Some model-theoretic results in the algebraic theory of quadratic forms

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1 Introduction

The theory of Witt rings of fields has led to different axiomatizations of quadratic form theory, for example abstract Witt rings, quadratic form schemes, quaternionic structures; each of these axiomatizations highlighting a particular point of view.

Recently, Dickmann and Miraglia have introduced a new axiomatization, the theory of special groups, which they have developed to get new results: see [DM], and [DM_2] in which Marshall's and Lam's conjectures are proven.
The category of special groups (with its morphisms) is naturally isomorphic to that of abstract Witt rings. Moreover, the theory of special groups is axiomatized by a finite set of formulae in a first-order language, and it is thus natural to look at it form the point of view of model theory.
This is the subject matter of this paper, in which we present some model-theoretic results concerning special groups of finite type (see definition 4.3):
We recall Feferman and Vaught's notion of generalized product and prove some results about them, and we introduce briefly the main concepts concerning special groups (sections 2 and 3).
In section 4, we interpret the operation of extension as a generalized product. Using this we characterize in section 6 the first-order theories of special groups of finite type by means of finite trees (proposition 6.2), and get some consequences concerning categoricity and saturation.
These results lead us to the problems of model-completeness and quantifier elimiation for reduced special groups of finite type: this is the content of sections 7 and 8, in which we characterize the elementary monomorphisms between such special groups (theorem 7.4), and we introduce an extended language in which they admit quantifier elimination (theorem 8.3).
We conclude this paper by the explicit computation of the (finite) Morley rank of special groups of finite type (section 9). [1]
Throughout this paper, the symbols ⧫ and ◊ will denote, respectively, the end of a proof, and the end of a proof within another.

2 Generalized products

As the model-theoretic notions used here are standard (except for the fact that we will use the same notation to denote a first-order structure and its underlying

[1] These results are part of a PhD thesis written under the direction of M. Dickmann. I would like to express here my gratitude to him for his many advice and remarks.

set), we only recall the notion of generalized product which appears in [FV]. For a given collection of first-order structures $\{A_i\}_{i\in I}$ of language L, this construction explicitly produces new first-order structures (in various languages), the usual product being a particular case.

Let $\{A_i\}_{i\in I}$ be a set of L-structures indexed by I and let S be the boolean algebra $< \mathcal{P}(I); \emptyset, \cup, \cap, ^-, \subseteq, R_1, \ldots, R_n, \ldots >$ of subsets of I, equipped with new relations R_i. We denote by $L_S = \{\emptyset, \cup, \cap, ^-, \subseteq, R_1, \ldots, R_n, \ldots\}$ the language of S. The underlying set of every generalized product of the A_i is the product $A = \prod_{i\in I} A_i$, the relations on it being the only objects that can be chosen more freely. We denote by $\bar{f}$ a tuple of elements in $\prod_{i\in I} A_i$, and by $\bar{f}_i$ the tuple in A_i consisting of the i-th coordinates of $\bar{f}$.
If θ is a L-formula, we write:

$$[\![\theta(\bar{f})]\!] = \{\ i \in I \mid A_i \models \theta(\bar{f}_i)\ \},$$

and for any L_S-formula Φ with m free variables, and any L-formulas $\theta_1, \cdots, \theta_m$, the sequence:

$$\zeta =< \Phi, \theta_1, \cdots, \theta_m >$$

is called an **acceptable sequence**. For an acceptable sequence ζ, we define:

$$Q_\zeta = \{\ \bar{f} = (\bar{f}_1, \ldots, \bar{f}_p) \in A^p \mid S \models \Phi([\![\theta_1(\bar{f})]\!], \cdots, [\![\theta_m(\bar{f})]\!])\ \}, \qquad (\clubsuit)$$

and the **generalized product of the A_i relative to the algebra S** is the structure:

$$< A; \{Q_\zeta \mid \zeta \text{ acceptable sequence}\} >$$

(we will also call generalized product every structure $< A; \{Q_\zeta\}_{\zeta\in\Delta} >$, where only a particular set of acceptable sequences has been chosen).
The results that we will use concerning generalized products are summarized in the following theorem:

Theorem 2.1 ([FV], theorem 5.1 and theorem 5.2, and [D], corollary 4.5.3, p. 361) *Generalized products preserve elementary equivalence, $L_{\infty\lambda}$-equivalence for every cardinal λ, and elementary extensions. More precisely:*
If $A_i \equiv B_i$ (respectively $A_i \equiv_{\infty\lambda} B_i$, $A_i \prec B_i$) for all $i \in I$, the generalized product of the A_i is elementarily equivalent to the generalized product of the B_i (respectively, $L_{\infty\lambda}$-equivalent, is an elementary substructure).

Now we prove some results about generalized products:

Lemma 2.2 *Let $M_1, \ldots, M_n$ be L-structures (n is finite), and N be a generalized product of $M_1, \ldots, M_n$ relative to the algebra S. Denote by $\tilde{L}$ the language of N. Then, for any $\tilde{L}$-formula $\varphi(\bar{x})$ there exist L-formulas $\theta_1(\bar{x}), \ldots, \theta_k(\bar{x})$, and a propositional formula F, all of these depending only on φ, such that, for all $\bar{a} = (\bar{a}_1, \ldots, \bar{a}_n) \in N$:*

$$N \models \varphi(\bar{a}) \;\Leftrightarrow$$
$$\models F(M_1 \models \theta_1(\bar{a}_1), \cdots, M_n \models \theta_1(\bar{a}_n), \cdots, M_1 \models \theta_k(\bar{a}_1), \cdots, M_n \models \theta_k(\bar{a}_n)),$$

where $M_i \models \theta_j(\bar{a}_i)$ denotes the truth value of $\theta_j(\bar{a}_i)$ in M_i.

Proof: Let $\varphi(\bar{x})$ be a $\tilde{L}$-formula. By [FV], theorem 3.1, there exists a L_S-formula Φ, and L-formulas $\theta_1, \cdots, \theta_k$, depending only on φ, such that, for all $\bar{a} \in N$:

$$N \models \varphi(\bar{a}) \Leftrightarrow S \models \Phi([\![\theta_1(\bar{a})]\!], \cdots, [\![\theta_k(\bar{a})]\!]).$$

But since the underlying set of S, $\mathcal{P}(\{1, \cdots, n\})$, is finite, so is $\Phi(S)$, the set of tuples of S satisfying Φ:

$$\Phi(S) = \{A_1, \cdots, A_l\}, \text{ with } A_1, \cdots, A_l \in \mathcal{P}(\{1, \cdots, n\})^k.$$

This gives:

$$\begin{aligned}
&S \models \Phi([\![\theta_1(\bar{a})]\!], \cdots, [\![\theta_k(\bar{a})]\!]) \Leftrightarrow ([\![\theta_1(\bar{a})]\!], \cdots, [\![\theta_k(\bar{a})]\!]) \in \{A_1, \cdots, A_l\} \\
&\quad \Leftrightarrow ([\![\theta_1(\bar{a})]\!], \cdots, [\![\theta_k(\bar{a})]\!]) = A_1 \vee \cdots \vee ([\![\theta_1(\bar{a})]\!], \cdots, [\![\theta_k(\bar{a})]\!]) = A_l \\
&\quad \Leftrightarrow ([\![\theta_1(\bar{a})]\!] = A_{1,1} \wedge \cdots \wedge [\![\theta_k(\bar{a})]\!] = A_{1,k}) \vee \cdots \\
&\qquad \cdots \vee ([\![\theta_1(\bar{a})]\!] = A_{l,1} \wedge \cdots \wedge [\![\theta_k(\bar{a})]\!] = A_{l,k}), \qquad (\star)
\end{aligned}$$

where $A_j = (A_{j,1}, \cdots, A_{j,k})$ for $j = 1, \cdots, l$. But for fixed θ and A:

$$[\![\theta(\bar{a})]\!] = A \Leftrightarrow \bigwedge_{i \in A} M_i \models \theta(\bar{a}_i) \wedge \bigwedge_{i \notin A} M_i \not\models \theta(\bar{a}_i).$$

Using this in $(\star)$, we get the formula F and the truth values of $\theta_j(\bar{a}_i)$ in M_i. Moreover, since Φ and $\theta_1, \cdots, \theta_k$ depend only on φ, so does F. ♦

Corollary 2.3 *Generalized products of a finite number of structures preserve λ-saturation, for any cardinal λ.*

Proof: Let $M_1, \ldots, M_n$ be λ-saturated L-structures, and N be a $\tilde{L}$-structure that is a generalized product of $M_1, \ldots, M_n$, relative to the algebra S. After replacing functions by their graphs, if necessary, we may suppose that L does not contain any function symbol.

Let L^+ be the language on N containing all possible relations given by the generalized product of $M_1, \cdots, M_n$, together with the projections $p_1, \cdots, p_n$ from N to $M_1, \cdots, M_n$ (to do this we have to identify M_i with the subset $c_1 \times \cdots \times c_{i-1} \times M_i \times c_{i+1} \times \cdots \times c_n$ of N, where the c_j are arbitrary (but fixed) elements of M_j, $j = 1, \cdots, n$). Then:

Fact 2.4 *If $i \in \{1, \cdots, n\}$, then M_i is definable as a L-structure in the L^+-structure N.*

The proof of the fact is a straightforward verification. ◊

Let N' be a λ-saturated elementary extension of N in the language L^+, and $M_i' = Im(P_i^{N'})$. Since $N \prec N'$, we have $N' = M_1' \times \cdots \times M_n'$ (as sets), and, using fact 2.4:

$$M_i \prec M_i' \text{ in the language } L, \text{ for } i = 1, \cdots, n.$$

As $\tilde{L} \subseteq L^+$, N' is naturally a $\tilde{L}$-structure. Lemma 2.2 gives that, for any $\tilde{L}$-formula φ there exist L-formulas $\theta_1, \cdots, \theta_k$ and a propositional formula F^φ, depending only on φ, such that for any $\bar{a} \in N$:

$$\begin{gathered} N \models \varphi(\bar{a}) \;\Leftrightarrow \\ \models F^\varphi(M_1 \models \theta_1(\bar{a}_1), \cdots, M_n \models \theta_1(\bar{a}_n), \cdots, M_1 \models \theta_k(\bar{a}_1), \cdots, M_n \models \theta_k(\bar{a}_n)). \end{gathered}$$

We get:

Fact 2.5 *N' is obtained from $M'_1, \cdots, M'_n$ in the same way that N is obtained from $M_1, \cdots, M_n$,* i.e., *for every $\bar{a} \in N'$:*

$$\begin{gathered} N' \models \varphi(\bar{a}) \\ \Leftrightarrow \\ \models F^\varphi(M'_1 \models \theta_1(\bar{a}_1), \cdots, M'_n \models \theta_1(\bar{a}_n), \cdots, M'_1 \models \theta_k(\bar{a}_1), \cdots, M'_n \models \theta_k(\bar{a}_n)). \end{gathered}$$

Proof: Routine verification, since the M'_i are definable in N' by the same L^+-formulas that define the M_i in N. $\diamondsuit$

Now we show the λ-saturation of N: Let $A \subseteq N$ with $card(A) < \lambda$, and let $p \in S_1(A)$ be a $\tilde{L}$-type in N. Since N' is λ-saturated, p is realized in N' by $(a_1, \cdots, a_n)$, with $a_i \in M'_i$.
Let $\varphi(x, \bar{\alpha}) \in p$ ($\bar{\alpha} \in A$ are the parameters). Lemma 2.2 applied to N gives a propositional formula G^φ and L-formulas $\theta_1^\varphi, \cdots, \theta_{l_\varphi}^\varphi$, depending only on φ, such that for all $x, \bar{y} \in N$,

$$\begin{gathered} N \models \varphi(x, \bar{y}) \;\Leftrightarrow \\ \models G^\varphi(M_1 \models \theta_1^\varphi(x_1, \bar{y}_1), \cdots, M_n \models \theta_1^\varphi(x_n, \bar{y}_n), \cdots, \\ M_1 \models \theta_{l_\varphi}^\varphi(x_1, \bar{y}_1), \cdots, M_n \models \theta_{l_\varphi}^\varphi(x_n, \bar{y}_n)); \end{gathered} \qquad (\clubsuit)$$

using fact 2.5 and assigning the values a to x and $\bar{\alpha}$ to $\bar{y}$ we get:

$$\begin{gathered} N' \models \varphi(a, \bar{\alpha}) \;\Leftrightarrow \\ \models G^\varphi(M'_1 \models \theta_1^\varphi(a_1, \bar{\alpha}_1), \cdots, M'_n \models \theta_1^\varphi(a_n, \bar{\alpha}_n), \cdots, \\ M'_1 \models \theta_{l_\varphi}^\varphi(a_1, \bar{\alpha}_1), \cdots, M'_n \models \theta_{l_\varphi}^\varphi(a_n, \bar{\alpha}_n)). \end{gathered}$$

Looking at the truth table of G^φ shows that this is equivalent to a formula of the form:

$$\begin{gathered} (M'_1 \models \varepsilon_{1,1}^\varphi \theta_1^\varphi(a_1, \bar{\alpha}_1) \;\wedge \cdots \wedge\; M'_n \models \varepsilon_{1,n}^\varphi \theta_{l_\varphi}^\varphi(a_n, \bar{\alpha}_n)) \;\vee \cdots \\ \vee\; (M'_1 \models \varepsilon_{k,1}^\varphi \theta_1^\varphi(a_1, \bar{\alpha}_1) \;\wedge \cdots \wedge\; M'_n \models \varepsilon_{k,n}^\varphi \theta_{l_\varphi}^\varphi(a_n, \bar{\alpha}_n)), \end{gathered} \qquad (\star)$$

where the $\varepsilon_{i,j}^\varphi$ are the empty symbol or $\neg$.
For example, we have (as $(\star)$ is a disjunction, and $a, \bar{\alpha}$ are fixed):

$$N' \models \varphi(a, \bar{\alpha}) \Leftrightarrow\; M'_1 \models \varepsilon_{1,1}^\varphi \theta_1^\varphi(a_1, \bar{\alpha}_1) \;\wedge \cdots \wedge\; M'_n \models \varepsilon_{1,n}^\varphi \theta_{l_\varphi}^\varphi(a_n, \bar{\alpha}_n),$$

i.e.:

$$\begin{gathered} N' \models \varphi(a, \bar{\alpha}) \;\Leftrightarrow \\ \varepsilon_{1,1}^\varphi \theta_1^\varphi(x, \bar{\alpha}_1) \in tp^{M'_1}(\bar{a}_1/p_1(A)), \cdots, \varepsilon_{1,n}^\varphi \theta_{l_\varphi}^\varphi(x, \bar{\alpha}_n) \in tp^{M'_n}(\bar{a}_n/p_n(A)). \end{gathered}$$

Applying this to every formula $\varphi \in p$, we get:

$$\begin{gathered} a \text{ is a realization of } p \Leftrightarrow \\ \{\varepsilon_{1,1}^\varphi \theta_1^\varphi(x, \bar{\alpha}_1\}_{\varphi \in p} \subseteq tp^{M'_1}(\bar{a}_1/p_1(A)), \cdots, \{\varepsilon_{1,n}^\varphi \theta_{l_\varphi}^\varphi(x, \bar{\alpha}_n)\}_{\varphi \in p} \subseteq tp^{M'_n}(\bar{a}_n/p_n(A)). \end{gathered}$$

Since the M_i are λ-saturated the types $tp^{M'_1}(a_1/p_1(A)), \cdots, tp^{M'_n}(a_n/p_n(A))$ are realized in $M_1, \cdots, M_n$, say by $b_1, \cdots, b_n$.
Unravelling the preceding equivalences, using $(\clubsuit)$, we obtain that $(b_1, \cdots, b_n)$ is a realization of p in N, which is therefore λ-saturated. $\blacklozenge$

The following will be of later use:

Lemma 2.6 *Let M be a generalized product of the L-structures $M_1, \cdots, M_n$, and $p_1, \cdots, p_n$ be the projections from $M = M_1 \times \cdots \times M_n$ onto $M_1, \cdots, M_n$, respectively. Then, for any $A \subset M$ and any $\bar{a} \in M$, $tp^M(\bar{a}/A)$ is determined by:*

$$tp^{M_1}(p_1(\bar{a}), p_1(A)), \cdots, tp^{M_n}(p_n(\bar{a}), p_n(A)).$$

Proof: Let $\bar{\alpha}, \bar{\beta} \in M$ be such that:

$$tp^{M_i}(p_1(\bar{\alpha}), p_i(A)) = tp^{M_i}(p_1(\bar{\beta}), p_i(A)), \text{ for all } i = 1, \cdots, n.$$

We will show that $tp^M(\bar{\alpha}/A) = tp^M(\bar{\beta}/A)$.

Let $\varphi(\bar{x}, \bar{a}) \in tp^M(\bar{\alpha}/A)$, with $\bar{a} \in A$. By lemma 2.2, there exist L-formulas $\theta_1(\bar{x}, \bar{y}), \ldots, \theta_k(\bar{x}, \bar{y})$, and a propositional formula F, depending only on φ, such that, for all $\bar{u}, \bar{v} \in M$:

$$\begin{gathered} M \models \varphi(\bar{u}, \bar{v}) \ \Leftrightarrow \\ \models F(M_1 \models \theta_1(p_1(\bar{u}), p_1(\bar{v})), \cdots, M_n \models \theta_1(p_n(\bar{u}), p_n(\bar{v})), \cdots \\ \cdots, M_1 \models \theta_k(p_1(\bar{u}), p_1(\bar{v})), \cdots, M_n \models \theta_k(p_n(\bar{u}), p_n(\bar{v}))). \end{gathered}$$

Since $M \models \varphi(\bar{\alpha}, \bar{a})$, we have:

$$\begin{gathered} \models F(M_1 \models \theta_1(p_1(\bar{\alpha}), p_1(\bar{a})), \cdots, M_n \models \theta_1(p_n(\bar{\alpha}), p_n(\bar{a})), \cdots \\ \cdots, M_1 \models \theta_k(p_1(\bar{\alpha}), p_1(\bar{a})), \cdots, M_n \models \theta_k(p_n(\bar{\alpha}), p_n(\bar{a}))), \end{gathered}$$

and the hypothesis gives:

$$\begin{gathered} \models F(M_1 \models \theta_1(p_1(\bar{\beta}), p_1(\bar{a})), \cdots, M_n \models \theta_1(p_n(\bar{\beta}), p_n(\bar{a})), \cdots \\ \cdots, M_1 \models \theta_k(p_1(\bar{\beta}), p_1(\bar{a})), \cdots, M_n \models \theta_k(p_n(\bar{\beta}), p_n(\bar{a}))), \end{gathered}$$

which implies $\varphi(\bar{x}, \bar{a}) \in tp^M(\bar{\beta}/A)$. ♦

3 Special groups

As mentioned in the introduction, the relevant reference concerning special groups is [DM], but in order to keep this paper as self-contained as possible, we recall some basic facts:

Definition 3.1 *A special group G is a group of exponent 2 (written multiplicatively) with a distinguished element -1 and a binary relation $\equiv$ on G^2, which verify the following axioms:*

SG_0) *$\equiv$ is an equivalence relation.*

SG_1) $\forall a, b\ (a, b) \equiv (b, a)$
(for $a, b \in G$, we denote by (a, b) the associated element of G^2).

SG_2) $\forall a\ (a, -a) \equiv (1, -1)$, *with* $-a = -1.a$.

SG_3) $\forall a, b, c, d\ (a, b) \equiv (c, d) \Rightarrow ab = cd$.

SG_4) $\forall a, b, c, d\ (a, b) \equiv (c, d) \Rightarrow (a, -c) \equiv (-b, d)$.

SG_5) $\forall a, b, c, d, x\ (a, b) \equiv (c, d) \Rightarrow (ax, bx) \equiv (cx, dx)$.

SG_6) **(3-transitivity)** $\forall a_1, a_2, a_3\ \forall b_1, b_2, b_3\ \forall c_1, c_2, c_3$
$(< a_1, a_2, a_3 > \equiv < b_1, b_2, b_3 > \wedge < b_1, b_2, b_3 > \equiv < c_1, c_2, c_3 >)$
$\Rightarrow < a_1, a_2, a_3 > \equiv < c_1, c_2, c_3 >$.
(The relation $\equiv$ between triples of elements of G is defined below.)

A **quadratic form of dimension n** *over G is a n-tuple $< a_1, \cdots, a_n >$ of elements of G.*

If F is a field of characteristic different from 2, and if we interpret G as $\boldsymbol{G(F)}= \dot{F}/\dot{F}^2$ and $\equiv$ as the isometry relation between quadratic forms of dimension 2, then the axioms $SG_0, \cdots, SG_6$ are satisfied (if we consider only quadratic forms in diagonal form, represented by the tuples of elements on the diagonal).
In this context, the meaning of SG_6 is clear: it asserts that isometry between quadratic forms of dimension 3 is transitive. In the special group context, we have to define isometry between quadratic forms of dimension greater than 2. This is done by the following definition, inspired by the inductive description of isometry in the field case (see [M_1], theorem 1.13, p. 16):

Definition 3.2 *We define a relation (still denoted by $\equiv$) between two n-tuples of elements of G, by induction on n:*

- $< a_1 > \equiv < b_1 > \Longleftrightarrow a_1 = b_1$.
- $< a_1, a_2 > \equiv < b_1, b_2 > \Longleftrightarrow (a_1, a_2) \equiv (b_1, b_2)$.
- $< a_1, \cdots, a_n > \equiv < b_1, \cdots, b_n > \Longleftrightarrow \exists x, y, z_3, \cdots, z_n$ *such that*
  $< a_1, x > \equiv < b_1, y > \wedge < a_2, \cdots, a_n > \equiv < x, z_3, \cdots, z_n > \wedge$
  $< b_2, \cdots, b_n > \equiv < y, z_3, \cdots, z_n >$.

Thus, a special group is a first-order structure in the language $\boldsymbol{L_{SG} = \{1, -1, ., \equiv\}}$. A morphism (respectively monomorphism, isomorphism) of special groups (SG-morphism for short) is just a L_{SG}-morphism (respectively monomorphism, isomorphism), in the usual model-theoretic sense.
With this notion of morphism, we can form the category of special groups, which turns out to be naturally isomorphic to that of abstract Witt rings, by a covariant functor (see [D_2]).

For a special group G and a quadratic form $< a_1, \cdots, a_n >$ over G, we define the set **of elements represented by** $< a_1, \cdots, a_n >$:

$$\mathbf{D_G} < \mathbf{a_1}, \cdots, \mathbf{a_n} > = \{b \in G \mid \quad \begin{array}{l} \exists b_2, \cdots, b_n \in G \\ < a_1, \cdots, a_n > \equiv < b, b_2, \cdots, b_n > \} \end{array}$$

(in the field case, this coincides with the set (modulo squares) of non-zero values of the quadratic form $< a_1, \cdots, a_n >$; see [M_1], corollary 1.5, p. 10).

Remark: For binary forms, isometry and representation are definable in terms of each other by quantifier-free positive formulas:

$$< a, b > \equiv < c, d > \Leftrightarrow (ab = cd \wedge ac \in D_G < 1, cd >),$$

$$a \in D_G < 1, b > \Leftrightarrow < a, ab > \equiv < 1, b > .$$

We will thus indistinctly work with isometry between forms of dimension 2, or with representation by binary forms of type $< 1, b >$.

A special group G verifying $-1_G \neq 1_G$ and $D_G < 1, 1 >= \{1\}$ is called **reduced**. A special group of the form $G(K)$, K a field, is reduced if and only if K is a Pythagorean field (every sum of squares is a square). The category of reduced special groups is isomorphic to the category of reduced abstract Witt rings (by restricting the covariant functor mentioned above), and to the category of abstract order spaces (by a contravariant functor; see [L] chapter 1, section 6, or [DM] chapter 3). A L_{SG}-substructure which satisfies the axioms of special groups is called a **special subgroup**.

Examples:

- If G is a group of exponent 2 with a distinguished element -1, the following relations between pairs of elements of G^2 define a special group structure on G:
  - The **trivial isometry**:

    $$< a, b > \equiv_t < c, d > \text{ if and only if } ab = cd.$$

    This isometry contains any other isometry relation on G.
  - The **fan isometry**, if $-1 \neq 1$, which is more easily described in terms of representation:

    $$D_{fan} < 1, a > = \begin{cases} G & \text{if } a = -1 \\ \{1, a\} & \text{otherwise} \end{cases}.$$

    A special group with the fan relation is always reduced. This isometry (denoted by $\equiv_{fan}$) is contained in any other isometry relation on G.
- If $\{-1, 1\}$ is the 2-element group of exponent 2, there is a unique isometry which turns it into a reduced special group. The resulting special group is that of any real closed field, and is denoted by $\mathbb{Z}_2$.

Unfortunately, there is no general theory of quotients for special groups. One simple example is given by the identity $f : (G, \equiv_{fan}) \longrightarrow (G, \equiv_t)$, which is a surjective SG-morphism, but $(G, \equiv_t)$ is not a quotient of $(G, \equiv_{fan})$.
There are nevertheless two cases in which a notion of quotient exists:

1. Quotients by Pfister subgroups (see [DM], chapter 2). This is the most important (and the most used) notion of a quotient, but it will not be used in this paper.
2. In this second case, the quotient is described as follows:

   Lemma 3.3 *Let G be a special group and K be a subgroup of G. Assume there is a group homomorphism $f : G \to K$ such that $f \restriction K = id_K$ and, for $a, b, c, d \in G$:*

   $$< a, b > \equiv_G < c, d > \implies < f(a), f(b) > \equiv_G < f(c), f(d) > .$$

   Then, the L_{SG}-structure $< K; \equiv_G \restriction K, f(-1_G) >$ is a special group and f is a special group morphism.
   Furthermore, f induces an isomorphism between this structure and $< G/ker(f); \equiv^*, -1_G/ker(f) >$, where, for $a, b, c, d \in G$:

   $< a/ker(f), b/ker(f) > \equiv^* < c/ker(f), d/ker(f) >$
   if and only if there are $a', b', c', d' \in G$ such that $aa', bb', cc', dd' \in ker(f)$ and $< a', b' > \equiv_G < c', d' >$.

   In particular, $< G/ker(f); \equiv^*, -1_G/ker(f) >$ is a special group.

Proof: Straightforward verification ♦

This isometry on K will be refered to as **the retract isometry with respect to f**.

We shall apply lemma 3.3 to the following situation:
Let $D_G = \bigcap_{g\in G} D_G < 1, g >$. Routine verification shows that the isometry $\equiv_G\restriction D_G$ is trivial. Hence, choosing -1_{D_G} to be any element of D_G, the L_{SG}-structure $< D_G; \equiv_G\restriction D_G, -1_{D_G} >$ is a special group, which is called **the trivial part of G**. Now, let $\bar{G}$ be a complement of D_G in G (as $\mathbb{F}_2$-vector spaces), and let $f : G \to \bar{G}$ be the projection onto $\bar{G}$. With $K = \bar{G}$, lemma 3.3 implies:

Fact 3.4 *With notation as above, let $-1_{\bar{G}}$ be the unique element of $\bar{G}$ such that $-1_G = -1_{D_G}.-1_{\bar{G}}$. Then, $< \bar{G}; \equiv_G\restriction \bar{G}, -1_{\bar{G}} >$ is a special group. Furthermore, $G \cong \bar{G} \times D_G$ as special groups.*

4 Constructions

There are two constructions which give new special groups: the product, which is the usual product of first-order structures, and the extension, which we now describe:

Definition 4.1 *If G is a special group, and H is a group of exponent 2, the group $G \times H$ can be turned into a special group (see [L], definition 1.10.1 and theorem 1.10.2, p. 93), written **$G[H]$**, by taking $-1_{G[H]} = -1_G \times 1_H$, and representation given by:*

$$D_{G[H]} < 1, gh >= \begin{cases} \{1, gh\} & \text{if } h \neq 1 \\ D_G < 1, g > \times\{1\} & \text{if } h = 1 \text{ and } g \neq -1 \\ G \times H & \text{if } h = 1 \text{ and } g = -1 \end{cases}.$$

This special group is called the **extension** *of G by H, and we will sometimes denote the elements of $G[H]$ by $g[h]$ rather than gh.*

The product is obviously a generalized product, and we can show that the extension is one too. Here there is a little problem: the definition of generalized product requires all factors to be of the same similarity type, but G is in the language L_{SG} and H is in the language $\{., 1\}$ of groups. In order to use the definition of generalized product we have to add a relation on H^4 and a constant on H, which may be taken arbitrarily (they do not appear in the expression of the generalized product):

Proposition 4.2 *Let G be a special group, and H be a group of exponent 2. Then $G[H]$ is a generalized product of the L_{SG}-structures G and H, where H is equipped with a new distinguished element, and a new quaternary relation (this relation and this element may be chosen freely).*

Proof: For convenience of notation, we set $G_0 = G$ and $G_1 = H$, as L_{SG}-structures, and use notation as in the beginning of section 2.
$S =< \mathcal{P}(\{0, 1\}); \emptyset, \cup, \cap, ^-, \subseteq>$ is the boolean algebra relatively to which we will express the generalized product.
Since $-1_{G[H]} = (-1_G, 1_H)$, it suffices to show that $R^{G[H]}$, the isometry relation seen as a subset of $G[H]^4$, is a relation of the generalized product. We have:

$$
\begin{aligned}
&< g_1h_1, g_2h_2 >\equiv_{G[H]}< g_3h_3, g_4h_4 >\Longleftrightarrow \\
&\quad (h_1 = h_2 \wedge h_3 = h_4 \wedge g_1 = -g_2 \wedge g_3 = -g_4) \vee \\
&\quad (h_1 = h_3 \wedge h_2 = h_4 \wedge g_1 = g_3 \wedge g_2 = g_4) \vee \\
&\quad (h_1 = h_4 \wedge h_2 = h_3 \wedge g_1 = g_4 \wedge g_2 = g_3) \vee \\
&\quad (h_1 = h_2 = h_3 = h_4 \wedge < g_1, g_2 >\equiv_G< g_3, g_4 >).
\end{aligned}
$$

$R^{G[H]}$ is thus the union of 4 subsets $A_1, \cdots, A_4$ of $G[H]^4$, and it is enough to show that each of these subsets is a relation in the generalized product. We check this for the first subset:

$$A_1 = \{(g_1h_1, \ldots, g_4h_4) \mid h_1 = h_2 \wedge h_3 = h_4 \wedge g_1 = -g_2 \wedge g_3 = -g_4)\}.$$

We have $A_1 = Q_{\zeta_1}$, where:

$$
\begin{aligned}
&\zeta_1 =< \Phi_1, \theta_1, \theta_1' >, \Phi_1(X_1, X_2) \sim X_1 \ni 1 \wedge X_2 \ni 0, \\
&\theta_1(x_1, \ldots, x_4) \sim x_1 = x_2 \wedge x_3 = x_4, \\
&\theta_1'(x_1, \ldots, x_4) \sim x_1 = -x_2 \wedge x_3 = -x_4.
\end{aligned}
$$

The verification is similar for the other 3 subsets, and is done without using the additional structure on H, which may then be chosen freely. ⧫

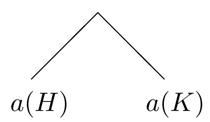
A special group is said to be **indecomposable** if it is not the product of two non-trivial (*i.e.* different from $\{1\}$) special groups. Otherwise it is said to be **decomposable**.

We close this section by introducing the notion of special group of finite type. It generalizes (and dualizes) the notion of abstract order space of finite type which appears in [ABR], chapter IV, 3.

Definition 4.3 *A special group of* **finite type** *is a special group built from finite special groups, using a finite number of times the operations of product and extension.*

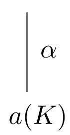
As in [ABR] (for abstract order spaces of finite type), to each special group of finite type, G, we associate a **tree**, denoted by $a(G)$, which describes its construction:

- The tree of a special group G which is neither an extension, nor a product, consists in one leaf: G.
- If H, K are special groups of finite type, the tree of $G = H \times K$ is:



.

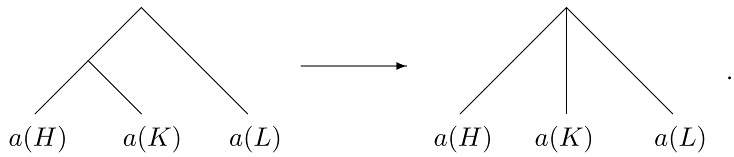
- The tree of an extension $G = K[H]$ is:



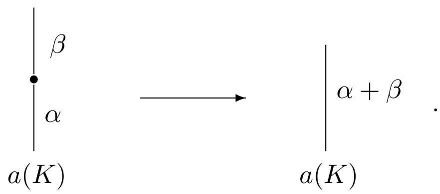
, where $\alpha = dim_{\mathbb{F}_2} H \geq 1$.

Trees are identified according to the following rules:

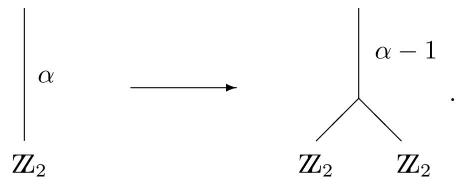
- Successive products as in the left diagram are denoted as in the one on the right (associativity of product):



- Successive extensions are reduced to one:



- Extensions from $\mathbb{Z}_2$ are not allowed. They are replaced as follows:



  This is possible because the special groups represented by these trees are isomorphic.

Modulo identification by the rules above, the tree of a special group of finite type is unique (see [M_1], theorem 5.23, p. 120, or, for the reduced case, [ABR], chapter IV, sections 3,4, and theorem 5.1).

The **pruned tree** of G is the tree of G with the following modification: each cardinal α labelling a vertical edge (*i.e.* corresponding to an extension) is replaced by $min(\alpha, \aleph_0)$.

We have

Corollary 4.4 *If G is a special group of finite type, then G is ω-stable and ω-categorical, and hence of finite Morley rank.*

Proof: Since G is of finite type, it is built in a finite number of steps, starting from finite special groups (which are ω-categorical and ω-stable) and using the following two operations: product of two special groups, extension of a special group by a group of exponent two. It follows from proposition 4.2 that these two operations are generalized products in the sense of [FV]. Using lemma 2.6, we see that a generalized product of a finite number of structures preserves ω-categoricity and ω-stability, and the result follows. ⧫

In particular, any special group of finite type, being stable, is of finite **chain length** (by [P_2], proposition 1.6 p. 23, or [H], theorem 5.7.2 p. 249), where the chain length is:

- The largest integer n such that there exist $a_0, \cdots, a_n$ such that:

$$D_G < 1, a_i > \subsetneqq D_G < 1, a_{i+1} >, \quad i = 0, \cdots, n-1,$$

if this integer exists.

- ∞ otherwise.

Moreover, using results of Marshall in [M_2], and the duality between reduced special groups and abstract order spaces that appears in [L], chapter 1, section 6, we know that the reduced special groups of finite chain length are the special groups of finite type that are built up starting out with the special group $\mathbb{Z}_2$. The preceding observation gives the converse: a reduced special group of finite type has finite chain length.
Thus, for G a reduced special group, the following are equivalent:

i) G is ω-stable.

ii) G is stable.

iii) G is of finite chain length.

iv) G is built up from $\mathbb{Z}_2$ applying a finite number of times the operations of product and extension.

v) G is of finite type.

Henceforth, all special groups will be of finite type.

5 Interpretations

In this section we show that each component in a finite product or in an extension is interpretable in the resulting special group. This will allow us to prove some model-theoretic properties by induction over the tree of a special group of finite type.

5.1 The product case

We have $G = G_1 \times \cdots \times G_n$, where the G_i are special groups of finite type, and we may suppose that each G_i is indecomposable.
By fact 3.4, we have $D_{G_i} = \{1\}$, or $G_i = D_{G_i}$, and in the latter case, $card(D_{G_i}) = 2$ (for, otherwise, it is easy to check that any decomposition of $G_i = D_{G_i}$ as a product of groups gives a decomposition of G_i as a product of special groups, contradicting the indecomposability of G_i).
Furthermore, we have $D_G = D_{G_1} \times \cdots \times D_{G_n}$, and (after suitable reindexing):

$$G = G_1 \times \cdots \times G_l \times \underbrace{G_{l+1} \times \cdots \times G_n}_{D_G}, \text{ as special groups,}$$

with $card(G_{l+1}) = \cdots = card(G_n) = 2$.
As seen before fact 3.4, D_G is definable in G, and if $\bar{G} = G_1 \times \cdots \times G_l$ and $\pi : G \to \bar{G}$ is the retract of the inclusion of $\bar{G}$ in G induced by $G = \bar{G} \times D_G$, then we see that $\bar{G}$ can be interpreted in G, because $\bar{G} = G/D_G$ as groups, and the isometry on $\bar{G}$ is the retract isometry with respect to π.
We now show that each G_i is definable in $\bar{G}$. To simplify notation, suppose that $l = 2$. So:

$$\bar{G} = G_1 \times G_2, \text{ with } D_{G_1} = \{1\} \text{ and } D_{G_2} = \{1\}.$$

For $a \in G_2$ we have:

$$D_{\bar{G}} < 1, (-1_{G_1}, a) >= D_{G_1} < 1, -1 > \times D_{G_2} < 1, a >= G_1 \times D_{G_2} < 1, a >,$$

and:

$$\bigcap_{a \in G_2} D_{\bar{G}} < 1, (-1_{G_1}, a) >= G_1 \times D_{G_2} = G_1 \times \{1\}.$$

Since G is ω-stable, so is $\bar{G}$, and there is no infinite decreasing chain of definable subgroups in $\bar{G}$ (see [P_2], proposition 1.6 p. 23, or [H], theorem 5.7.2 p. 249). So:

$$\bigcap_{a \in G_2} D_{\bar{G}} < 1, (-1_{G_1}, a) > = \bigcap_{i=1}^{m} D_{\bar{G}} < 1, (-1_{G_1}, a_i) >= G_1 \times \{1\}.$$

G_1 is definable with parameters in $\bar{G}$, and thus is interpretable with parameters in G. The same argument applies to G_2.

We then get the following lemma:

Lemma 5.1 *Let G be a special group, written $G = G_1 \times \cdots \times G_l \times D_G$ as a product of special groups.*

- *If $G \prec G'$, then $G' = G'_1 \times \cdots \times G'_l \times D_{G'}$ with $G_i \prec G'_i$ for all $i = 1, \ldots, l$, and $D_G \prec D_{G'}$ (i.e. $D_G = D_{G'}$, since D_G is finite).*
  Moreover, if G_i is indecomposable, then G'_i is indecomposable.
- *If $G' \prec G$, then $G' = G'_1 \times \cdots \times G'_l \times D_G$, as special groups.*
  Moreover, after suitable reindexing, $G'_i \prec G_i$, for $i = 1, \cdots, l$.

Proof: Since D_G is definable without parameters in G, we have $card(D_G) = card(D_{G'})$, and $D_{G'} = D_G$ (they are both finite).
As seen above, the special group $\bar{G}$ is interpretable in G. Moreover, $\bar{G}'$ is interpretable in the same way in G' (D_G and $D_{G'}$ are definable by the same formula in G, G'). This gives, using $G \prec G'$, respectively $G' \prec G$:

$$\bar{G} \prec \bar{G}' \text{ (respectively } \bar{G}' \prec \bar{G}).$$

We are thus reduced to the case $G = G_1 \times \cdots \times G_l$, with $D_G = \{1\}$, and we know that the G_i are definable in G with parameters, say by the formulas $\varphi_1(\bar{a}, x)$, $\ldots$, $\varphi_l(\bar{a}, x)$, respectively.
Now we have to distinguish between the cases $G \prec G'$ and $G' \prec G$:

- If $G \prec G'$. The proof is straightforward, using the preceding observations about the definability of $G_1, \cdots, G_l$ in G.
- If $G' \prec G$. The following formula is first-order and is satisfied in G:

  $$\exists \bar{z}\ [\textstyle\bigwedge_{i=1}^{l} \text{“}\varphi_i(\bar{z}, G) \text{ is a special group”} \wedge$$
  $$\text{“}G = \varphi_1(\bar{z}, G) \times \cdots \times \varphi_l(\bar{z}, G) \text{ as } L_{SG}\text{-structures”}].$$

  By elementary equivalence, G' satisfies this formula, then:

  $$G' = \varphi_1(\bar{b}, G') \times \cdots \times \varphi_l(\bar{b}, G') \text{ as special groups },$$

  where $\bar{b}$ is a realization for $\bar{z}$ in G'.
  The first item gives a decomposition of G: $G = G''_1 \times \cdots \times G''_l \times D_G$, with $G'_i \prec G''_i$, for $i = 1, \cdots, l$, and the unicity of such a decomposition (see [M_1], corollary 5.10, p. 104) gives $G''_i \cong G_i$ (after a suitable reindexing), and thus $G'_i \prec G_i$. ⧫

5.2 The extension case

We have $G = G_1[H]$, where G_1 is a special group of finite type, and H is a group of exponent 2, with $card(H) \geq 2$.

Definition 5.2 (see [M$_1$], p. 114) *Let G be any special group different from $\{-1,1\}$, and $a \in G$. We say that a is* **basic** *if $D_G < 1, a > \neq \{1, a\}$ or $D_G < 1, -a > \neq \{1, -a\}$, and that a is* **rigid** *otherwise. If $G = \{-1,1\}$, -1 and 1 are called basic. We denote by $\boldsymbol{B_G}$ the set of all basic elements of G.*

We then have (see [M$_1$], theorem 5.18, 5.19 and corollary 5.20, pp. 115 to 117, for proofs in the abstract Witt rings setting, and [DM], ex. 1.14, p. 14 for proofs for special groups):

Proposition 5.3 *B_G, with the structure induced by G, is a special group, and $G \cong B_G[H]$, where $H \cong G/B_G$ (as groups). Moreover:*
- *If $G = G'[H']$, with G' a special subgroup of G, then $B_G \subseteq G'$.*
- *If G' is a subgroup of G such that $B_G \subseteq G'$, then G' is a special group (with the structure induced by G), and $G \cong G'[L]$ with $L \cong G/G'$ (as groups).*

Using definition 5.2, one sees that if $G = G_1 \times \cdots \times G_n$ is a product of special groups, then every element of G is basic, except if $G = \mathbb{Z}_2 \times \mathbb{Z}_2$, because $Z_2 \times \mathbb{Z}_2 \cong \mathbb{Z}_2[H]$, where H is the 2-element group of exponent 2. Hence $\mathbb{Z}_2 \times Z_2 \cong \mathbb{Z}_2[H]$ is the only product isomorphic to an extension.
Moreover, being basic is clearly a first-order property, so B_G is definable (without parameters) in G, and, as G is of finite type, so is B_G.

We conclude this section by a lemma that will be useful at the end of this paper:

Lemma 5.4 *If G is a special group of finite type and H is a group of exponent 2, then every definable subset of $B_G[H]$ is definable in the product of special groups $B_G \times H$, where H is endowed with the fan isometry, and any element in $H \setminus \{1\}$ as -1_H.*

Proof: Let $K = \varphi(B_G[H], \bar{m})$ be a definable subset of $B_G[H]$, with $\bar{m} \in B_G[H]$. Proposition 4.2 says that $B_G[H]$ is a generalized product of B_G and H, if H is endowed with the fan structure, as in the statement. Lemma 2.2 applies and gives L_{SG}-formulas $\theta_1, \cdots, \theta_k$ and a propositional formula F, depending only on φ, such that, for all $a, \bar{m} \in B_G[H]$:

$$\begin{array}{c} B_G[H] \models \varphi(a, \bar{m}) \text{ if and only if} \\ \models F(B_G \models \theta_1(a_1, \bar{m}_1), \cdots, B_G \models \theta_k(a_1, \bar{m}_1), \\ H \models \theta_1(a_2, \bar{m}_2), \cdots, H \models \theta_k(a_2, \bar{m}_2)). \end{array} \qquad (\star)$$

The argument at the begining of section 5.1 shows that the special groups B_G and H are both definable in $B_G \times H$. Furthermore, it is easy to construct L_{SG}-formulas whose interpretation in $B_G \times H$ are the predicates required to represent $B_G[H]$ as a generalised product (*i.e.* the predicates $A_1, \cdots, A_4$ in the proof of proposition 4.2). It follows that there is an L_{SG}-formula $\psi(x, \bar{y})$ such that, for all $a, \bar{m} \in B_G \times H$:

$$\begin{array}{c} \models F(B_G \models \theta_1(a_1, \bar{m}_1), \cdots, B_G \models \theta_k(a_1, \bar{m}_1), \\ H \models \theta_1(a_2, \bar{m}_2), \cdots, H \models \theta_k(a_2, \bar{m}_2)) \\ \text{if and only if} \\ B_G \times H \models \psi(a, \bar{m}). \end{array}$$

Using $(\star)$, this gives:

$$B_G[H] \models \varphi(a, \bar{m}) \text{ if and only if } B_G \times H \models \psi(a, \bar{m}),$$

for all $a, \bar{m} \in B_G \times H$. ♦

6 Categoricity and saturation

6.1 Categoricity

Lemma 6.1 *If G is a special group of finite type, and $G \prec \tilde{G}$, then $G \equiv_{\infty\omega} \tilde{G}$, and G, $\tilde{G}$ have the same pruned tree.*

Proof: Since G is of finite type, we can proceed by induction on its tree:

- If G is finite, obviously we have $G \cong \tilde{G}$.
- If $G = B_G[H]$ is an extension (with $B_G \neq G$).
  B_G is definable in G, so $B_{\tilde{G}} \neq \tilde{G}$ and $\tilde{G} = B_{\tilde{G}}[\tilde{H}]$. By the induction hypothesis, $B_G \equiv_{\infty\omega} B_{\tilde{G}}$ in L_{SG}, and B_G, $B_{\tilde{G}}$ have the same pruned tree.
  Moreover, $H \cong G/B_G \prec \tilde{G}/B_{\tilde{G}} \cong \tilde{H}$ as groups, so $card(\tilde{H}) = card(H)$ if $card(H) < \aleph_0$, and $card(\tilde{H}) \geq \aleph_0$ if $card(H) \geq \aleph_0$.
  This implies that $H \equiv \tilde{H}$ in the language $\{1, .\}$ (hence $H \equiv_{\infty\omega} \tilde{H}$), and that G and $\tilde{G}$ have the same pruned tree.
  But $B_G[H]$ is obtained from B_G in L_{SG} and H in the language $\{1, .\}$, by a generalized product, and $B_{\tilde{G}}[\tilde{H}]$ is obtained in the same way from $B_{\tilde{G}}$ and $\tilde{H}$ (the additionnal structure on H, $\tilde{H}$, required for this generalized product, can be chosen to be definable in $\{1, .\}$ (*e.g.*, the fan structure), and we still have $H \equiv_{\infty\omega} \tilde{H}$ in this expanded language). Since generalized products preserve $L_{\infty\omega}$-equivalence, we have $B_G[H] \equiv_{\infty\omega} B_{\tilde{G}}[\tilde{H}]$ as special groups.
- If G is a product. We keep the same notation as in subsection 5.1:

  $$G = G_1 \times \cdots \times G_l \times \underbrace{G_{l+1} \times \cdots \times G_n}_{D_G}, \text{ as special groups.}$$

  By lemma 5.1, we have: $\tilde{G} = \tilde{G}_1 \times \cdots \times \tilde{G}_l \times D_{\tilde{G}}$, with $G_1 \prec \tilde{G}_1, \ldots, G_l \prec \tilde{G}_l$, and $D_G \cong D_{\tilde{G}}$.
  By induction, G_i and $\tilde{G}_i$ have the same pruned tree and are $L_{\infty\omega}$-equivalent, for $i = 1, \cdots, l$. Since products preserve $L_{\infty\omega}$-equivalence we have $G \equiv_{\infty\omega} \tilde{G}$, and G, $\tilde{G}$ have the same pruned tree. ♦

Proposition 6.2 *If G_1 is a special group of finite type and G_2 is a special group such that $G_2 \equiv G_1$, then G_2 is of finite type, and the following are equivalent:*

i) $G_1 \equiv G_2$.

ii) $G_1 \equiv_{\infty\omega} G_2$.

iii) G_1 and G_2 have the same pruned tree.

In particular, $Th(G)$ is ω-categorical.

Proof: We show that G_2 is of finite type by induction on the tree of G_1:

- If G_1 is finite, then $G_1 \cong G_2$.

- If $G_1 = K_1 \times \cdots \times K_n$ is a product. Let G' be a common elementary extension of G_1 and G_2. The first part of lemma 5.1 gives $G' = K'_1 \times \cdots \times K'_n$, with $K_i \prec K'_i$ for $i = 1, \cdots, n$. Moreover, as $G_2 \prec G'$, the second part of lemma 5.1 gives $G_2 = L_1 \times \cdots \times L_n$, with $L_i \prec K'_i$, for $i = 1, \cdots, n$.
  Thus $L_i \equiv K_i$, and by induction the L_i are of finite type, which implies that G_2 is of finite type.
- If $G_1 = B_{G_1}[H]$. By elementary equivalence, $G_2 = B_{G_2}[H']$, with $B_{G_1} \equiv B_{G_2}$, and by induction B_{G_2} is of finite type, which implies that G_2 is of finite type.

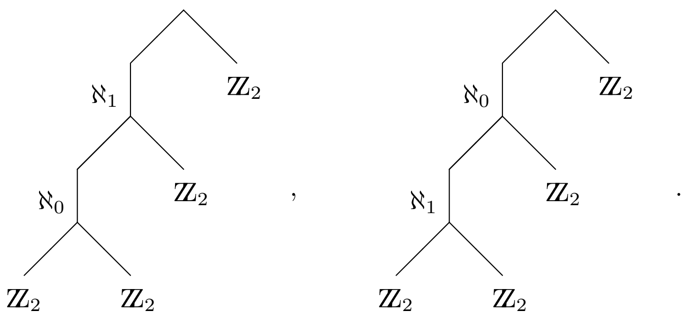
We show now the equivalences:
i)⇒ii) Let G'_1, G'_2 be ω-saturated elementary extensions of G_1, G_2 respectively. We have $G'_1 \equiv G'_2$, and by ω-saturation $G'_1 \equiv_{\infty\omega} G'_2$:

$$\begin{array}{ccc} G_1 & \prec & G'_1 \\ & & |||_{\infty\omega} \\ G_2 & \prec & G'_2 \end{array} .$$

The conclusion follows from the previous lemma.
ii)⇒i) Obvious.
i)⇒iii) Consider an elementary extension G' of G_1 and G_2. By lemma 6.1: G_1, G', and G_2, G' have the same pruned tree.
iii)⇒i) By induction on the tree, using the Feferman-Vaught theorem for products and extensions. ♦

Remarks:

- Completing proposition 6.2, we remark that there are (reduced) special groups G of finite type such that $Th(G)$ is not $\aleph_1$-categorical: the following special groups have the same pruned tree and thus are elementarily equivalent, they have the same cardinality ($\aleph_1$), but they are not isomorphic:



- Proposition 6.2 implies that if G_1 and G_2 are two special groups of finite type, and if $W(G_1)$ and $W(G_2)$ are their associated abstract Witt rings, we have: $G_1 \equiv G_2$ in L_{SG} if and only if $W(G_1) \equiv W(G_2)$ in L_G, the language of rings with a predicate G, interpreted by G_1 in $W(G_1)$ and by G_2 in $W(G_2)$:
  If $W(G_1) \equiv W(G_2)$ in L_G, then G_1 is definable in $W(G_1)$ and G_2 is definable in the same way in $W(G_2)$, which implies $G_1 \equiv G_2$ in L_{SG}.
  If $G_1 \equiv G_2$ in L_{SG}, we have $G_1 \equiv_{\infty\omega} G_2$, and thus $W(G_1) \equiv W(G_2)$ in L_G, because for any formula φ in L_G, there is a formula φ^* in $(L_{SG})_{\infty\omega}$ such that, for $i = 1, 2$:
  $$W(G_i) \models \varphi \Leftrightarrow G_i \models \varphi^*.$$

The formula φ^* is constructed by induction on φ, as follows:

- If φ is atomic, φ is either "$a \in G$", which is always true in G_i, or an equality between two sums of elements of G_i: $a_1+\cdots+a_n = b_1+\cdots+b_m$. We may assume $n \geq m$, and by definition of Witt rings, this equality in $W(G_i)$ is equivalent to the following isometry in G_i:

$$< a_1, \cdots, a_n > \equiv < b_1, \cdots, b_m > \oplus \frac{n-m}{2} < -1, 1 > .$$

  We then take for φ^* the L_{SG}-formula which describes this isometry.
- If $\varphi = \neg\varphi_1$, then $\varphi^* = \neg\varphi_1^*$.
- If $\varphi = \varphi_1 \wedge \varphi_2$, then $\varphi^* = \varphi_1^* \wedge \varphi_2^*$.
- If $\varphi = \exists f \ \varphi_1(f)$, we take:

$$\varphi^* = \bigvee_{n\in\omega} \exists x_1, \cdots, x_n \ \varphi_1^*(< x_1, \cdots, x_n >).$$

6.2 Saturation

We start with the simple remark that every countable special group of finite type is saturated: if G is such a special group, $Th(G)$ is ω-categorical and has thus a countable saturated model, which is unique up to isomorphism by ω-categoricity. This can be refined in the following way:

Definition 6.3 • *If G is an infinite special group of finite type, its tree contains a finite number of extensions by infinite groups of exponent 2, say $H_1, \ldots, H_n$. Define:*

$$\boldsymbol{ext(G)} = min\{card(H_i) \mid i = 1, \ldots, n\}$$

(it is the least infinite cardinal labelling a vertical edge in the tree of G).
• *If G is a finite special group, we take $\boldsymbol{ext(G)} = \infty$, and we assume that ∞ is larger than any cardinal.*

We have the following result:

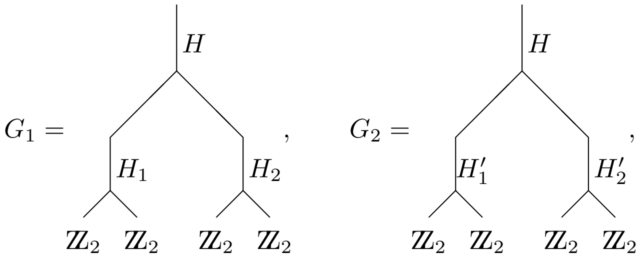
Proposition 6.4 *Let G be a special group of finite type. Then for any infinite cardinal λ:*

G is λ-saturated if and only if $ext(G) \geq \lambda$.

Proof: Easy induction on the tree of G, using that interpretability and generalized products of a finite number of structures both preserve λ-saturation, and that a λ-saturated structure has cardinal greater or equal than λ. ⧫

7 Model-completeness

The question arises naturally whether the first-order theory of a special group (or of a reduced special group) of finite type is model-complete. This is false in general, as shown by the following counter-example:
Let G_1 and G_2 be the following reduced special groups:



with $card(H_1) = card(H_2) = card(H) = \aleph_0$, and $H_1' = H_1 \times H$, $H_2' = H_2 \times H$. Take then:

$$\begin{array}{ccc} G_1 & \stackrel{f}{\longrightarrow} & G_2 \\ (a[h_1] \times b[h_2])[h] & \mapsto & (a[h_1 \times h] \times b[h_2 \times h])[1], \end{array}$$

where $a, b \in \mathbb{Z}_2 \times \mathbb{Z}_2$.

One can verify that f is a SG-monomorphism, and we know that $G_1 \equiv G_2$ because these two special groups have the same pruned tree, but f is not elementary:
Take $h \in H \setminus \{1\}$, and let $a = (1[1] \times 1[1])[h]$. We have:
$D_{G_1} < 1, a >= \{1, a\}$, but $f(a) = (1[1 \times h] \times 1[1 \times h])[1]$, and then:
$D_{G_2} < 1, f(a) >= \{1, 1[1 \times h]\} \times \{1, 1[1 \times h]\} \times 1$, which is different from $\{1, f(a)\}$.

We thus see that f is not elementary because it does not respect an extension of the tree (the last extension in the tree of G_1). In fact this is more general, and we will see that a SG-monomorphism between two reduced special groups of finite type (*i.e.* of finite chain length) is elementary if and only if it respects the extensions in the following sense:

Definition 7.1 *If G is a reduced special group of finite chain length, we introduce a predicate A which will be interpreted in G in the following way (by induction on the construction of G):*

- *If $G = \mathbb{Z}_2$ then $A^G = \emptyset$.*
- *If $G \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \cong \mathbb{Z}_2[\{1, h\}]$, where $\{1, h\}$ is the group of exponent 2 with two elements, then $A^G = \emptyset$ (this actually means that we chose to consider G as a product rather than an extension; cf. below).*

In all other cases:

- *If $G = G_1 \times \cdots \times G_n$ then:*
$$A^G = (A^{G_1} \times 1 \cdots \times 1) \cup \cdots \cup (1 \times \cdots \times A^{G_n}).$$
- *If $G = K[H]$ with $K = B_G$ then:*
$$A^G = (A^K \times 1) \cup (G \setminus K).$$

If $f : G \longrightarrow G'$ is a morphism of reduced special groups of finite type, we say that f **respects the extensions** *if f preserves the predicate A.*

Remarks:

- One verifies easily that for any reduced special group of finite type, G, A^G is definable with parameters in G. With this, we show that two reduced special groups of finite type, elementarily equivalent in L_{SG}, are elementarily equivalent in $L_{SG} \cup \{A\}$ (by taking a common elementary extension and using the same kind of argument as in proposition 6.2).

- If $a \in A^G$, the definition of A^G tells us that (upon removing some "$\times 1$" in the expression of a, if necessary) $a \in K[H] \setminus K$, where K is a special group that is not an extension, and whose tree is a subtree of that of G.
  Moreover, for any such $K[H]$, we have $card(K[H]) > 4$:
  Suppose $card(K[H]) \leq 4$. This implies $K[H] = \mathbb{Z}_2[\{1,h\}]$. But we have chosen to see $\mathbb{Z}_2[\{1,h\}]$ as the product $\mathbb{Z}_2 \times \mathbb{Z}_2$, which means that $A^{K[H]} = \emptyset$; a contradiction since $a \in A^{K[H]}$.
- All these $K[H]$ will be refered to as **extensions of the tree of G**.

Lemma 7.2 *Let G be a reduced special group of finite type, $a \in G$, $K[H]$ be an extension of the tree of G, and $b_1, b_2 \in K[H] \setminus K$ be such that $b_1, b_2 \in D_G < 1, a >$ and $b_1 \neq b_2$.*
Then $D_G < 1, a > \supseteq K[H]$.

Proof: Easy verification, by induction on the tree of G. ♦

Lemma 7.3 *Let G be a special group of finite type, and $K_1[H_1]$, $K_2[H_2]$ be extensions of the tree of G. Then one (and only one) of the following occur:*

- $K_1[H_1] = K_2[H_2]$.
- $K_1[H_1] \subseteq K_2$ *or* $K_2[H_2] \subseteq K_1$.
- $\forall a \in K_1[H_1]$, $\forall b \in K_2[H_2]$:

$$D_G < 1, ab > = D_{K_1[H_1]} < 1, a > \; \times D_{K_2[H_2]} < 1, b > .$$

Proof: Straightforward, by induction on the tree of G. ♦

We come now to the main result of this section, which characterizes the elementary monomorphisms between reduced special groups of finite type:

Theorem 7.4 *If G and G' are two elementarily equivalent reduced special groups of finite type, and $f : G \to G'$ is a monomorphism of special groups which respects extensions, then f is elementary.*

Remark: The converse of theorem 7.4 is easily verified: if G, G' are special groups of finite type and $f : G \longrightarrow G'$ is an elementary L_{SG}-monomorphism, then f preserves the predicate A (see the first remark after definition 7.1).

The rest of section 7 is devoted to the proof of this theorem. Before starting it, if $K[H]$ is an extension of the tree of G', we will say that this extension is **used** if there exists $g \in A^G$ such that $f(g) \in K[H] \setminus K$.

Lemma 7.5 *Let G and G' be reduced special groups of finite type (not necessarily elementarily equivalent) and $f : G \to G'$ be a monomorphism in the language $\{1, ., \equiv\}$.*
If $a \in G$ is such that $D_G < 1, a >$ contains k distinct extensions $K[H]$ of the tree of G, then $D_{G'} < 1, f(a) >$ contains at least k distinct extensions of the tree of G', and these extensions are all used.

Proof of the lemma: Induction on k:

- $k = 1$. Let $K[H]$ be the extension which is contained in $D_G < 1, a >$. Since G is reduced, we have $-1 \neq 1$. In particular, if $h \in H \setminus \{1\}$, then $1_K.h, -1_K.h \in (K[H] \setminus K) \subseteq A^G$, and:
  $f(1_K.h)$, $f(-1_K.h) \in D_{G'} < 1, f(a) >$, with $f(1_K.h)$, $f(-1_K.h) \in A^{G'}$ (f respects the extensions).
  So there exist two extensions, $K'_1[H'_1]$ and $K'_2[H'_2]$, in the tree of G', such that $f(1_K.h) \in K'_1[H'_1] \setminus K'_1$ and $f(-1_K.h) \in K'_2[H'_2] \setminus K'_2$. This gives:
  $D_{G'} < 1, f(1_K.h) >= \{1, f(1_K.h)\}$, and
  $D_{G'} < 1, f(-1_K.h) >= \{1, f(-1_K.h)\}$.

  Suppose now $K'_1[H'_1] \neq K'_2[H'_2]$, and consider the two remaining cases given by lemma 7.3:

  - $\star$ If $K'_1[H'_1] \subseteq K'_2$ or $K'_2[H'_2] \subseteq K'_1$. Suppose, for example, $K'_1[H'_1] \subseteq K'_2$. Then $f(1_K.h)f(-1_K.h) \in K'_2[H'_2] \setminus K'_2$, and:
    $D_{G'} < 1, f(1_K.h)f(-1_K.h) >= \{1, f(1_K.h)f(-1_K.h)\}$, has 2 elements. But, since $D_G < 1, -1_K >\supseteq K[H]$, we have:
    $D_{G'} < 1, f(1_K.h)f(-1_K.h) >= D_{G'} < 1, f(-1_K) >\supseteq f(K[H])$, which implies $card(K[H]) \leq 2$. This contradicts the remark after definition 7.1, that $card(K[H]) > 4$.
  - $\star$ Otherwise, we have:

    $$\begin{aligned} D_{G'} &< 1, f(1_K.h)f(-1_K.h) >= \\ &= D_{K'_1[H'_1]} < 1, f(1_K.h) > \times D_{K'_2[H'_2]} < 1, f(-1_K.h) > \\ &= \{1, f(1_K.h)\} \times \{1, f(-1_K.h)\}. \end{aligned}$$

    As in the preceding case, we have:
    $D_{G'} < 1, f(1_K.h)f(-1_K.h) >= D_{G'} < 1, f(-1_K) >\supseteq f(K[H])$, which gives $card(K[H]) \leq 4$, contradicting the remark after definition 7.1.

  We then have $K'_1[H'_1] = K'_2[H'_2]$, and $f(1_K.h), f(-1_K.h) \in K'_1[H'_1] \setminus K'_1$. Lemma 7.2 says that $K'_1[H'_1] \subseteq D_{G'} < 1, f(a) >$ (and the extension $K'_1[H'_1]$ is obviously used).

- $k + 1$, with $k \geq 1$. We proceed now by induction on G (omitting the case $G = \mathbb{Z}_2$, for which the hypotheses are not verified).
  - $\star$ The extension case: $G = B_G[H]$, with $H \neq \{1\}$.
    Firstly, the case $a \in B_G[H] \setminus B_G$ is impossible, for otherwise $D_G < 1, a >$ $= \{1, a\}$, which cannot contain two distinct extensions. Hence, $a \in B_G$, and we have to distinguish two cases:

    • The extension $B_G[H]$ is not among the $k + 1$ extensions contained in $D_G < 1, a >$. In this case, $D_{B_G} < 1, a >$ contains $k + 1$ extensions, and, considering $f \restriction B_G$, induction applied to B_G gives us that $D_{G'} < 1, f(a) >$ contains $k+1$ extensions, which are all used. The proof is finished in this case.

    • The extension $B_G[H]$ is one of the $k + 1$ extensions contained in $D_G < 1, a >$. Then $a = -1_G$, and $D_G < 1, a > = G$. Consider the restriction of f to B_G. It is a $\{1, ., \equiv\}$-monomorphism which respects extensions.
    Remark that, as G contains at least 2 extensions ($k+1 \geq 2$), B_G contains

at least one extension, and thus decomposes as a product:

$$B_G = G_1 \times \cdots \times G_n.$$

Let $\varepsilon_i = (1, \cdots, 1, -1_{G_i}, 1, \cdots, 1) \in B_G$, and let k_i be the number of extensions contained in $D_G < 1, \varepsilon_i >= 1 \times \cdots \times 1 \times G_i \times 1 \times \cdots \times 1$. Then $k_1 + \cdots + k_n = k$.

Fact 7.6 *$D_{G'} < 1, f(a) >$ contains k used extensions: $K'_1[H'_1], \cdots, K'_k[H'_k]$, and each of these extensions is included in one $D_{G'} < 1, f(\varepsilon_i) >$.*

Proof: $a = -1_G \in B_G$, and $D_{B_G} < 1, a >$ contains k extensions. The induction hypothesis gives that $D_{G'} < 1, f(a) >$ contains k extensions of the tree of G', which are all used.

We now consider $D_G < 1, \varepsilon_i >$. Induction on the tree of G, applied to $f \restriction B_G$, gives that every $D_{G'} < 1, f(\varepsilon_i) >$ contains k_i extensions, which are all used. Moreover, since f is a $\{1, ., \equiv\}$-monomorphism, for all $i \neq j$, $D_{G'} < 1, f(\varepsilon_i) >$ and $D_{G'} < 1, f(\varepsilon_j) >$ have no used extension in common (this would imply $D_{B_G} < 1, \varepsilon_i > \cap D_{B_G} < 1, \varepsilon_j > \neq \{1\}$, a contradiction).

Since $k_1 + \cdots + k_n = k$, and $D_{G'} < 1, f(\varepsilon_i) > \subseteq D_{G'} < 1, f(a) >$, we may assume that every extension $(K'_1[H'_1], \cdots, K'_k[H'_k])$ that is contained in $D_{G'} < 1, f(a) >$ is included in one of the $D_{G'} < 1, f(\varepsilon_i) >$ (replacing, if necessary, the $K'_1[H'_1], \cdots, K'_k[H'_k]$ by the k used extensions included in $\bigcup_{i=1}^{n} D_{G'} < 1, f(\varepsilon_i) >$). $\Diamond$

To show that $D_{G'} < 1, f(a) >$ contains $k + 1$ used extensions, it only remains to prove the following:

Fact 7.7 *$D_{G'} < 1, f(a) >$ contains an used extension (of the tree of G') other than $K'_1[H'_1], \cdots, K'_k[H'_k]$.*

Proof: Let $h \in H \setminus \{1\}$, and consider $1.h, -1.h \in B_G[H] \setminus B_G$. Because f respects the extensions, there exist two extensions $\tilde{K}_1[L'_1]$, $\tilde{K}_2[L'_2]$ of $A^{G'}$ such that $f(1.h) \in \tilde{K}_1[L'_1] \setminus \tilde{K}_1$ and $f(-1.h) \in \tilde{K}_2[L'_2] \setminus \tilde{K}_2$. The same argument as in the case $k = 1$ shows that these two extensions are equal, and thus: $f(1.h), f(-1.h) \in \tilde{K}_1[L'_1] \setminus \tilde{K}_1$. Since we also have $f(1.h), f(-1.h) \in D_{G'} < 1, f(a) >$, lemma 7.2 gives $D_{G'} < 1, f(a) > \supseteq \tilde{K}_1[L'_1]$.

Finally this extension is none of $K'_1[H'_1], \cdots, K'_k[H'_k]$, because this would imply $f(1.h) \in D_{G'} < 1, f(\varepsilon_i) >$ for some $i \in \{1, \cdots, n\}$, and then $1.h \in D_G < 1, \varepsilon_i > \subseteq B_G$, contradicting the choice of $1.h$. $\Diamond$

So $D_{G'} < 1, f(a) >$ contains $k + 1$ used extensions.

⋆ The product case: $G = G_1 \times \cdots \times G_n$. Then $a = (a_1, \cdots, a_n)$, and: $D_G < 1, a >= D_{G_1} < 1, a_1 > \times \cdots \times D_{G_n} < 1, a_n >$.
Each $D_{G_i} < 1, a_i >$ contains k_i extensions of the tree of G, for $i = 1, \cdots, n$, with $k_1 + \cdots + k_n = k + 1$.
Then $f \restriction G_i : G_i \longrightarrow G'$ is a $\{1, ., \equiv\}$-monomorphism and induction on G gives that each $D_{G'} < 1, f(a_i) >$ contains k_i extensions of the tree of G', which are all used.
Moreover, if $i \neq j$, then $D_G < 1, a_i > \cap D_G < 1, a_j > = \{1\}$. Since f is a $\{1, ., \equiv\}$-monomorphism, this gives that $D_{G'} < 1, f(a_i) > \cap D_{G'} < 1, f(a_j) >$ does not contain any used extension. So:

$D_{G'} < 1, f(a_1) > \cup \cdots \cup D_{G'} < 1, f(a_n) >$ contains $k_1 + \cdots + k_n = k+1$ extensions of the tree of G', which are used.
But f is a SG-morphism, $D_G < 1, a_i > \subseteq D_G < 1, a >$, and G' is reduced; hence, $D_{G'} < 1, f(a_i) > \subseteq D_{G'} < 1, f(a) >$, for all $i = 1, \cdots, n$.
Thus, $D_{G'} < 1, f(a) >$ contains $k+1$ extensions of the tree of G', which are all used. ♦

Proof of theorem 7.4: By induction on the tree of G:

- If G is finite, then f is an isomorphism.
- If $G = G_1 \times \cdots \times G_n \times G_{n+1}$, we may assume that each $G_1, \ldots, G_n$ is an extension and that G_{n+1} is finite and does not contain any extension.
  G and G' have the same pruned tree, so $G' = G'_1 \times \cdots \times G'_n \times G'_{n+1}$, with $G_i \equiv G'_i$ for all $i = 1, \cdots, n+1$.
  Fix some notation for later use: If $i \in \{1, \cdots, n\}$, we denote $\varepsilon_i = (1_{G_1}, \cdots, 1_{G_{i-1}}, -1_{G_i}, 1_{G_{i+1}}, \cdots, 1_{G_{n+1}})$, and we denote by ε'_i the corresponding element of G'.
  Then we have $G_1 = D_G < 1, \varepsilon_1 >, \cdots, G_n = D_G < 1, \varepsilon_n >$, and let k_i be the number of extensions contained in G_i.
  The proof in this case is organized as a succession of facts:

  Fact 7.8 *Each $D_{G'} < 1, f(\varepsilon_i) >$ contains exactly k_i extensions, and they are all used. Moreover, each extension of G' is used and is included in some $D_{G'} < 1, f(\varepsilon_i) >$, for $i \in \{1, \cdots, n\}$.*

  Proof: By lemma 7.5, we know that every $D_{G'} < 1, f(\varepsilon_i) >$ contains at least k_i used extensions. Moreover, if $D_{G'} < 1, f(\varepsilon_i) >$ contains one extension, then this extension is used by no other $D_{G'} < 1, f(\varepsilon_j) >$, where $j \neq i$ (because f is a monomorphism and $D_G < 1, \varepsilon_i > \cap D_G < 1, \varepsilon_j >= \{1\}$). The conclusion follows, since both G' and G contain $k_1 + \cdots + k_n$ extensions. ♢

  Fact 7.9 *Let $K[H]$ be an extension of the tree of G', such that $K[H] \subseteq D_{G'} < 1, f(\varepsilon_i) >$. If $K'[H']$ is another extension of the tree of G', which is below or above $K[H]$ (*i.e.* which verifies $K'[H'] \subseteq K$, or $K[H] \subseteq K'$, respectively), then $K'[H'] \subseteq D_{G'} < 1, f(\varepsilon_i) >$.*

  Proof: This is obvious if $K'[H']$ is below $K[H]$, and this implies that:
  If $D_{G'} < 1, f(\varepsilon_i) >$ contains an extension (call it E), it contains all the extensions above E in the tree. Indeed, if an extension above E is in $D_{G'} < 1, f(\varepsilon_j) >$ with $j \neq i$, then the extension E of $D_{G'} < 1, f(\varepsilon_i) >$ (which is used, by the previous fact), is below an extension in $D_{G'} < 1, f(\varepsilon_j) >$. By the first observation, this extension E is then in $D_{G'} < 1, f(\varepsilon_j) >$, and since it is used and f is a monomorphism, we have $D_G < 1, \varepsilon_i > \cap D_G < 1, \varepsilon_j > \neq \{1\}$, which is impossible. ♢

  We show now that this implies:

  Fact 7.10 *There exists a permutation $\sigma \in S_n$ such that:*
  $$\forall i = 1, \ldots, n, \quad D_{G'} < 1, f(\varepsilon_i) > \supseteq G'_{\sigma(i)}. \qquad (\star)$$

  Proof: Let $i \in \{1, \cdots, n\}$. Since $D_G < 1, \varepsilon_i > = G_i$ contains an extension, so does $D_{G'} < 1, f(\varepsilon_i) >$; call E this extension. We have $E \subseteq G'_j$ for one $j \in \{1, \cdots, n\}$. But we have just seen that $D_{G'} < 1, f(\varepsilon_i) >$ contains all the extensions that are below or above E in the tree of G'. In particular, since

G'_j is an extension, we have $G'_j \subseteq D_{G'} < 1, f(\varepsilon_i) >$, and by fact 7.8, this last extension is used, *i.e.* contains an element $f(a)$ with $a \in A^G$.
Remark that, by fact 7.8, we may assume that there exists $k \in \{1, \cdots, n\}$ such that $a \in D_G < 1, \varepsilon_k >$, and, since f is a monomorphism and $D_G < 1, \varepsilon_i > \cap D_G < 1, \varepsilon_k >= \{1\}$ whenever $i \neq k$, we have necessarily $i = k$. Thus we have $\sigma : \{1, \cdots, n\} \longrightarrow \{1, \cdots, n\}$ such that for all $i \in \{1, \cdots, n\}$, $D_{G'} < 1, f(\varepsilon_i) > \supseteq G'_{\sigma(i)}$.
But we have just seen that the last extension of any $G'_{\sigma(i)}$ is used by some $f(a)$, with $a \in D_G < 1, \varepsilon_i >$. Moreover f is injective and $D_G < 1, \varepsilon_i > \cap D_G < 1, \varepsilon_j > = \{1\}$ if $i \neq j$. This implies that σ is bijective (otherwise, if $\sigma(i) = \sigma(j) = k$, with $i \neq j$, then $G'_k \subseteq D_{G'} < 1, f(\varepsilon_i) >$, $D_{G'} < 1, f(\varepsilon_j) >$. But G'_k is an extension containing an element $f(a)$, with $a \in D_G < 1, \varepsilon_i >$, which gives $f(a) \in D_{G'} < 1, f(\varepsilon_j) >$, a contradiction). $\Diamond$

Remarks:

- Observe that fact 7.10 does not say anything concerning G_{n+1} or G'_{n+1}, because the arguments we have employed involve extensions, and G_{n+1}, G'_{n+1} do not contain any.
- This latter fact implies, in particular, that every extension in the tree of G' is contained in some $D_{G'} < 1, f(a) >$, for some $a \in G \setminus \{-1\}$.

Fact 7.11 *For all* $i = 1, \ldots, n$ $f(\varepsilon_i) = \varepsilon'_{\sigma(i)}$, *and* $f(\varepsilon_{n+1}) = \varepsilon'_{n+1}$.

Proof: Let $i \neq j \in \{1, \ldots, n\}$. Using $(\star)$ in fact 7.10, we have:

$$f(\varepsilon_i) = a_1 \times \cdots \times a_{\sigma(i)-1} \times -1 \times a_{\sigma(i)+1} \times \cdots \times a_n \times a_{n+1},$$
$$f(\varepsilon_j) = b_1 \times \cdots \times b_{\sigma(j)-1} \times -1 \times b_{\sigma(j)+1} \times \cdots \times b_n \times b_{n+1}.$$

By the choice of $\varepsilon_i, \varepsilon_j$: $D_G < 1, \varepsilon_i >$, $D_G < 1, \varepsilon_j > \subseteq D_G < 1, \varepsilon_i\varepsilon_j >$, and, applying f (since G' is reduced):

$$D_{G'} < 1, f(\varepsilon_i) >, \, D_{G'} < 1, f(\varepsilon_j) > \subseteq D_{G'} < 1, f(\varepsilon_i\varepsilon_j) >,$$

which implies:

$$G'_{\sigma(i)}, \, G'_{\sigma(j)} \subseteq D_{G'} < 1, f(\varepsilon_i\varepsilon_j) > .$$

But, as $f(\varepsilon_i\varepsilon_j) = a_1b_1 \times \cdots \times a_{\sigma(j)-1}b_{\sigma(j)-1} \times a_{\sigma(j)}.-1 \times \cdots \times a_{\sigma(i)-1}b_{\sigma(i)-1} \times -1.b_{\sigma(i)} \times \cdots \times a_nb_n \times a_{n+1}b_{n+1}$ (assuming for instance $\sigma(j) < \sigma(i)$), we must have $a_{\sigma(j)} = 1$, and $b_{\sigma(i)} = 1$. Since this is true for any $i, j = 1, \ldots, n$, we have, for $i = 1, \ldots, n$:
$f(\varepsilon_i) = \varepsilon'_{\sigma(i)}.\alpha_i$, with $\alpha_i \in G'_{n+1}$.
Consider now ε_{n+1} and ε'_{n+1}. We have:

$$\begin{aligned}\varepsilon'_1.\cdots.\varepsilon'_{n+1} = -1_{G'} &= f(-1_G) = f(\varepsilon_1.\cdots.\varepsilon_{n+1}) \\ &= \varepsilon'_1.\cdots.\varepsilon'_n.(\alpha_1.\cdots.\alpha_n).f(\varepsilon_{n+1}).\end{aligned}$$

This gives $f(\varepsilon_{n+1}) = \varepsilon'_{n+1}(\alpha_1.\cdots.\alpha_n)$. But we know that $G_{n+1} = D_G < 1, \varepsilon_{n+1} >$, which implies:

$$f(G_{n+1}) \subseteq D_{G'} < 1, f(\varepsilon_{n+1}) > = D_{G'} < 1, \varepsilon'_{n+1}(\alpha_1.\cdots.\alpha_n) > \subseteq G'_{n+1}$$

(the last inclusion because $\varepsilon'_{n+1}(\alpha_1.\cdots.\alpha_n) \in G'_{n+1}$).
Moreover, since $card(G_{n+1}) = card(G'_{n+1})$, we have $f(G_{n+1}) = G'_{n+1}$, *i.e.*, using the last line of inclusions: $D_{G'} < 1, f(\varepsilon_{n+1}) >= G'_{n+1}$. This entails $f(\varepsilon_{n+1}) = \varepsilon'_{n+1}$.
We conclude this part by showing that $\alpha_i = 1$ for all $i = 1, \cdots, n$. We know

that $f(\varepsilon_i) = \varepsilon'_i.\alpha_i$, with $\alpha_i \in G'_{n+1}$.
If $\alpha_i \neq 1$, then $D_{G'} < 1, f(\varepsilon_i) > \cap\, G'_{n+1} \supsetneqq \{1\}$, *i.e.*:
$D_{G'} < 1, f(\varepsilon_i) > \cap\ f(G_{n+1}) \supsetneqq \{1\}$, which contradicts $D_G < 1, \varepsilon_i > \cap\, G_{n+1} = \{1\}$, since f is a monomorphism. $\lozenge$

Then we define σ on $\{1, \cdots, n+1\}$ by setting $\sigma(n+1) = n+1$.
Summarizing, we have $\sigma \in S_{n+1}$ such that for any $i = 1, \cdots, n+1$, $f(G_i) \subseteq G'_{\sigma(i)}$. To apply the induction hypothesis, we only need show that $f(G_i) \equiv G'_{\sigma(i)}$. Let $i_0 \in \{1, \ldots, n+1\}$. Since σ is a product of cycles, let $\{i_0, i_1, \ldots, i_k\}$ be the cycle containing i_0. To illustrate the argument, we do the proof for $\{i_0, i_1, \ldots, i_k\} = \{1, 2, 3\}$; the general proof goes exactly the same way.

Since f is a monomorphism which preserves the predicate A, we have the following inclusions of $L_{SG} \cup \{A\}$-structures:

$$G_1 \hookrightarrow G'_2,\ G_2 \hookrightarrow G'_3,\ G_3 \hookrightarrow G'_1,$$

It follows that $G_1 \models Th_\forall(G'_2)$, $G_2 \models Th_\forall(G'_3)$, $G_3 \models Th_\forall(G'_1)$ (the theories are in the language $L_{SG} \cup \{A\}$). Since $G_i \equiv G'_i$ ($i = 1, 2, 3$) in L_{SG}, the first item in the remark after definition 7.1 shows that these equivalences also hold in $L_{SG} \cup \{A\}$. We have then:

$$G_1 \models Th_\forall(G_2),\ G_2 \models Th_\forall(G_3),\ G_3 \models Th_\forall(G_1),$$

in the language $L_{SG} \cup \{A\}$.
We now invoke the following general model-theoretic result (see [H], corollary 6.5.3, p. 295): if M and N are L-structures such that $M \models Th_\forall(N)$, then M can be embedded in a L-structure N' elementarily equivalent to N. Applying this, we get the following chain of inclusions:

$$G_1 \hookrightarrow G_2^{(1)} \hookrightarrow G_3^{(1)} \hookrightarrow G_1^{(2)} \hookrightarrow G_2^{(2)} \hookrightarrow G_3^{(2)} \hookrightarrow G_1^{(3)} \hookrightarrow \cdots \qquad (\spadesuit)$$

where the inclusions are monomorphisms in the language $L_{SG} \cup \{A\}$, and $\forall i, k\ G_i^{(k)} \equiv G_i$ in the language $L_{SG} \cup \{A\}$. In particular $G_i^{(k)}$ and G_i have the same pruned tree. The induction hypothesis then applies to the $G_i^{(k)}$, and proves that the monomorphism:

$$G_i^{(k)} \hookrightarrow G_i^{(k+1)},$$

given by the chain $(\spadesuit)$ is elementary (because this monomorphism is a $L_{SG} \cup \{A\}$-monomorphism, and thus respects the extensions).
Then, denoting by $\mathcal{G}$ the inductive limit of the chain $(\spadesuit)$, we obtain that every $G_i^{(k)}$ is an elementary substructure of $\mathcal{G}$. This proves that $G_1, G_2, G_3, G'_1, G'_2, G'_3$ are elementarily equivalent.

We have then $f \restriction G_i : G_i \hookrightarrow G'_{\sigma(i)}$ for $i = 1, \ldots, n+1$, with $G_i \equiv G'_{\sigma(i)}$, and $f \restriction G_i$ respects the extensions. The induction hypothesis gives that $f \restriction G_i$ is an elementary map from G_i to $G'_{\sigma(i)}$, and then $f : G \hookrightarrow G'$ is elementary, since products preserve elementary inclusion.

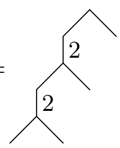
- $G = B_G[H]$, with $H \neq \{1\}$.
  As G and G' have the same pruned tree, $G' = B_{G'}[H']$ with $H' \neq \{1\}$, and $B_{G'} \equiv B_G$. Since B_G is defined by an existential formula, we have $f(B_G) \subseteq B_{G'}$. By induction, $f \restriction B_G : B_G \longrightarrow B_{G'}$ is elementary.
  We now show that $f(H) \subseteq G' \setminus B_{G'}$. Let $h \in H$. We know that $f(h)$ is

in an extension of G'. This extension is $G' \setminus B_{G'}$: otherwise $f(h)$ would be in an extension of $B_{G'}$. But B_G and $B_{G'}$ are elementarily equivalent products of special groups; then, by induction, the second item in the remarks after fact 7.10 applies, and we know that every extension of $B_{G'}$ is included in $D_{G'} < 1, f(a) >$, for some $a \in B_G$, $a \neq -1$. This implies $f(h) \in D_{G'} < 1, f(a) >$ and thus $h \in D_G < 1, a >$ (f is a monomorphism), which is impossible because $D_G < 1, a > \subseteq B_G$ and $h \in G \setminus B_G$.

So, up to isomorphism, we may assume $f(H) \subseteq H'$. Moreover, we know that H and H' are either both infinite or of the same finite cardinality (because G and G' are elementarily equivalent). So $f(H) \prec H'$, in the language $\{1, .\}$.
Since extensions are generalized products, we obtain that $f(G) = f(B_G)[f(H)]$ is an elementary substructure of $G' = B_{G'}[H']$. ♦

8 Quantifier elimination

The theory of a special group of finite type does not necessarily admit quantifier elimination in the language L_{SG}, since there are monomorphisms between elementarily equivalent special groups which are not elementary. The question is then to look for a language in which it would eliminate quantifiers, and the obvious first attempt is the language $L_{SG} \cup \{A\}$, where A is the relation defined in definition 7.1.
However, we cannot get quantifier elimination in this language, as shown by the following example of a reduced special group of finite type, whose theory (in the language $L_{SG} \cup \{A\}$) does not admit elimination of quantifiers (since in this example all special groups are reduced, the leaves in their trees are all $\mathbb{Z}_2$, and are omitted):

$G =$ 

.

Remark that $cl(G) = 4$ and that every substructure of G has chain length at most 4. Indeed, for spaces of orderings of finite type the chain length is the number of leaves in the tree, see [ABR], proposition 3.7, p. 98. This is also the case for reduced special groups of finite type by the isomorphism of categories mentioned in section 3. To show that $T = Th(G)$ in $L_{SG} \cup \{A\}$ does not admit quantifier elimination, we use the following criterion (recall that T is model-complete by theorem 7.4):

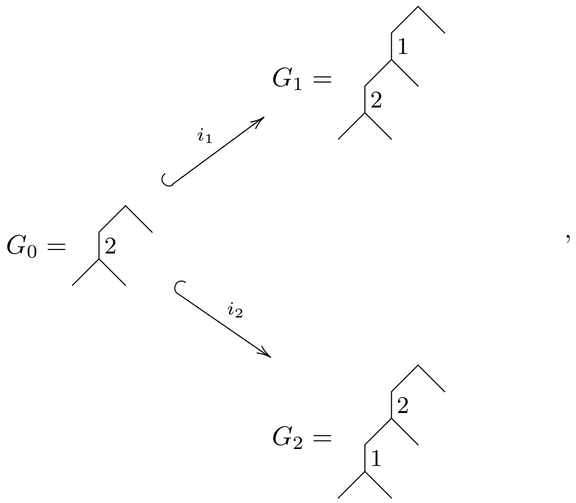
Proposition 8.1 ([CK], proposition 3.5.19, p. 202) *Let T be a model-complete theory. Then the following are equivalent:*

1. *T is a model-completion of $T_\forall$.*

2. *$T_\forall$ has the amalgamation property.*

3. *T admits elimination of quantifiers.*

We consider the following diagram, where the G_i, $i = 0, 1, 2$, are models of $T_\forall(G)$:

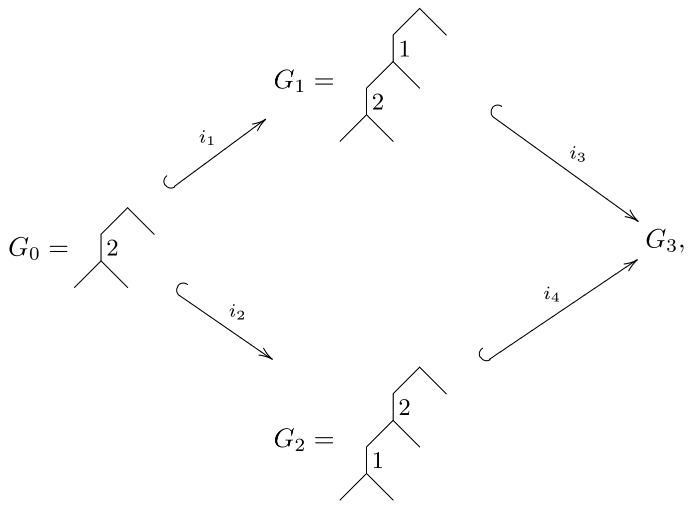


where, denoting by H_i the group of exponent 2 with 2^i elements:

$$\begin{array}{rcl} i_1 : (\mathbb{Z}_2 \times \mathbb{Z}_2)[H_2] \times \mathbb{Z}_2 & \longrightarrow & ((\mathbb{Z}_2 \times \mathbb{Z}_2)[H_2] \times \mathbb{Z}_2)[H_1] \times \mathbb{Z}_2 \ , \\ (\alpha, \beta)[h] \times \gamma & \longmapsto & ((\alpha, \beta)[h] \times \gamma)[1] \times \gamma \end{array}$$

$$\begin{array}{rcl} i_2 : (\mathbb{Z}_2 \times \mathbb{Z}_2)[H_2] \times \mathbb{Z}_2 & \longrightarrow & ((\mathbb{Z}_2 \times \mathbb{Z}_2)[H_1] \times \mathbb{Z}_2)[H_2] \times \mathbb{Z}_2 \ . \\ (\alpha, \beta)[h] \times \gamma & \longmapsto & ((\alpha, \alpha)[1] \times \beta)[h] \times \gamma \end{array}$$

It is easy to check that i_1 and i_2 are $L_{SG} \cup \{A\}$-monomorphisms.
To show that $Th(G)$ does no admit quantifier elimination in the language $L_{SG} \cup \{A\}$, we show that the above diagram cannot be completed as follows:



where i_3, i_4 are $L_{SG} \cup \{A\}$-monomorphisms: if this were the case, then we would have $cl(G_3) \geq 5$, contradicting that G_3 should be a model of $T_\forall$:
Take, in G_0:

$a_0 = 1, a_1 = (1, -1)[1] \times 1, a_2 = (-1, -1)[1] \times 1$, and $a_3 = (-1, -1)[1] \times -1$. We have $a_0 \in D_{G_0} < 1, a_1 >, a_1 \in D_{G_0} < 1, a_2 >, a_2 \in D_{G_0} < 1, a_3 >$.
Since, for $j = 1, 2$, i_j is a monomorphism, the images of these elements under i_j verify the same relations in G_j. But, for $a = ((-1, -1)[1] \times -1)[1] \times 1 \in G_1$ and $b = ((-1, 1)[1] \times -1)[1] \times 1 \in G_2$ we have:
$i_1(a_2) \in D_{G_1} < 1, a >$ and $a \in D_{G_1} < 1, i_1(a_3) >$,
$i_2(a_1) \in D_{G_2} < 1, b >$ and $b \in D_{G_2} < 1, i_2(a_2) >$.
The commutativity of the diagram would imply:
$i_4 \circ i_2(a_0) \in D_{G_3} < 1, i_4 \circ i_2(a_1) >, i_4 \circ i_2(a_1) \in D_{G_3} < 1, i_4(b) >$,
$i_4(b) \in D_{G_3} < 1, i_4 \circ i_2(a_2) >, i_4 \circ i_2(a_2) = i_3 \circ i_1(a_2) \in D_{G_3} < 1, i_3(a) >$,
$i_3(a) \in D_{G_3} < 1, i_3 \circ i_1(a_3) >$.
This contradicts the fact that the chain length of G_3 is at most 4. ♦

We can nevertheless expand the language L_{SG} to obtain quantifier elimination for reduced special groups of finite type:
If G is such a special group, we define the language $L^+(Th(G))$ by induction on the tree of G (this language depends only on the theory of G) by adding unary function symbols at each step of the construction of G:

- If $G = \mathbb{Z}_2$ then $L^+(Th(\mathbb{Z}_2)) = L_{SG}$.
- If $G = G_1 \times G_2$ then, denoting by p_i the projection from G to G_i given by this product, we take:
$$L^+(Th(G)) = L_{SG} \cup \{p_1, p_2\} \cup \{p \times 1 \mid p \text{ function symbol in } L^+(Th(G_1))\} \cup \{1 \times p \mid p \text{ function symbol in } L^+(Th(G_2))\},$$
with $\begin{array}{rcl} p \times 1 : G_1 \times G_2 & \to & G_1 \times G_2, \\ (g_1, g_2) & \mapsto & (p(g_1), 1) \end{array}$ and $\begin{array}{rcl} 1 \times p : G_1 \times G_2 & \to & G_1 \times G_2 \\ (g_1, g_2) & \mapsto & (1, p(g_2)). \end{array}$
- If $G = B_G[H]$ with $card(H) \geq 2$, and if π is the projection from G onto B_G, we take:
$$L^+(Th(G)) = L_{SG} \cup \{\pi\} \cup \{p \times 1 \mid p \text{ function symbol in } L^+(Th(B_G))\},$$
with $\begin{array}{rcl} p \times 1 : B_G[H] & \to & B_G[H] \\ (g_1, g_2) & \mapsto & (p(g_1), 1). \end{array}$

Before proceeding further, here is the criterion for quantifier elimination that we will use:

Proposition 8.2 *Let T be a first-order theory in the language L. Then the following are equivalent:*

1. *T admits quantifier elimination in L.*
2. *For all $\mathcal{A}, \mathcal{B} \models T$, for all $\mathcal{C} \subseteq \mathcal{B}$ ($\mathcal{C}$ may not be a model of T), all L-monomorphisms $f : \mathcal{C} \to \mathcal{A}$, all $\bar{c} \in \mathcal{C}$, and all primitive L-formulas $\varphi(v_1, \ldots, v_n)$:*
$$\mathcal{B} \models \varphi(\bar{c}) \Rightarrow \mathcal{A} \models \varphi(f(\bar{c})).$$

Recall that a primitive formula is a formula of the form:
$$\varphi(\bar{v}) = \exists x \ \bigwedge_{i=1}^{k} \psi_i(\bar{v}, x),$$

where the ψ_i are atomic formulas or negations of atomic formulas. Moreover, the same statement with $\mathcal{C} \subseteq \mathcal{A}, \mathcal{B}$, and f the inclusion of $\mathcal{C}$ in $\mathcal{A}$ is still equivalent to condition 8.2(2). We use this last form to get:

Theorem 8.3 *Let G be a reduced special group of finite chain length. Then G admits quantifier elimination in the language $L^+(Th(G))$.*

Proof: By induction on the tree of G:

- $G = \mathbb{Z}_2$: $Th(G)$ admits elimination of quantifiers in $L^+(Th(\mathbb{Z}_2)) = L_{SG}$, because $\mathbb{Z}_2 = \{-1, 1\}$ and $-1, 1 \in L_{SG}$.
- $G = G_1 \times \cdots \times G_n$, where each $Th(G_i)$ admits quantifier elimination in $L^+(Th(G_i))$, for $i = 1, \ldots, n$.
  Let $\mathcal{A}, \mathcal{B} \models Th(G)$, $\mathcal{C}$ be a $L^+(Th(G))$-substructure of $\mathcal{A}$ and $\mathcal{B}$, and $\varphi(\bar{c})$ be a $L^+(Th(G))$-primitive formula with parameters in $\mathcal{C}$.
  Let G_0 be a prime model of $Th(G)$ (if G is infinite, it is the countable model of $Th(G)$, since $Th(G)$ is ω-categorical). Then:

  $$G_0 \prec \mathcal{A} \text{ and } f : G_0 \hookrightarrow \mathcal{B},$$

  where f is an elementary monomorphism.
  As G and G_0 have the same pruned trees, G_0 is a product $G_{0,1} \times \cdots \times G_{0,n}$ and, taking $\varepsilon_1 = -1_{G_{0,1}} \times 1 \times \cdots \times 1, \cdots, \varepsilon_n = 1 \times \cdots \times 1 \times -1_{G_{0,n}}$, we get $G_0 = D_{G_0} < 1, \varepsilon_1 > \times \cdots \times D_{G_0} < 1, \varepsilon_n >$ as special groups. This implies:

  $$\mathcal{A} = D_{\mathcal{A}} < 1, \varepsilon_1 > \times \cdots \times D_{\mathcal{A}} < 1, \varepsilon_n > \text{ and:}$$

  $$\mathcal{B} = D_{\mathcal{B}} < 1, f(\varepsilon_1) > \times \cdots \times D_{\mathcal{B}} < 1, f(\varepsilon_n) >$$

  as special groups, with $D_{\mathcal{A}} < 1, \varepsilon_i > \equiv D_{\mathcal{B}} < 1, f(\varepsilon_i) > \equiv G_i$, for all $i = 1, \cdots, n$.

  Consider $\bar{c} \in \mathcal{C}$. In $\mathcal{A}$ we have $\bar{c} = \bar{c}_1 \times \cdots \times \bar{c}_n$, with $\bar{c}_i \in D_{\mathcal{A}} < 1, \varepsilon_i >$, and $\bar{c}_i = p_i(\bar{c}) \in \mathcal{C}$ because $\mathcal{C}$ is a $L^+(Th(G))$-substructure. This gives $\mathcal{C} \models \bar{c}_i = p_i(\bar{c})$, and thus $\mathcal{B} \models \bar{c}_i = p_i(\bar{c})$ (since $\mathcal{C} \subseteq \mathcal{A} \cap \mathcal{B}$).
  As $\mathcal{B} \models Im(p_i) = D < 1, f(\varepsilon_i) >$, we obtain $\bar{c} = \bar{c}_1 \times \cdots \times \bar{c}_n$ in $\mathcal{B}$, with $\bar{c}_i \in D_{\mathcal{B}} < 1, f(\varepsilon_i) >$.

  To complete the proof in this case, using proposition 8.2 it is enough to show that $\mathcal{A} \equiv \mathcal{B}$ in the language $L^+(Th(G)) \cup \{\bar{c}\}$. To do this, we check easily that the structure $< \mathcal{A}; \bar{c} >$ is a generalized product of the structures $< D_{\mathcal{A}} < 1, \varepsilon_i >; \bar{c}_i >$ in the languages $L^+(Th(G_i))$ with suitable additional constants. Likewise, $< \mathcal{B}; \bar{c} >$ is obtained from the structures $< D_{\mathcal{B}} < 1, f(\varepsilon_i) >; \bar{c}_i >$ by the same generalized product. Thus we need only show that for all $i = 1, \cdots, n$, $D_{\mathcal{A}} < 1, \varepsilon_i > \equiv D_{\mathcal{B}} < 1, f(\varepsilon_i) >$ in the language $L^+(Th(G_i)) \cup \{\bar{c}_i\}$:
  Let $\theta(\bar{c}_i)$ be a $L^+(Th(G_i))$-formula. We have:

  $$\begin{aligned} D_{\mathcal{A}} < 1, \varepsilon_i > \models \theta(\bar{c}_i) &\Leftrightarrow D_{\mathcal{A}} < 1, \varepsilon_i > \models \theta'(\bar{c}_i) \\ &\Leftrightarrow \mathcal{C} \models \theta'(\bar{c}_i) \\ &\Leftrightarrow D_{\mathcal{B}} < 1, f(\varepsilon_i) > \models \theta'(\bar{c}_i) \\ &\Leftrightarrow D_{\mathcal{B}} < 1, f(\varepsilon_i) > \models \theta(\bar{c}_i), \end{aligned}$$

  where:

- θ' is a quantifier-free $L^+(Th(G_i))$-formula which verifies (by induction hypothesis) $G_i \models \forall \bar{x}\ \theta(\bar{x}) \leftrightarrow \theta'(\bar{x})$. This is also true in $D_{\mathcal{A}} < 1, \varepsilon_i >$ and $D_{\mathcal{B}} < 1, f(\varepsilon_i) >$ by elementary equivalence, and justifies the first and last equivalences.
- The second and third equivalences are true because θ' is a quantifier-free formula, and $\bar{c}_i \in \mathcal{A} \cap \mathcal{B}$.

The proof is complete in the case of products.

- G is an extension, $G = B_G[H]$ with $card(H) \geq 2$, and by induction B_G admits elimination of quantifiers in $L^+(Th(B_G))$. Let $\mathcal{A}$, $\mathcal{B} \models Th(G)$, $\mathcal{C}$ be a $L^+(Th(G))$-substructure of $\mathcal{A}$, $\mathcal{B}$, and $\varphi(\bar{c})$ be a $L^+(Th(G))$-primitive formula with parameters $\bar{c} = (c_1, \cdots, c_n) \in \mathcal{C}$.
  We have $\mathcal{A} = B_{\mathcal{A}}[H_{\mathcal{A}}]$, $\mathcal{B} = B_{\mathcal{B}}[H_{\mathcal{B}}]$, and $\bar{c} = \bar{c}_1 \times \bar{c}_2$ in $\mathcal{A}$, with $\bar{c}_1 \in B_{\mathcal{A}}$ and $\bar{c}_2 \notin B_{\mathcal{A}}$. We denote by π the projection from $\mathcal{A} = B_{\mathcal{A}}[H_{\mathcal{A}}]$ onto $B_{\mathcal{A}}$.
  As $\pi \in L^+(Th(G))$, it is easy to show (as in the case of products) that:
  $\bar{c}_1 \in B_{\mathcal{B}}$ and $\bar{c} = \bar{c}_1 \times \bar{d}_2$ in $\mathcal{B}$, with $\bar{d}_2 \notin B_{\mathcal{B}}$.
  To show $\mathcal{A} \equiv \mathcal{B}$ in $L^+(Th(G)) \cup \{\bar{c}\}$, using generalized products (as in the product case), it is enough to prove:
  $$B_{\mathcal{A}} \equiv B_{\mathcal{B}} \text{ in } L^+(Th(B_G)) \cup \{\bar{c}_1\} \text{ and}$$
  $$(H_{\mathcal{A}}, \bar{c}_2) \equiv (H_{\mathcal{B}}, \bar{d}_2) \text{ in the language } \{1, .\}. \tag{$\star$}$$
  Since the argument proving the first assertion is similar to that employed in the case of products, we consider only the second item:
  $H_{\mathcal{A}} \equiv H_{\mathcal{B}}$ in $\{1, .\}$ and in such a structure (a vector space over $\mathbb{F}_2$), two elements different from 1 have the same type.
  Thus ($\star$) will be true if $\forall i = 1, \cdots, k$, $(c_{2,i} = 1 \Leftrightarrow d_{2,i} = 1)$ (where $\bar{c}_2 = (c_{2,1}, \cdots, c_{2,k})$ and $\bar{d}_2 = (d_{2,1}, \cdots, d_{2,k})$). But:
  $$\begin{aligned} c_{2,i} = 1 \quad &\Leftrightarrow \quad \mathcal{A} \models c_i = \pi(c_i), \text{ with } c_i = c_{1,i} \times c_{2,i} \\ &\Leftrightarrow \quad \mathcal{B} \models c_i = \pi(c_i), \text{ because } \bar{c} \in \mathcal{A} \cap \mathcal{B} \\ &\Leftrightarrow \quad d_{2,i} = 1, \text{ by choice of } d_2. \end{aligned}$$
  This completes the proof. ♦

Remarks:

- The language $L^+(Th(G))$ depends only on the pruned tree of G, and we can show (by induction on the pruned tree), that the projections in $L^+(Th(G))$ are axiomatized by a finite number of $L^+(Th(G))$-first-order formulas, *i.e.* there exists a $L^+(Th(G))$-formula φ such that $G \models \varphi$ if and only if the function symbols in $L^+(Th(G))$ verify the properties required in the construction of $L^+(Th(G))$ before proposition 8.2.
- Theorem 8.3 has been proved for reduced special groups of finite chain length (which are the special groups of finite type built up from $\mathbb{Z}_2$). This was used at one point at the beginning of the proof, for the case $G = \mathbb{Z}_2$, namely, that $\mathbb{Z}_2$ admits quantifier elimination in L_{SG}. The proof remains correct for special groups of finite type that are built from finite special groups which admit quantifier elimination in L_{SG}.

9 Morley rank

We have seen in corollary 4.4 and in proposition 6.2 that if G is any special group of finite type, its theory is ω-stable, ω-categorical, and of finite Morley rank. We now compute this rank, by induction on the tree of G:
• If G is finite, $RM(G) = 0$.

• If G is a product $G_1 \times \cdots \times G_n$, we use the following result: if K_1, K_2 are groups definable in a ω-categorical, ω-stable structure, then $RM(K_i) = RU(K_i) < \aleph_0$, for $i = 1, 2$ (by [CHL], theorem 5.1), and $RU(K_1 \times K_2) = RU(K_1) + RU(K_2)$ (see [P$_2$], theorem 6.1, p. 182).
With the notations of fact 3.4, we have $G = \bar{G}_1 \times \cdots \times \bar{G}_n \times D_{G_1} \times \cdots \times D_{G_n}$, with $D_G = D_{G_1} \times \cdots \times D_{G_n}$ finite, and we get:

$$\begin{array}{rcl} RM(G) & = & RM(\bar{G}_1 \times \cdots \times \bar{G}_n \times D_G) \\ & = & RM(\bar{G}_1) + \cdots + RM(\bar{G}_n) + RM(D_G) \\ & = & RM(\bar{G}_1) + \cdots + RM(\bar{G}_n) \text{ (since } D_G \text{ is finite).} \end{array}$$

But $G_i = \bar{G}_i \times D_{G_i}$. The same argument as above shows that $RM(G_i) = RM(\bar{G}_i) + RM(D_{G_i}) = RM(\bar{G}_i)$, since D_{G_i} is finite, and we have:

$$RM(G) = RM(G_1) + \cdots + RM(G_n).$$

• If $G = B_G[H]$, we know that G is a generalized product of B_G and H (if we add a quaternary relation R^H and a constant -1_H in H). With this, we show:

Proposition 9.1 *If $G = B_G[H]$:*

$$RM(G) = \begin{cases} RM(B_G) + 1 & \textit{if } H \textit{ is infinite} \\ RM(B_G) & \textit{otherwise} \end{cases}.$$

Proof: We start with a simple observation:

$$G = \bigcup_{h \in H} B_G \times \{h\},$$

and $B_G \times \{h\}$ is definable in $G = B_G[H]$. So if H is infinite:

$$RM(G) \geq RM(B_G) + 1 \tag{1}$$

(the $B_G \times \{h\}$ are disjoint and have the same Morley rank).
We will now show:

$$RM(G) \leq RM(B_G) + RM(H), \tag{2}$$

where the Morley rank of H is that of H as vector space over $\mathbb{F}_2$, *i.e.*:

$$RM(H) = \begin{cases} 1 & \text{if } H \text{ is infinite} \\ 0 & \text{otherwise} \end{cases}.$$

We have seen in lemma 5.4 that every definable subset of $B_G[H]$ is definable in the product $B_G \times H$, if H is equipped with the fan isometry, and with an element different from 1 as -1_H (this relation and this constant are both definable (with parameters in H), in the language $\{1, .\}$).
This gives:

$$RM(B_G[H]) \leq RM(B_G \times H).$$

The computation done for the product case applies and gives $RM(B_G \times H) = RM(B_G) + RM(H)$, i.e.:

$$RM(B_G[H]) \leq RM(B_G) + RM(H).$$

As the additional structure on H is definable in the language $\{1, .\}$ for groups, we have:

$$RM(\ < H; 1, -1_H, ., R^H > \ ) = \begin{cases} 1 & \text{if } H \text{ is infinite} \\ 0 & \text{otherwise} \end{cases},$$

which conludes the proof, using (1) and (2). ♦

Remarks: For any special group of finite type, it is thus easy to compute its Morley rank from its (pruned) tree.
Furthermore, if $h(G)$ is the height of the tree of G, and $p(G)$ is the maximal number of factors appearing in a product in the decomposition of G as products and extensions, one verifies that the Morley rank of G is bounded as follows:

$$RM(G) \leq p(G)^{h(G)}.$$

If G is reduced, this bound can be expressed in terms of $cl(G)$, the chain length of G, and we get $RM(G) \leq cl(G)^{2cl(G)}$.

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