$ over Y.

(2) Let p be a form over G and assume that q is anisotropic. Assume also that for every finite subspace Y of X_G such that q is anisotropic over Y and every morphism of special groups $f : G_Y \to H$, we have $D(f \circ \pi_Y(p)) \subseteq D(f \circ \pi_Y(q))$. Then p is a subform of q.

This allows us to reduce certain questions to finite reduced special groups (i.e. the special groups G_Y corresponding to the finite subspaces Y appearing in the above lemma), which are special groups of fields, by [10, theorem 4.10].

The next two results are the analogues of [6, lemma 4.5, theorem 5.8] and [14, chapter 4 theorem 5.4 (ii), (i)].

Proposition 17 *Let G be a reduced special group and let φ and ψ be quadratic forms over G, with $\dim\varphi \geq 2$, $\varphi \not\equiv \langle -1, 1\rangle$. Let $\lambda_\varphi : G \to G_\varphi$ be a GPS of φ over G. Assume $1 \in D\varphi$, ψ anisotropic, and $\lambda_\varphi(\psi)$ hyperbolic. Then for every $\alpha \in D\psi$, $\alpha\varphi$ is a subform of ψ.*

Proof: Let $n = \dim\varphi$ and let $\alpha \in D\psi$. To show that $\alpha\varphi$ is a subform of ψ it is enough to show that $i(\psi - \alpha\varphi) \geq n$. Assume that $i(\psi - \alpha\varphi) < n$. By lemma 16 1. there is a finite subspace Y_0 of X_G such that $i(\psi - \alpha\varphi) < n$ on Y_0, and since ψ is anisotropic, there is a finite subspace Y_1 of X_G such that ψ is anisotropic over Y_1. Moreover, by Pfister's local-global principle there is $\sigma \in X_G$ such that $\sigma(\varphi) \not\equiv \langle -1, 1\rangle$. Let Y be the finite subspace generated by $Y_0 \cup Y_1 \cup \{\sigma\}$. Then over Y: ψ is anisotropic, $i(\psi - \alpha\varphi) < n$, and $\varphi \not\equiv \langle -1, 1\rangle$. In terms of special groups, if π_Y denotes the projection from G onto G_Y: $\pi_Y(\psi)$ is anisotropic, $i(\pi_Y(\psi) - \pi_Y(\alpha)\pi_Y(\varphi)) < n$, and $\pi_Y(\varphi) \not\equiv \langle -1, 1\rangle$.

Since G_Y is finite, by [10, theorem 4.10] there is a field K such that $G_Y = SG(K)$. Let $K(\pi_Y(\varphi))$ be a generic partial splitting field of $\pi_Y(\varphi)$, and let $\xi : SG(K) \to SG(K(\pi_Y(\varphi)))$ be the morphism of special groups induced by the field extension $K(\pi_Y(\varphi)) \mid K$. Since $\xi \circ \pi_Y(\varphi)$ is isotropic, there is an elementary extension L of $SG(K(\pi_Y(\varphi))$ and a morphism of special groups $\tau : G_\varphi \to L$, such that the following diagram commutes

$$\begin{array}{ccc} G \xrightarrow{\pi_Y} G_Y = SG(K) & \xrightarrow{\xi} & SG(K(\pi(\varphi))) \\ \downarrow{\scriptstyle \lambda_\varphi} & & \downarrow{\scriptstyle \lambda} \\ G_\varphi & \xrightarrow{\tau} & L \end{array}$$

Since $\lambda_\varphi(\psi)$ is hyperbolic, $\xi\circ\pi_Y(\psi)$ is hyperbolic, and we can apply [14, chapter 4 theorem 5.4 (ii)] to $\pi_Y(\varphi)$ and $\pi_Y(\psi)$. We get that $\pi_Y(\alpha)\pi_Y(\varphi)$ is a subform of $\pi_Y(\psi)$, and consequently $i(\pi_Y(\psi) - \pi_Y(\alpha)\pi_Y(\varphi)) \geq n$, a contradiction. □

Proposition 18 *Let G be a reduced special group and let φ be an anisotropic form of dimension at least 2 over G. Let $\lambda_\varphi : G \to G_\varphi$ be a GPS of φ over G. Assume $1 \in D\varphi$ and $\lambda_\varphi(\varphi)$ hyperbolic. Then φ is a Pfister form.*

PROOF: Assume φ is not a Pfister form, and let ρ be a Pfister form over G, of maximal dimension with respect to the property of being a subform of φ (such a form exists since $1 \in D(\varphi)$). Then there is a form γ over G, of dimension at least 1, such that $\varphi \equiv \rho \oplus \gamma$. Let $a \in D\gamma$, and consider the Pfister form $q = \langle 1, a\rangle \otimes \rho = \rho \oplus a\rho$. We show that q is a subform of φ, therefore reaching a contradiction.

By lemma 16 2. it suffices to show that if Y is a finite subspace of X_G such that $\pi_Y(\varphi)$ is anisotropic, and $f : G_Y \to H$ is a morphism of special groups, then $D(f \circ \pi_Y(q)) \subseteq D(f \circ \pi_Y(\varphi))$.

We start by observing that $\pi_Y(\varphi)$ is a Pfister form: Let K be a field such that $G_Y = SG(K)$ and let $K(\pi_Y(\varphi))$ be a generic splitting field of $\pi_Y(\varphi)$. Since $\pi_Y(\varphi)$ is isotropic over $SG(K(\pi_Y(\varphi)))$ there is an elementary extension L of $SG(K(\pi_Y(\varphi)))$ and a morphism of special groups $G_\varphi \to L$ such that the following diagram is commutative:

$$\begin{array}{ccc} G \xrightarrow{\pi_Y} G_Y = SG(K) & \longrightarrow & SG(K(\pi_Y(\varphi))) \\ \downarrow{\scriptstyle \lambda_\varphi} & & \downarrow{\scriptstyle \curlywedge} \\ G_\varphi & \longrightarrow & L \end{array}$$

Since $\lambda_\varphi(\varphi) \sim 0$ it follows that $\pi_Y(\varphi)$, which is not hyperbolic over K, becomes hyperbolic over $K(\pi_Y(\varphi))$. By [14, chapter 4 theorem 5.4 (i)], $\pi_Y(\varphi)$ is then a scalar multiple of a Pfister form, hence a Pfister form since it represents 1.

Let now $x \in D(f \circ \pi_Y(q)) = D(f \circ \pi_Y(\rho) \oplus f \circ \pi_Y(a) \cdot f \circ \pi_Y(\rho))$. Then there are $\alpha, \beta \in D(f \circ \pi_Y(\rho))$ such that $x \in D\langle \alpha, f \circ \pi_Y(a)\beta\rangle = \beta D\langle \alpha\beta, f \circ \pi_Y(a)\rangle$. But $\alpha\beta \in D(f \circ \pi_Y(\rho))$ since $f \circ \pi_Y(\rho)$ is a Pfister form, and $f \circ \pi_Y(a) \in D(f \circ \pi_Y(\gamma))$. It follows that $D\langle \alpha\beta, f \circ \pi_Y(a)\rangle \subseteq D(f \circ \pi_Y(\rho) \oplus f \circ \pi_Y(\gamma)) = D(f \circ \pi_Y(\varphi))$. Moreover $\beta \in D(f \circ \pi_Y(\rho)) \subseteq D(f \circ \pi_Y(\varphi))$ and $f \circ \pi_Y(\varphi)$ is a Pfister form. So $\beta D\langle \alpha\beta, f \circ \pi_Y(a)\rangle \subseteq D(f \circ \pi_Y(\varphi))$ and thus $x \in D(f \circ \pi_Y(\varphi))$, proving the desired inclusion. □

The next theorem mimics [6, lemma 4.4] and [14, chapter 4 theorem 5.4 (iv)].

Theorem 19 *Let G be a reduced special group and let φ be a Pfister form over G with $\dim \varphi \geq 2$. Let $\lambda_\varphi : G \to G_\varphi$ be a GPS of φ over G. Assume ψ is an anisotropic quadratic form over G such that $\lambda_\varphi(\psi)$ is hyperbolic. Then $\psi \equiv \varphi \otimes p$ for some form p over G.*
In particular, for the corresponding abstract Witt rings:

$$\ker(\lambda_\varphi : W(G) \to W(G_\varphi)) = \varphi \cdot W(G).$$

PROOF: Since φ is a Pfister form, it represents 1 and $\lambda_\varphi(\varphi)$ is hyperbolic. Let $\alpha \in D\psi$. By proposition 17, either $\psi \equiv \alpha\varphi$, which proves the result, or $\psi \equiv \alpha\varphi \oplus \psi_1$, where ψ_1 is a form over G. Then ψ_1 satisfies the same hypotheses as ψ, and we conclude by induction on the dimension of ψ. □

With this, we can compare the theories $\overline{T_{\varphi,G}}^{pp}$, for φ anisotropic Pfister form over G. The first observation is that if p and q are quadratic forms of dimension at least 2 over a special group G, then $\overline{T_{q,G}}^{pp} \subseteq \overline{T_{p,G}}^{pp}$ implies $\overline{T_{q,G}}^{pp} = \overline{T_{p,G}}^{pp}$, since the theories $\overline{T_{q,G}}^{pp}$ and $\overline{T_{p,G}}^{pp}$ are complete with respect to pp-formulas with parameters in G, and only consist of such formulas or their negations.

So we are left with the question of determining when two such theories are equal. The next corollary corresponds to [6, theorem 4.2 (i)]:

Corollary 20 *Let G be a reduced special group and let φ and ψ be anisotropic Pfister forms over G, of dimension at least 2. Then $\overline{T_{\varphi,G}}^{pp} = \overline{T_{\psi,G}}^{pp}$ if and only if $\varphi \equiv \psi$.*

PROOF: We prove the non-trivial implication. Let $\lambda_\varphi : G \to G_\varphi$ be a GPS of φ over G. Then it also a GPS of ψ over G, and thus, since ψ is a Pfister form, $\lambda_\varphi(\psi) \sim 0$. By theorem 19, there is a form q_1 over G such that $\psi \equiv q_1 \otimes \varphi$. Similarly, reversing the roles of φ and ψ, we find a form q_2 over G such that $\varphi \equiv q_2 \otimes \psi$. Then $\psi \equiv q_1 \otimes q_2 \otimes \psi$, which implies $q_1 = \langle a_1 \rangle$ and $q_2 = \langle a_2 \rangle$ for some $a_1, a_2 \in G$. Then $\psi \equiv a_1\varphi$. In particular $a_1 \in D_G(\psi)$ and, since ψ is a Pfister form, $a_1\psi \equiv \psi$. Thus $\psi \equiv a_1\psi \equiv \varphi$. □

The example given in [6, example 4.1 (i)] also applies in our context: If ψ is a neighbour of a Pfister form φ (i.e. ψ is a subform of φ and $\dim \psi > \frac{1}{2} \dim \varphi$), then for every morphism of special groups $f : G \to K$, $f(\varphi)$ is isotropic if and only if $f(\psi)$ is isotropic. By lemma 11 it means that $\overline{T_{\varphi,G}}^{pp} = \overline{T_{\psi,G}}^{pp}$.

Remark 21 *Assume G is a reduced special group and $\langle 1, d \rangle$ is an anisotropic form over G (i.e. $d \neq -1$). Let $\lambda_d : G \to G_d$ be a GPS of $\langle 1, d \rangle$. Then*

$\ker \lambda_d = \{1, -d\}$. *Indeed let* $a \neq 1$. *Then*

$$\begin{aligned} a \in \ker \lambda_d &\Leftrightarrow \langle \lambda_d(a), -1\rangle \equiv \langle -1, 1\rangle \\ &\Leftrightarrow \lambda_d(\langle a, -1\rangle) \equiv \langle -1, 1\rangle \\ &\Leftrightarrow \exists \alpha \in G\ \langle a, -1\rangle \equiv \alpha\langle 1, d\rangle \\ &\quad \textit{by theorem 19} \\ &\Leftrightarrow \exists \alpha \in G\ \langle 1, -a\rangle \equiv -\alpha\langle 1, d\rangle \\ &\Leftrightarrow \langle 1, -a\rangle \equiv \langle 1, d\rangle \\ &\quad \textit{by multiplying both sides by } -\alpha \textit{ since} \\ &\quad -\alpha \in D\langle 1, -a\rangle,\ \textit{i.e.}\ -\alpha\langle 1, -a\rangle \equiv \langle 1, -a\rangle \\ &\Leftrightarrow a = -d. \end{aligned}$$

4 On the class of models of $T_{q,G}$

We return to the general case of non-necessarily reduced special groups.

If G is a special group and q is a form over G of dimension at least 2, we denote by $\mathbb{K}_{q,G}$ the class of models of $T_{q,G}$.
We begin with the following straightforward observation, which we state without proof:

Lemma 22 *Let G be a special group and let q be a form over G. Let $\lambda : G \to K$ and $i : K \to L$ be morphisms of special groups, with i pure over $\lambda(G)$. Then λ is a GPS of q over G if and only if $i \circ \lambda$ is a GPS of q over G.*

4.1 Functorial properties

We start with a straightforward result about inductive limits (for a presentation of these, see [5] pp. 50-51).

Proposition 23 *Let G be a special group and let q be a form over G, of dimension at least 2. Let $(G_j, f_{jk} \mid j \leq k \in J)$ be an inductive system of special groups over a right-directed poset J, such that each G_j is in $\mathbb{K}_{q,G}$ and each f_{jk} is an $L_{SG}(G)$-monomorphism. Let G' be the inductive limit of this system. Then $G' \in \mathbb{K}_{q,G}$.*

PROOF: This is clear since the theory $T_{q,G}$ is $\forall\exists$, by [5, theorem 2.4.6] (the part on page 52). $\square$

Proposition 24 *Let $L_{SG}(\bar{a})$ be the language L_{SG} expanded by new constants $a_1, \ldots, a_n$, with $n \geq 2$. Let $(G_j, f_{jk} \mid j \leq k \in J)$ and $\mathcal{K} = (K_j, g_{jk} \mid j \leq k \in J)$ be two inductive systems of special groups over a right-directed poset J, where the f_{jk} are $L_{SG}(\bar{a})$-morphisms and the g_{jk} are L_{SG}-morphisms. Denote by G and K their respective inductive limits.*
For each $i \in I$ let q_i be the quadratic form $\langle a_1^{G_i}, \ldots, a_n^{G_i} \rangle$, and assume that there is a morphism of special groups $\lambda_i : G_i \to K_i$, such that λ_i is a GPS of q_i over G_i and, for every $i \leq j \in J$ the following diagram commutes:

$$\begin{array}{ccc} G_i & \xrightarrow{f_{ij}} & G_j \\ \lambda_i \downarrow & & \downarrow \lambda_j \\ K_i & \xrightarrow[g_{ij}]{} & K_j \end{array} \tag{4}$$

Then the canonical morphism $\lambda : G \to K$ is a GPS of $\langle a_1^G, \ldots, a_n^G \rangle$ over G.

PROOF: Since the diagram (4) commutes, we can assume that, in the inductive system $\mathcal{K}$, the g_{jk} are actually $L_{SG}(\bar{a})$-morphisms, where the interpretation of $\bar{a}$ in K_i is $\lambda_i(\bar{a}^{G_i})$. The morphism $\lambda : G \to K$ is then an $L_{SG}(\bar{a})$-morphism, and it follows that $K \models T^0_{q,G}$ (where the interpretation of the elements of G is given by λ). Let now $\neg\varphi(\bar{g}) \in T_{q,G} \setminus T^0_{q,G}$, so

$$SG \cup \mathrm{pp}^G(G) \cup \text{“}q \text{ isotropic”} \nvdash \varphi(\bar{g}). \tag{5}$$

Assume by way of contradiction that $K \models \varphi(\lambda(\bar{g}))$. Let $i_0 \in I$ be such that $\bar{g} \in G_{i_0}$. Then there is $j_0 \geq i_0$ such that $K_{j_0} \models \varphi(\lambda_{j_0}(f_{i_0 j_0}(\bar{g})))$, i.e. by definition of λ_{j_0}: $SG \cup \mathrm{pp}^{G_{j_0}}(G_{j_0}) \cup \text{“}q_{j_0} \text{ isotropic”} \vdash \varphi(f_{i_0 j_0}(\bar{g}))$.
But since $\mathrm{pp}^{G_{j_0}}(G_{j_0}) \subseteq \mathrm{pp}^G(G)$ (via the canonical map $G_{j_0} \to G$), we get $SG \cup \mathrm{pp}^G(G) \cup \text{“}q \text{ isotropic”} \vdash \varphi(\bar{g})$, contradicting (5). $\square$

Proposition 25 *Let G be a special group and let q be a form over G, of dimension at least 2. Assume $\mathcal{G} = (G_j, f_{jk} \mid j \leq k \in J)$ is a projective system of topological groups, each equipped with an Hausdorff topology, over a right-directed poset J, such that*

(1) $G_j \in \mathbb{K}_{q,G}$ and the relation $\equiv$ is a closed subset of G_j^4, for every $j \in J$;
(2) The morphisms f_{jk} are continuous $L_{SG}(G)$-morphisms for every $j \leq k \in J$.

Then the projective limit G' of $\mathcal{G}$ is in $\mathbb{K}_{q,G}$.

PROOF: The exact same proof as [2, lemma 6] gives us that the canonical inclusion of G' in $\prod_{i \in I} G_i$ is pure in the language $L_{SG}(G)$. Since by proposition

30 1., the $L_{SG}(G)$-structure $\prod_{i \in I} G_i$ is a model of $T_{q,G}$, it follows that G' is model of $T_{q,G}$. $\square$

Proposition 26 *Let $\sigma : G_1 \to G_2$ be a morphism of special groups, and let q be a quadratic form over G_1, of dimension at least 2. Then for every GPS $\lambda_q : G_1 \to G_{1q}$ of q over G_1 there exist a GPS $\lambda_{\sigma(q)} : G_2 \to G_{2\sigma(q)}$ of $\sigma(q)$ over G_2 and a morphism of special groups $\sigma_q : G_{1q} \to G_{2\sigma(q)}$ such that the following diagram is commutative.*

$$\begin{array}{ccc} G_1 & \xrightarrow{\sigma} & G_2 \\ \lambda_q \downarrow & & \downarrow \lambda_{\sigma(q)} \\ G_{1q} & \xrightarrow[\sigma_q]{} & G_{2\sigma(q)} \end{array}$$

Proof: To get the desired conclusion we just need to take for $G_{2\sigma(q)}$ any model of $\Omega = T_{\sigma(q),G_2} \cup \mathrm{pp}^{G_{1q}}(G_{1q}) \cup \{c_{\sigma(g)} = c_{\lambda_q(g)} \mid g \in G_1\}$, so we have to check that the theory Ω is consistent. Consider a finite subset of Ω. It is included in

$$A := T_{\sigma(q),G_2} \cup \{\varphi(c_{\lambda_q(\bar{g})}, c_{\bar{h}_1})\} \cup \{c_{\lambda_q(\bar{g})} = c_{\sigma(\bar{g})}\},$$

where $\varphi(c_{\lambda_q(\bar{g})}, c_{\bar{h}_1}) \in \mathrm{pp}^{G_{1q}}(G_{1q})$, $\bar{g} \in G_1$, and $\bar{h}_1 \in G_{1q} \setminus \lambda_q(G_1)$.

To find a model of A, it is enough to find a model of

$$B := T_{\sigma(q),G_2} \cup \{\exists \bar{x}\ \varphi(c_{\lambda_q(\bar{g})}, \bar{x})\} \cup \{c_{\lambda_q(\bar{g})} = c_{\sigma(\bar{g})}\}.$$

Let $\mu : G_2 \to K$ be a GPS of $\sigma(q)$ over G_2. Fix the following interpretation in K of the constants $c_{\sigma(g)}$ and $c_{\lambda_q(g)}$, for $g \in G$:

$$c^K_{\sigma(g)} = \mu \circ \sigma(g), \quad c^K_{\lambda_q(g)} = \mu \circ \sigma(g)$$

(the second part of this interpretation: $c^K_{\lambda_q(g)} = \mu \circ \sigma(g)$, is well-defined: Let g_1, $g_2 \in G_1$ be such that $\lambda_q(g_1) = \lambda_q(g_2)$. Then $T^0_{q,G_1} \vdash c_{g_1} = c_{g_2}$. But K is a model of T^0_{q,G_1} if the interpretation of the elements of G_1 is given by $\mu \circ \sigma$. Then $K \models \mu \circ \sigma(g_1) = \mu \circ \sigma(g_2)$).
We claim that K is then a model of B. It is clear that it is a model of $T_{\sigma(q),G_2} \cup \{c_{\lambda_q(\bar{g})} = c_{\sigma(\bar{g})}\}$. Consider now $\exists \bar{x}\ \varphi(c_{\lambda_q(\bar{g})}, \bar{x})$. We have to show that $K \models \exists \bar{x}\ \varphi(\mu \circ \sigma(\bar{g}), \bar{x})$.

We know that $G_{1q} \models \exists \bar{x}\ \varphi(\lambda_q(\bar{g}), \bar{x})$, so $T^0_{q,G_1} \vdash \exists \bar{x}\ \varphi(c_{\bar{g}}, \bar{x})$. But K is a model of T^0_{q,G_1} if the interpretation of the elements of G_1 is given by $\mu \circ \sigma$. Then $K \models \exists \bar{x}\ \varphi(\mu \circ \sigma(\bar{g}), \bar{x}))$. $\square$

Proposition 27 *Let G be a special group and let q be a form over G of dimension at least 2. Let $\lambda_q : G \to G_q$ and $\mu_q : G \to H_q$ be two GPS of q over G.*

Then there is a model L of $T_{q,G}$ and pure morphisms of special groups $f : H_q \to L$ and $g : G_q \to L$ such that the following diagram is commutative.

$$\begin{array}{ccc} G & \xrightarrow{\lambda_q} & G_q \\ {\scriptstyle \mu_q}\downarrow & & \downarrow{\scriptstyle g} \\ H_q & \xrightarrow[f]{} & L \end{array}$$

In particular the class $\mathbb{K}_{q,G}$ has the joint embedding property.

Proof: Let

$$\begin{aligned} T = T_{q,G} &\cup \mathrm{pp}^{H_q}(H_q) \cup \text{neg-pp}^{H_q}(H_q) \\ &\cup \mathrm{pp}^{G_q}(G_q) \cup \text{neg-pp}^{G_q}(G_q) \cup \{c_{\mu_q(g)} = c_{\lambda_q(g)} \mid g \in G\}. \end{aligned}$$

A model of T will provide both the special group L, the two pure morphisms, and the commutativity of the diagram. We show its existence by compactness. Consider a finite subset of T. It is included in

$$\begin{aligned} A = T_{q,G} &\cup \{\varphi(c_{\mu_q(\bar g)}, c_{\bar h})\} \cup \{\neg\psi_1(c_{\mu_q(\bar g)}, c_{\bar h}), \ldots, \neg\psi_r(c_{\mu_q(\bar g)}, c_{\bar h})\} \\ &\cup \{\varphi'(c_{\lambda_q(\bar g)}, c_{\bar k})\} \cup \{\neg\psi'_1(c_{\lambda_q(\bar g)}, c_{\bar k}), \ldots, \neg\psi'_s(c_{\lambda_q(\bar g)}, c_{\bar k})\}, \end{aligned}$$

where $\bar g \in G$, $\bar h \in H_q \setminus \mu_q(G)$, $\bar k \in G_q \setminus \lambda_q(G)$, $\varphi(c_{\mu_q(\bar g)}, c_{\bar h}) \in \mathrm{pp}^{H_q}(H_q)$, $\neg\psi_i(c_{\mu_q(\bar g)}, c_{\bar h}) \in \text{neg-pp}^{H_q}(H_q)$, $\varphi'(c_{\lambda_q(\bar g)}, c_{\bar k}) \in \mathrm{pp}^{G_q}(G_q)$, and $\neg\psi'_j(c_{\lambda_q(\bar g)}, c_{\bar k}) \in \text{neg-pp}^{G_q}(G_q)$.

Since $H_q \models \exists \bar x\ \varphi(\mu_q(\bar g), \bar x)$, we get, applying proposition 12, that $G_q \models \exists \bar x\ \varphi(\lambda_q(\bar g), \bar x)$. Let $\bar\alpha \in G_q$ be such that $G_q \models \varphi(\lambda_q(\bar g), \bar\alpha)$. Similarly we get $H_q \models \exists \bar x\ \varphi'(\mu_q(\bar g), \bar x)$, and find $\bar\beta \in H_q$ such that $H_q \models \varphi'(\mu_q(\bar g), \bar\beta)$.

We consider now the structure $M := H_q \times G_q$, together with the following interpretations:

$$\begin{aligned} &c^M_{\mu_q(g)} = c^M_{\lambda_q(g)} = (\mu_q(g), \lambda_q(g)), \text{ for every } g \in G, \\ &c^M_{\bar h} = \bar h \times \bar\alpha, \\ &c^M_{\bar k} = \bar\beta \times \bar k \end{aligned}$$

(the first line of interpretations is well defined since by proposition 12, for g_1, $g_2 \in G$: $\lambda_q(g_1) = \lambda_q(g_2)$ if and only if $\mu_q(g_1) = \mu_q(g_2)$).
By proposition 30 1., M, together with the interpretation of the elements of G given by $\mu_q \times \lambda_q$, is a model of $T_{q,G}$. From $H_q \models \varphi(\mu_q(\bar g), \bar h)$ and $G_q \models \varphi(\lambda_q(\bar g), \bar\alpha)$ follows $M \models \varphi(c_{\mu_q(\bar g)}, c_{\bar h})$. From $H_q \not\models \psi_i(\mu_q(\bar g), \bar h)$ follows $M \not\models \psi_i(c_{\mu_q(\bar g)}, c_{\bar h})$, for $i = 1, \ldots, r$. Similarly we get $M \models \varphi'(c_{\lambda_q(\bar g)}, c_{\bar k})$ and $M \not\models \psi'_j(c_{\lambda_q(\bar g)}, c_{\bar k})$, for $j = 1, \ldots, s$. So M is a model of A. $\square$

Proposition 28 *Let $i : G_1 \to G_2$ be a pure morphism of special groups, let q be a form over G_1 of dimension at least 2, and let $\lambda : G_2 \to K$ be a GPS of $i(q)$ over G_2. Then $\lambda \circ i$ is a GPS of q over G_1.*

PROOF: We have to show that K, together with the interpretation of the elements of G_1 given by $\lambda \circ i$, is a model of T_{q,G_1}. It is clear that it is a model of T^0_{q,G_1}. Let $\psi(\bar{g})$ be a pp-formula with parameters in G_1, such that

$$T^0_{q,G_1} \not\vdash \psi(c_{\bar{g}}). \tag{6}$$

Assume $K \models \psi(\lambda \circ i(\bar{g}))$. Then by definition of K, $T^0_{q,G_2} \vdash \psi(c_{i(\bar{g})})$, so by compactness there is a formula $\varphi(c_{i(\bar{g}_1)}, c_{\bar{g}_2}) \in \mathrm{pp}^{G_2}(G_2)$, with $\bar{g}_1 \in G_1$ and $\bar{g}_2 \in G_2 \setminus i(G_1)$, such that $SG \cup \{\varphi(c_{i(\bar{g}_1)}, c_{\bar{g}_2})\} \cup$ "$i(q)$ isotropic" $\vdash \psi(c_{i(\bar{g})})$. This yields

$$SG \cup \{\exists \bar{x}\ \varphi(c_{\bar{g}_1}, \bar{x})\} \cup \text{“}q \text{ isotropic”} \vdash \psi(c_{\bar{g}}). \tag{7}$$

But, since i is pure, $G_1 \models \exists \bar{x}\ \varphi(\bar{g}_1, \bar{x})$, so $\exists \bar{x}\ \varphi(c_{\bar{g}_1}, \bar{x}) \in \mathrm{pp}^{G_1}(G_1)$, which, together with (7), contradicts (6). □

Corollary 29 *Let G be a special group and let q be a form over G. Let H be a group of exponent 2, and let i be the inclusion of G into $G[H]$. If a morphism of special groups $\lambda : G[H] \to K$ is a GPS of q over $G[H]$, then $\lambda \circ i$ is a GPS of q over G.*

Proposition 30 *Let G be a special group, let q be a form over G, and let $\lambda : G \to K$ be a morphism of special groups.*

(1) Assume λ is a GPS of q over G, and let $\mu : G \to H$ be any morphism of special groups such that $\mu(q)$ is isotropic in H. Then $\lambda \times \mu : G \to K \times H$, $x \mapsto (\lambda(x), \mu(x))$ is a GPS of q over G.
(2) Let H be a group of exponent 2 and let i be the inclusion of K into $K[H]$. Then $i \circ \lambda : G \to K[H]$ is a GPS of q over G if and only if λ is a GPS of q over G.

PROOF:

(1) An straightforward verification shows that $K \times H$, together with the interpretation of the elements of G given by the map $\lambda \times \mu$, is a model of $T_{q,G}$.
(2) It is a direct application of lemma 22 (i is pure since the canonical projection from $K[H]$ onto K is a retract of i that is a morphism of special groups). □

In the following special case, we can explicitly compute the generic splitting morphism of a simple quadratic form:

Proposition 31 *Let G be a special group, let H be a group of exponent 2, and let $h \in H \setminus \{1\}$. Write $H = \{1, h\} \times H'$ for some subgroup H' of H, and define a morphism of groups $\tau : G[H] \to G[H']$ by $\tau \restriction G = Id_G$, $\tau(h) = 1$ and $\tau(h') = h'$ for every $h' \in H'$.*
Then τ is a morphism of special groups, $\tau(\langle 1, -h\rangle)$ is isotropic, and for every morphism of special groups $f : G[H] \to L$ such that $f(\langle 1, -h\rangle)$ is isotropic, there is a morphism of special groups $\theta : G[H'] \to L$ such that the following diagram is commutative:

$$\begin{array}{ccc} G[H] & \xrightarrow{\tau} & G[H'] \\ {\scriptstyle f}\downarrow & \swarrow{\scriptstyle \theta} & \\ L & & \end{array}$$

In particular, $\tau : G[H] \to G[H']$ is a GPS of $\langle 1, -h\rangle$ over $G[H]$.

PROOF: That τ is a morphism of special groups and $\tau(\langle 1, -h\rangle)$ is isotropic is clear. Assume $f : G[H] \to L$ is a morphism of special groups such that $f(\langle 1, -h\rangle)$ is isotropic, i.e. $f(h) = 1$. Define $\theta : G[H'] \to L$ by $\theta(g) = f(g)$ for every $g \in G$ and $\theta(h') = f(h')$ for every $h' \in H'$. Then $\theta \circ \tau = f$, and we only have to check that θ is a morphism of special groups: Let $x \in D_{G[H']}\langle 1, z\rangle$ with $x, z \in G[H']$. We distinguish three cases:

- $z \in G \setminus \{-1\}$. Then $x \in D_G\langle 1, z\rangle$, so $f(x) \in D_L\langle 1, f(z)\rangle$, i.e. $\theta(x) \in D_L\langle 1, \theta(z)\rangle$.
- $z = gh'$ with $g \in G$ and $h' \in H'$, $h' \neq 1$. Then $x \in \{1, gh'\}$, and necessarily $\theta(x) \in D_L\langle 1, \theta(z)\rangle$.
- $z = -1$. Then $\theta(z) = -1$ and $\theta(x) \in D_L\langle 1, \theta(z)\rangle$. □

Corollary 32 *The theory $T_{q,G}$ is not complete in general.*

PROOF: Choose G and q such that $T_{q,G}$ has a finite model K. For instance, by proposition 31, we can take $G = \mathbb{Z}_2[H]$, where H is any finite group of exponent 2, and $q = \langle 1, -h\rangle$ with $h \in H \setminus \{1\}$. Then $K \times K$ is also a model of $T_{q,G}$ by proposition 30 1., and is not elementarily equivalent to K. □

Proposition 28 proved that a GPS still remains a GPS if we take a pure restriction of its domain (that still contains the coefficients of the quadratic form). We consider now the "reverse" question: If G is a special group and q is a form over G of dimension at least 2, is there a theory whose models are GZSG of q over G, and are still GZSG of q over G', whenever G' is a pure extension of G, of "small enough" cardinality?

For a special group G and a form q of dimension at least 2 over G, we define the following $L_{SG}(G)$-theory:

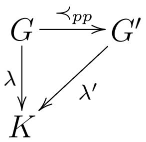
$$T^+_{q,G} = T_{q,G} \ \cup$$

$$\{\exists \bar{g}_2\ \varphi(\bar{g},\bar{g}_2) \wedge \bigwedge_{i=1}^{k} \neg\psi_i(\bar{g},\bar{g}_2) \mid \varphi,\psi_i \text{ are all pp-formulas, } \bar{g}\in G,$$
$$\text{there are } G \prec_{pp} G_2 \text{ and } \bar{g}_2 \in G_2 \text{ such that } G_2 \models \varphi(\bar{g},\bar{g}_2) \text{ and,}$$
$$\text{for all } 1\le i\le k,\ SG \cup \mathrm{pp}^{G_2}(G_2) \cup \text{“}q \text{ isotropic”} \not\vdash \psi_i(\bar{g},\bar{g}_2)\}.$$

The second part of the condition (there are...) can also be written as follows: There are $G \prec_{pp} G_2$ and $\bar{g}_2 \in G_2$ such that $G_2 \models \varphi(\bar{g},\bar{g}_2)$, and for every $1 \le i \le k$, there is a morphism of special groups $f_i : G_2 \to L_i$ such that , $L_i \models$ “$f_i(q)$ isotropic”, and $L_i \not\models \psi_i(f(\bar{g}), f_i(\bar{g}_2))$.

Proposition 33 *Let G be a special group and let q be a form over G, of dimension at least 2. Then*

(1) $T^+_{q,G}$ *has a model;*
(2) *If $K \models T^+_{q,G}$ is α^+-saturated, then for every $G \prec_{pp} G'$ with $|G'| \le \alpha$, K is a GZSG of q over G'.*
More precisely, if $\lambda : G \to K$ is the morphism of special groups induced by the interpretation in K of the constants for the elements of G, there is a morphism of special groups $\lambda' : G' \to K$, such that λ' is a GPS of q over G' and the following diagrams commutes:



Proof:

(1) Let $n \in \mathbb{N}$ and let $F_1(\bar{g}),\ldots,F_n(\bar{g}) \in T^+_{q,G} \setminus T_{q,G}$. If we find a model of $T_{q,G} \cup \{F_1(\bar{g}),\ldots,F_n(\bar{g})\}$ the result will follow by compactness. By definition of $T^+_{q,G}$, each F_i is of the form

$$F_i(\bar{g}) = \exists \bar{g}_2\ \varphi_i(\bar{g},\bar{g}_2) \wedge \bigwedge_{j=1}^{k_i} \neg\psi_{ij}(\bar{g},\bar{g}_2).$$

By choice of F_i, for every $i = 1,\ldots,n$, we have:
- a pure extension G_i of G and $\bar{g}_{2,i} \in G_i$ such that $G_i \models \varphi_i(\bar{g},\bar{g}_{2,i})$;
- for every $j = 1,\ldots,k_i$, a morphism of special groups $f_{ij} : G_i \to L_{ij}$ such that $L_{ij} \models$ “$f_{ij}(q)$ isotropic” and $L_{ij} \not\models \psi_{ij}(f_{ij}(\bar{g}), f_{ij}(\bar{g}_{2,i}))$.

Consider now, for $1 \le i \le n$, $\lambda_i : G_i \to K_i$ a GPS of q over G_i. Then we have a morphism of special groups $\theta : K_i \to L_{ij}$ such that the following diagram commutes (replacing the L_{ij} by sufficiently saturated elementary extensions, if necessary, cf. proposition 8):

$$\begin{array}{ccc} G \prec_{pp} G_i & \xrightarrow{f_{ij}} & L_{ij} \\ \downarrow \lambda_i & \nearrow \theta_{ij} & \\ K_i & & \end{array}$$

By proposition 28, λ_i is also a GPS of q over G. So $K_i \models T_{q,G}$, and $K_i \models F_i(\lambda_i(\bar{g}))$: indeed $G_i \models \varphi_i(\bar{g}, \bar{g}_{2,i})$ so $K_i \models \varphi_i(\lambda(\bar{g}), \lambda(\bar{g}_{2,i}))$, and $K_i \not\models \psi_{ij}(\lambda_i(\bar{g}), \lambda_i(\bar{g}_{2,i}))$, since otherwise, applying θ_{ij}, we would get $L_i \models \psi_{ij}(f_{ij}(\bar{g}), f_{ij}(\bar{g}_{2,i}))$, a contradiction.

Let $K := \prod_{i=1}^n K_i$ and

$$\begin{aligned} \lambda : \quad \prod_{i=1}^n G_i \quad &\rightarrow K \\ (x_1, \ldots, x_n) &\mapsto (\lambda_1(x_1), \ldots, \lambda_n(x_n)). \end{aligned}$$

We claim that $K \models T_{q,G} \cup \{F_1(\lambda(\bar{g})), \ldots, F_n(\lambda(\bar{g}))\}$ (we see G as a subset of $\prod_{i=1}^n G_i$ via the diagonal inclusion). It is clear that $K \models T_{q,G}$ since the class of models of $T_{q,G}$ is closed under products (see proposition 30 1.). Consider now, for instance, the formula $F_1(\lambda(\bar{g})) = \exists \bar{a}\ \varphi_1(\lambda(\bar{g}), \bar{a}) \wedge \bigwedge_{j=1}^{k_1} \neg\psi_{1j}(\lambda(\bar{g}), \bar{a})$. Let $p_l : K \rightarrow K_l$ be the canonical projection, for $l = 1, \ldots, n$. We find $\bar{a} \in K$ such that $K \models \varphi_1(\lambda(\bar{g}), \bar{a}) \wedge \bigwedge_{j=1}^{k_1} \neg\psi_{1j}(\lambda(\bar{g}), \bar{a})$ by determining the values of $p_1(\bar{a}), \ldots, p_n(\bar{a})$. We know that $K_1 \models F_1(\lambda_1(\bar{g}))$ so there is $\bar{a}_1 \in K_1$ such that $K_1 \models \varphi_1(\lambda_1(\bar{g}), \bar{a}_1) \wedge \bigwedge_{j=1}^{k_1} \neg\psi_{1j}(\lambda_1(\bar{g}), \bar{a}_1)$. We take $p_1(\bar{a}) = \bar{a}_1$. Observe now that, with this choice of $p_1(\bar{a})$, for any values of $p_2(\bar{a}), \ldots, p_n(\bar{a})$ we have $K \models \bigwedge_{j=1}^{k_1} \neg\psi_{1j}(\lambda(\bar{g}), \bar{a})$ (since $K_1 \models \bigwedge_{j=1}^{k_1} \neg\psi_{1j}(\lambda_1(\bar{g}), \bar{a}_1)$).
Consider $i \in \{2, \ldots, n\}$. We have $G_1 \models \varphi_1(\bar{g}, \bar{g}_{2,1})$, so $G_1 \models \exists \bar{x}\ \varphi_1(\bar{g}, \bar{x})$. Since every G_k is pure over G, it follows that $G_i \models \exists \bar{x}\ \varphi_1(\bar{g}, \bar{x})$ and we get $K_i \models \exists \bar{x}\ \varphi_1(\lambda_i(\bar{g}), \bar{x})$. We define $p_i(\bar{a})$ to be such an $\bar{x}$ in K_i, and we have $K \models \varphi_1(\lambda(\bar{g}), \bar{a})$

(2) We have to show that $K \models T_{q,G'}$, where the interpretation of the (constants corresponding to the) elements of G is given by λ. Since K is α^+-saturated and $|G'| \leq \alpha$, it is enough to show that every finite subset of $T_{q,G'}$ can be satisfied in K. Let $SG \cup$ "q isotropic" $\cup \{\varphi(\bar{g}, \bar{g}_2)\} \cup \{\neg\psi_1(\bar{g}, \bar{g}_2), \ldots, \neg\psi_n(\bar{g}, \bar{g}_2)\}$ be such a finite subset, with $\bar{g} \in G$, $\bar{g}_2 \in G' \setminus G$, $\varphi(\bar{g}, \bar{g}_2) \in \mathrm{pp}^{G_2}(G_2)$, and for every $1 \leq i \leq n$: ψ_i is a pp-formula such that $SG \cup \mathrm{pp}^{G_2}(G_2) \cup$ "q isotropic" $\not\vdash \psi_i(\bar{g}, \bar{g}_2)$. It is clear that $K \models SG \cup$ "q isotropic". We want K to be a model of $\exists \bar{g}_2\ \varphi(\bar{g}, \bar{g}_2) \wedge \bigwedge_{i=1}^n \neg\psi_i(\bar{g}, \bar{g}_2)$. But this formula is in $T_{q,G}^+$, so is satisfied in K, where $\bar{g}$ is interpreted by $\lambda(\bar{g})$. □

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