Axiomatization of local-global principles for pp-formulas in spaces of orderings

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Two important results in quadratic form theory, Pfister's local-global principle and the isotropy theorem (see [5]), can be stated more generally for spaces of orderings (an abstract version of real spectras of formally real fields), for which they are expressed as local-global principles:
A property of quadratic forms (expressed as a so-called positive-primitive formula) holds if and only if it holds locally (at every single ordering for Pfister's local-global principle, at every finite subspace for the isotropy theorem).

In his paper [7], Marshall introduces a much broader local-global principle that could be satisfied by spaces of orderings, and showed how several important questions about quadratic forms and real algebraic geometry would follow from it. He asks, for a space of orderings (X, G) (the unexplained terminology will be introduced later):

"Is it true that any positive-primitive formula $\Phi(\bar{g})$ with parameters $\bar{g}$ in G which holds in every finite subspace of (X, G) necessarily holds in (X, G)?" (LG)

No counter-examples are known. In this paper we work in the theory of reduced special groups (a first-order theory dual to that of spaces of orderings) and consider, for a given pp-formula $\Phi(\bar{g})$ with parameters in G, the following instance of the above question:

"Is it true that if $\Phi(\bar{g})$ holds in every finite subspace of (X, G) it necessarily holds in (X, G)?" $(\mathrm{LG}(\Phi, \bar{g}))$

After the development of the necessary tools, we show that the property "$\Phi(\bar{g})$ is false in some finite subspace" is first-order axiomatizable, and is also equivalent to the property "$\Phi(\bar{g})$ is false in some finite subspace of bounded cardinality", the bound being a function of the number of free and bounded variables in $\Phi(\bar{x})$, c.f. Theorem 2. The axiomatizability of the property $(\mathrm{LG}(\Phi, \bar{g}))$ is then a consequence (c.f. Theorem 1).

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In section 3 we conclude with consequences and related results. We get a bound on the cardinality of the subspaces appearing in the isotropy theorem (the bound depending only on the dimension of the quadratic form. Note that a sharper bound already exists in the special case of the isotropy theorem, see [1] theorem 6.4 p. 114). We show that every profinite and every pseudo finite reduced special group satisfies (LG). Moreover the class of reduced special groups satisfying (LG($\Phi, \bar{g}$)) (for some interpretation of $\bar{g}$) is closed under direct limits and the class of reduced special groups satisfying (LG) is closed under saturated quotients (in other words the class of spaces of orderings satisfying (LG) is closed under subspaces).

1 Preliminaries

We use the terminology of special groups and abstract spaces of orderings, as described in [2] and [1, 6]. We briefly recall the most important notions.

If (X, G) is a space of orderings, G is a group of exponent 2 with distinguished element -1, and X is a closed subset of $\mathrm{Hom}(G, \{-1, 1\}) \subseteq \{-1, 1\}^G$ equipped with the topology induced by the product topology (of the trivial topology on $\{-1, 1\}$), which separates the points of G. X satisfies other properties, in particular $\sigma(-1) = -1$ for every $\sigma \in X$. X is then compact, Hausdorff and totally disconnected.

The sets $X(a_1, \ldots, a_n) = \{\sigma \in X \mid \sigma(a_i) = 1 \; i = 1, \ldots, n\}$ (for $n \in \mathbb{N}$, $a_1, \ldots, a_n \in G$) are clopen sets which form a basis for the topology on X.

A subspace Y of (X, G) is any intersection of sets of the form $X(a_1, \ldots, a_n)$, and if $\Delta = \bigcap\{\ker \sigma \mid \sigma \in Y\}$ we have $Y = \Delta^\perp \cap X := \{\sigma \in X \mid \Delta \subseteq \ker \sigma\}$.

The main example of space of orderings is the space of orderings of a formally real field K, where $G = \dot{K}/\sum \dot{K}^2$ and X is the set of orderings on K.

The language of special groups is $L_{SG} = \{1, -1, ., \equiv\}$, while a special group is an L_{SG}-structure G where G is a group of exponent 2 satisfying some axioms (expressible as first-order L_{SG}-formulas), and $\equiv$ is a binary relation on G^2.

The classical example of a special group is the special group of a field K, where $G = \dot{K}/\dot{K}^2$, and $\equiv$ is interpreted as the isometry between diagonal quadratic forms of dimension 2. The special group of any real closed field contains exactly two elements 1 and -1 and is denoted by $\mathbb{Z}_2$.

A special group G is reduced if for every $a \in G$, $\langle a, a\rangle \equiv \langle 1, 1\rangle$ implies $a = 1$.

There is an isomorphism of categories between spaces of orderings and reduced special groups (see [2], chapter 3), which can be described for the objects as follows:

- If (X, G) is a space of orderings, the binary relation on G^2 defined by $\langle a, b\rangle \equiv \langle c, d\rangle$ if and only if $\forall\sigma \in X_G\ \sigma(a) + \sigma(b) = \sigma(c) + \sigma(d)$ (sum in $\mathbb{Z}$), turns G into a reduced special group.

- If G is a reduced special group, then (X_G, G) is a space of orderings, with $X_G := \{\sigma : G \to \mathbb{Z}_2 \mid \sigma \text{ morphism of special groups}\}$.

Concerning subspaces, if Y is a subspace of X_G, we define $Y^\perp = \bigcap_{\sigma\in Y} \ker\sigma$ and $G_Y = G/Y^\perp$. Then (Y, G_Y) is a space of orderings, and we denote by G_Y the corresponding special group. The canonical projection $\pi_Y : G \to G_Y$ is a morphism of special groups.

Let (X, G) be a space of orderings with associated reduced special group G. A quadratic form $\langle a_1, \dots, a_n\rangle$ of dimension n over (X, G) (or over G) is a n-tuple of elements of G.

Using as a definition the inductive description of isometry for fields, it is possible to define an isometry $\equiv_G$ between forms $\langle a_1, \dots, a_n\rangle$, $\langle b_1, \dots, b_n\rangle$ of dimension n over G, by a first-order L_{SG}-formula with parameters $a_1, \dots, a_n$, $b_1, \dots, b_n$.
This isometry coincides with the one defined as follows (Pfister's local- global principle; see [2], proposition 3.7):

$$\langle a_1, \dots, a_n\rangle \equiv_X \langle b_1, \dots, b_n\rangle \text{ if and only if } \forall\sigma \in X_G\ \sum_{i=1}^{n} \sigma(a_i) = \sum_{i=1}^{n} \sigma(b_i) \in \mathbb{Z}.$$

We conclude these preliminaries with some notations about (first-order) L_{SG}-formulas with parameters $\bar{g}$ in some reduced special group G.

A term $t(\bar{g})$ with parameters $\bar{g}$ is a product of $1, -1$ and some elements of $\bar{g}$. Atomic formulas are of the form $t_1(\bar{g}) = t_2(\bar{g})$ and $\langle t_1(\bar{g}), t_2(\bar{g})\rangle \equiv \langle t_3(\bar{g}), t_4(\bar{g})\rangle$, where $t_1(\bar{g}), t_2(\bar{g}), t_3(\bar{g}), t_4(\bar{g})$ are terms.
A positive-primitive (pp for short) formula $\Phi(\bar{g})$ with parameters $\bar{g}$ is of the form $\exists\bar{x}\ \vartheta(\bar{g}, \bar{x})$, where $\vartheta(\bar{g}, \bar{x})$ is a finite conjunction of atomic formulas in $\bar{g}, \bar{x}$.
The isometry $\langle a_1, \dots, a_n\rangle \equiv_G \langle b_1, \dots, b_n\rangle$ is expressible by a pp-formula, as is the property "$\langle a_1, \dots, a_n\rangle$ is isotropic".
If (X, G) is a space of orderings with associated special group G, and if $\Phi(\bar{g})$ is a L_{SG}-formula, the notation $X \models \Phi(\bar{g})$ will stand for $G \models \Phi(\bar{g})$. Similarly, for a subspace Y of (X, G), $Y \models \Phi(\bar{g})$ will stand for $G_Y \models \Phi(\pi_Y(\bar{g}))$.

We will use the following direct consequence of Pfister's local-global principle: If (X, G) is a space of orderings and $\vartheta(\bar{g})$ is a conjunction of atomic formulas with parameters in G, then $G \models \vartheta(\bar{g})$ if and only if for every $\sigma \in X$ $\mathbb{Z}_2 \models \vartheta(\sigma(\bar{g}))$. This gives, for a subspace Y of (X, G): $G_Y \models \vartheta(\pi_y(\bar{g}))$ if and

only if for every $\sigma \in Y$ $\mathbb{Z}_2 \models \vartheta(\sigma(\bar{g}))$.
We can apply this to a pp-formula $\Phi(\bar{g})$ with parameters in G. If $\Phi(\bar{g})$ is of the form $\exists \bar{x}\ \vartheta(\bar{g}, \bar{x})$ as above, we get $Y \models \Phi(\bar{g})$ if and only if there exists $\bar{b} \in G$ such that for every $\sigma \in Y$ $\mathbb{Z}_2 \models \vartheta(\sigma(\bar{g}), \sigma(\bar{b}))$.
Remark that this last equivalent characterization of $Y \models \Phi(\bar{g})$ also makes sense if Y is not a subspace of X, but only a subset. We will thus use it as the definition of $Y \models \Phi(\bar{g})$ when $\Phi(\bar{g})$ is a pp-formula and Y is a subset of X.

2 Main results

2.1 Model-theoretic tools

Definition 1 *Let L be a first-order language and M and N be L-structures. Let Γ be a set of L-formulas. We say that:*

1. *A map $f : M \to M$ is a Γ-morphism if for every formula $\gamma(\bar{x}) \in \Gamma$ and every $\bar{m} \in M$:*
$$M \models \gamma(\bar{m}) \Rightarrow N \models \gamma(f(\bar{m})).$$
Remark that f is not necessarily an L-morphism.

2. *Γ is called* **uniform** *if it satisfies the following two conditions:*
   (a) *Whenever $\gamma(x_1, \ldots, x_n) \in \Gamma$ and $y_1, \ldots, y_n$ are variables in L then $\gamma(y_1, \ldots, y_n) \in \Gamma$.*
   (b) *Γ is closed by conjunctions.*

3. *$\exists\Gamma$ and $\forall\Gamma$ will denote the set of existential closures of formulas of Γ, and the set of universal closures of formulas of Γ, respectively.*

The proof of the following theorem is a simple model theoretic observation, but the statement is the central idea of our model theoretic arguments.

Theorem 1 *Let I be an index set. For each $i \in I$ let Γ_i be a uniform set of L-formulas and let M_i be an L-structure. Then the following are equivalent:*

1. *$\bigcup_{i \in I} (\exists\Gamma_i \cap Th(M_i))$ is consistent.*

2. *There are an L-structure N and Γ_i-morphisms $f_i : M_i \longrightarrow N$.*

PROOF. 2.⇒1. By 2., the L-structure N is a model of $\bigcup_{i\in I}(\exists\Gamma_i \cap Th(M_i))$; even without the assumption that the Γ_i are uniform.

1.⇒2. We may assume that $M_i \cap M_j = \emptyset$ for all $i \neq j$. For $i \in I$ let $D_i := \{\gamma(\underline{\bar{a}}) \mid \gamma(\bar{x}) \in \Gamma_i,\ \bar{a} \in M \text{ and } M \models \gamma(\bar{a})\}$. By Gödel, it is enough to show that

$$\Phi := \bigcup_{j\in J} D_j$$

is a consistent set of $L(\bigcup_{i\in I} M_i)$-sentences for all finite $J \subseteq I$. Since the sets Γ_i are uniform it is enough to show that $\Phi_0 := \{\gamma_j(\underline{a_j^1},\ldots,\underline{a_j^{k_j}}) \mid j \in J\}$ is consistent, where $\gamma_j(x_1,\ldots,x_{k_j}) \in \Gamma_j$ with mutually distinct variables $x_1,\ldots,x_{k_j}$ and $a_j^i \in M_j$ with $M_j \models \gamma_j(a_j^1,\ldots,a_j^{k_j})\}$. The set $\{\exists x_1 \ldots x_{k_j}\ \gamma_j \mid j \in J\}$ is consistent by 1. Let N be an L-structure together with elements $\{b_j^i \mid j \in J,\ 1 \leq i \leq k_j\}$ such that that $N \models \gamma_j(b_j^1,\ldots,b_j^{k_j})$. If we interpret the constants $\underline{a_j^i}$ as b_j^i we have a model of Φ_0. □

Lemma 1 *Let L be a first-order language and let $\mathbb{K}$ be a class of L-structures. Then the following are equivalent, for an L-structure M:*

1. *M is a model of $Th(\mathbb{K})$.*
2. *M is an elementary substructure of an ultraproduct of elements of K.*

PROOF. Let $\mathbb{K}_0$ be the class of all L-structures which can be elementary embedded into an ultraproduct of elements of $\mathbb{K}$. Obviously, $Th(\mathbb{K}_0) = Th(\mathbb{K})$. By the well known test for axiomatizability of structure classes ([3] 9.5.10), $\mathbb{K}_0$ is axiomatizable. Hence $\mathbb{K}_0 = Mod(Th(\mathbb{K}_0)) = Mod(Th(\mathbb{K}))$ as desired. □

Proposition 1 *Let T be an L-theory, let $\mathbb{K}$ be a class of models of T and let Γ be a uniform set of L-formulas. Then for each model M of T the following are equivalent:*

1. *$M \models T_\Gamma$, where*

$$T_\Gamma := Th(\mathbb{K}) \cap \{\forall x_1 \ldots x_n\ \neg\gamma(x_1,\ldots,x_n) \mid \gamma(x_1,\ldots,x_n) \in \Gamma\}.$$

2. *There is a Γ-morphism $M \longrightarrow N$, where N is an ultraproduct of elements of $\mathbb{K}$.*

In other words $T \cup T_\Gamma$ axiomatizes the class of all T-models for which there is a Γ-morphism into an ultraproduct of elements of $\mathbb{K}$.

PROOF. 2. $\Rightarrow$ 1. Let $\gamma(x_1,\ldots,x_n) \in \Gamma$ such that $\forall x_1 \ldots x_n\ \neg\gamma \in Th(\mathbb{K})$ and let $f : M \longrightarrow N$ be a Γ-morphism, where N is an ultraproduct of elements from $\mathbb{K}$. Suppose $M \models \neg\forall x_1 \ldots x_n\ \neg\gamma$, thus $\exists x_1 \ldots x_n\ \in \exists_M\Gamma$. Hence there is some $b \in N^n$ with $N \models \gamma(b)$, in contradiction to $N \models Th(\mathbb{K})$.

1. $\Rightarrow$ 2. Conversely let $\tilde{T} = Th(\mathbb{K})$ (observe that $\tilde{T}$ contains T). Let $\Gamma_1 := \Gamma$ and $M_1 := M$. Let $\Gamma_2 := \tilde{T}$ and let M_2 be a model of $\tilde{T}$. Then $\exists\Gamma_2 = \tilde{T}$. If $\Phi \in \exists\Gamma_1 \cap Th(M_1)$, then $\tilde{T} \not\vdash \neg\Phi$ by 1, so $(\exists\Gamma_1 \cap Th(M_1)) \cup \exists\Gamma_2$ is consistent and by 1 there are Γ_i-morphism $M_i \to N$, where N is an L-structure. A Γ_2-morphism $M_2 \to N$ is just a model N of $\tilde{T}$ with an arbitrary map $f : M_2 \to N$ (there are no free variables in Γ_2!).

Since N is a model of $T = Th(\mathbb{K})$, it can be embedded into an ultraproduct of elements of $\mathbb{K}$ by lemma 1. □

Corollary 1 *Let T be an L-theory and let Γ be a uniform set of L-formulas. Fix $n \in \mathbb{N}$ and suppose for each model M of T there is a given subset $P_M \subseteq M^n$ such that the following two properties hold.*

1. *For all ultraproducts $M = \prod_{i\in I} M_i/\mathcal{U}$ of models M_i of T we have*
$$\prod_{i\in I} P_{M_i}/\mathcal{U} \subseteq P_M;$$
2. *If $f : M \to N$ is a Γ-morphism and $M, N \models T$, then $f^{-1}(P_N) \subseteq P_M$.*

Then there is a set $\{\psi_i(\bar{y}) \mid i \in I\}$ of L-formulas in n free variables $\bar{y} = (y_1,\ldots,y_n)$, where each ψ_i is of the form $\exists\bar{x}\ \gamma(\bar{y},\bar{x})$ with $\gamma \in \Gamma$ such that for all models M of T
$$P_M = \bigcap_{i\in I} M^n \setminus \psi_i[M^n].$$

PROOF. Let $\bar{c} = (c_1,\ldots,c_n)$ be n new constants. We work in the language $L(\bar{c})$. Let
$$\Gamma_1 = \{\gamma(x_1,\ldots,x_k,c_1,\ldots,c_n) \mid \gamma(x_1,\ldots,x_k,y_1,\ldots,y_n) \in \Gamma\}.$$

Hence Γ_1 is obtained from Γ by replacing some variables in Γ by some of the new constants $c_1,\ldots,c_n$ (in particular, Γ_1 contains Γ). Let $\mathbb{K}$ be the class of all $L(\bar{c})$-structures $(M,\bar{a})$ such that $M \models T$ and $\bar{a} \in P_M$. By assumption 1, $\mathbb{K}$ is closed under ultraproducts. By assumption 2, if $f : (M,\bar{a}) \to (N,\bar{b})$ is a Γ_1-morphism, $(N,\bar{b}) \in \mathbb{K}$ and $M \models T$, then $(M,\bar{a}) \in \mathbb{K}$. Hence if $(M,\bar{a})$ is an $L(\bar{c})$-structure, such that M is a model of T, then $(M,\bar{a}) \in \mathbb{K}$ if and only if there is a Γ_1-morphism $(M, \bar{a})$ into an ultraproduct of elements from $\mathbb{K}$. By 1, the class $\mathbb{K}$ is axiomatized by T together with the set

$$Th(\mathbb{K}) \cap \{\forall x_1 \dots x_k \ \neg\gamma(x_1, \dots, x_k, c_1, \dots, c_n) \mid \gamma(x_1, \dots, x_k, y_1, \dots, y_n) \in \Gamma\}.$$

Let $\{\gamma_i(\bar{x}_i, \bar{y}) \mid i \in I\}$ be an enumeration of all $\gamma(x_1, \dots, x_k, y_1, \dots, y_n) \in \Gamma$ such that $\forall x_1 \dots x_k \ \neg\gamma(x_1, \dots, x_k, c_1, \dots, c_n) \in Th(\mathbb{K})$. Then the $\psi_i(\bar{y}) := \exists x_1 \dots x_k \ \gamma_i(x_1, \dots, x_k, y_1, \dots, y_n)$ have the required properties. □

So far we have the model theoretic facts we need in order to show our main theorem 1. The results below are used in the section on applications.

First another consequence of 1.

Corollary 2 *Let Γ be a uniform set of L-formulas and let $f : M \to N$ be a map between L-structures. Then the following properties are equivalent:*

1. *For all $\gamma(\bar{x}, \bar{y}) \in \Gamma$ and all $\bar{a} \in M$ we have $N \models \exists\bar{x} \ \gamma(\bar{x}, f(\bar{a})) \Rightarrow M \models \exists\bar{x} \ \gamma(\bar{x}, \bar{a})$.*
2. *There is an elementary extension $M' \succ M$ and a Γ-morphism $\mu : N \to M'$ such that $\mu \circ f$ is the inclusion $M \to M'$.*

Proof. 2. $\Rightarrow$ 1. Clear.
1. $\Rightarrow$ 2. We apply 1 to the language $L(M)$ which expands L and has constants $\underline{a}$ for every $a \in M$. Let M_1 be the $L(M)$-structure which expands M and interprets $\underline{a}$ as a. Let M_2 be the $L(M)$-structure which expands N and interprets $\underline{a}$ as $f(a)$. Let $\Gamma_1 := Th(M_1)$ and let $\Gamma_2 := \{\gamma(\bar{x}, \underline{\bar{a}}) | \gamma(\bar{x}, \bar{y}) \in \Gamma, \ \bar{a} \in M\}$. Then Γ_1, Γ_2 are uniform sets of $L(M)$-formulas and by assumption on f, $(\exists\Gamma_1 \cap Th(M_1)) \cup (\exists\Gamma_2 \cap Th(M_2))$ is consistent. By 1 there is an $L(M)$-structure M_0, a Γ_1-morphism $i : M_1 \to M_0$ and a Γ_2-morphism $\mu : M_2 \to M_0$. We may identify M_1 with the image of i and we can take M' to be the underlying L-structure of M_0. Since i is a Γ_1-morphism, M' is an elementary extension of M. Since μ is a Γ_2-morphism, μ is a Γ-morphism and $\mu \circ f$ is the inclusion $M \to M'$. □

The next definition comes from the classical notion in modules:

Definition 2 *Let L be a first-order language.*

1. *A positive-primitive formula (pp for short) with parameters $\bar{a}$ in some L-structure is a formula of the form $\exists\bar{x} \ \vartheta(\bar{x}, \bar{a})$, where $\vartheta(\bar{x}, \bar{a})$ is a conjunction of atomic formulas.*

2. *If $f : M \to N$ is an morphism of L-structures, f is called pure if for every pp-formula $\Phi(\bar{m})$ with parameters in M, $N \models \Phi(f(\bar{m}))$ implies $M \models \Phi(\bar{m})$.*

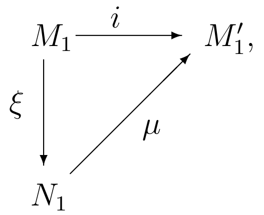
Remark: A pure L-morphism is a morphism that satisfies one of the equivalent conditions of corollary 2 if Γ is the set of conjunctions of atomic L-formulas.

Proposition 2 *Let L be a first-order language and let C be a finite set of constants from L. Let $\xi : M \to N$ be a pure morphism of L-structures, and let $\mathcal{F}$ be a set of (not necessarily definable) functions from M to C^M, where C^M is the set of interpretations in M of the elements of C.*

Let $L(\mathcal{F})$ be the language obtained from L by adding unary function symbols $\underline{f}$ for every function $f \in \mathcal{F}$.

Then we can expand N into an $L(\mathcal{F})$-structure such that ξ is a pure $L(\mathcal{F})$-morphism. Moreover, if the induced morphism $\xi \restriction C^M : C^M \to C^N$ is an L-isomorphism, then for every $f \in \mathcal{F}$ which is an L-morphism, we can choose the interpretation $\underline{f}^N$ in N such that $\underline{f}^N$ is an L-morphism.

Proof. Let Γ be the set of conjunctions of atomic L-formulas, M_1 be the $L(\mathcal{F})$-structure obtained from M by adding the functions in $\mathcal{F}$, and N_1 be any expansion of N which is an $L(\mathcal{F})$-structure.
Then $f : M_1 \to N_1$ is a map satisfying the first property of corollary 2. Hence there exist an $L(\mathcal{F})$-elementary extension M_1' of M_1 and a Γ-morphism (i.e. an L-morphism) $\mu : N_1 \to M_1'$ such that the following diagram is commutative:



where i denotes the elementary embedding of M_1 in M_1'.

We now change the interpretation in N of the elements of $\mathcal{F}$. For $f \in \mathcal{F}$ and $n \in N$ we have $\underline{f}^{M_1'} \circ \mu(n) = c_n^{M_1'}$, for some $c_n \in C$ ($c_n^{M_1'}$ denotes the interpretation of c_n in M_1'). We then interpret f in N by $\underline{f}^N(n) = c_n^N$ (the interpretation of c_n in N). This turns N into a new $L(\mathcal{F})$-structure, and ξ and μ become $L(\mathcal{F})$-morphisms.

- For ξ: Let $m \in M$. Then $\underline{f}^M(m) = c^M$ for some $c \in C$, which gives $\underline{f}^{M'}(i(m)) = c^{M'}$ and applying the definition of $\underline{f}^N$ we get $\underline{f}^N(\xi(m)) = c^N$. So $\xi(\underline{f}^M(m)) = \underline{f}^N(\xi(m))$.

- For μ: Let $n \in N$. Then $\underline{f}^N(n) = c^N$, for some $c \in C$ such that $\underline{f}^{M'} \circ \mu(n) = c^{M'}$. So $\mu(\underline{f}^N(n)) = \mu(c^N) = c^{M'}$.

Since i is an elementary $L(\mathcal{F})$-morphism, we get that ξ is a pure $L(\mathcal{F})$-morphism.

We now check the last assertion of the proposition. Assume that $\xi \restriction C^M : C^M \to C^N$ is an isomorphism. Since i induces an isomorphism of L-structures from $C^M = C^{M_1}$ to $C^{M_1'}$, we can assume that M_1, M_1' and N all contain a copy of C^{M_1}.

Let $f \in \mathcal{F}$ such that f is an L-morphism. Then $\underline{f}^{M_1'}$ is an L-morphism. Since we identify $C^{M_1'}$ and C^N with C^{M_1}, the above definition of $\underline{f}^N$ becomes $\underline{f}^N(n) = \underline{f}^{M_1'} \circ \mu(n)$, and $\underline{f}^N$ is an L-morphism since $\underline{f}^{M_1'}$ and μ are. □

2.2 Testing pp-formulas on subspaces of bounded cardinality

Lemma 2 *Let G be a special group and let H be a group of exponent 2. Let $a\delta \in G[H]$, where $a \in G$ and $\delta \in H \setminus \{1\}$. Then there exists an automorphism of special groups $\xi : G[H] \to G[H]$ such that $\xi(a\delta) \in H$.*

Proof. Let π_H be the canonical projection from $G[H]$ onto H and let H' be a subgroup of $G[H]$ such that $GH' = GH$, $H' \cap G = \{1\}$, $a\delta \in H'$ and $\pi_H(h') = 1 \Leftrightarrow h' = 1$ for every $h' \in H'$ (if $\{\delta, h_2, h_3, \ldots\}$ is a basis of H, take $\{a\delta, h_2, h_3, \ldots\}$ as a basis of H'). Let $\lambda : H \to H'$ be an automorphism of groups such that $\lambda(\delta) = a\delta$. Define:

$$\begin{array}{rcl} G[H] & \xrightarrow{\xi} & G[H]. \\ gh & \mapsto & g\lambda(h) \end{array}$$

ξ is a morphism of groups and $\xi(-1) = -1$. We check that ξ respects the isometry:
Let $gh, g'h' \in G[H]$ such that $gh \in D\langle 1, g'h'\rangle$. We have to consider three cases:

- $h' = 1$ and $g = -1$. Then $\xi(g'h') = -1.1$, so $D\langle 1, \xi(g'h')\rangle = G[H]$, which gives $\xi(gh) \in D\langle 1, \xi(g'h')\rangle = G[H]$.

- $h' = 1$ and $g' \neq -1$. Then $D\langle 1, g'h'\rangle = D_G\langle 1, g'\rangle \times \{1\}$ and the hypothesis gives $g \in D_G\langle 1, g'\rangle$ and $h = 1$. Since $D\langle 1, \xi(g'h')\rangle = D\langle 1, g'.1\rangle = D_G\langle 1, g\rangle \times \{1\}$ we have $\xi(gh) \in D\langle 1, \xi(g'h')\rangle$.

- $h' \neq 1$. Then $D\langle 1, g'h'\rangle = \{1, g'h'\}$, so $gh = 1$ or $gh = g'h'$. In either case we get $\xi(gh) \in D\langle 1, \xi(g'h')\rangle$.

Then ξ is a morphism of special groups.
We check in a similar way that ξ^{-1} is a morphism of special groups. $\square$

The following result, due to M. Marshall, can be found pp. 12-13 in [8] which is yet unpublished. We include it here for completeness.

Lemma 3 (Marshall) *Let $\Phi(\bar{z})$ be a pp-formula with at most m quantifiers, and let $\bar{h} \in \Delta$, where Δ is a group of exponent 2 generated by $\bar{h}$. Suppose $l(\bar{z}) = n$.*
Then there exist pp-formulas $\Phi_{\bar{\delta}}(\bar{z})$ for $\bar{\delta} \in \Delta^n$ such that each $\Phi_{\bar{\delta}}(\bar{z})$ has at most m quantifiers and for every reduced special group G and every $\bar{g} \in G$ we have:

$$G[\Delta] \not\models \Phi(\bar{g}\bar{h})$$
$$\textit{if and only if}$$
$$\textit{for every } \bar{\delta} \in \Delta,\ G \not\models \Phi_{\bar{\delta}}(\bar{g})).$$

Proof. Let π_G, π_H be the canonical projections from $G[\Delta]$ onto G, Δ respectively. Say that $\Phi(\bar{z})$ has the form $\exists \bar{x}\ \vartheta(\bar{z}, \bar{x})$, where $\vartheta(\bar{z}, \bar{x}) = \bigwedge_{i=1}^{k} \gamma_i(\bar{z}, \bar{x})$ with $\gamma_1, \ldots, \gamma_k$ atomic formulas.
For simplicity we can assume that all the γ_i are of the form $t_i(\bar{z}, \bar{x}) \in D\langle 1, s_i(\bar{z}, \bar{x})\rangle$, since an equality $a = 1$ is given by $a \in D\langle 1, 1\rangle$.
For $\bar{\delta} \in \Delta^n$ let:

$$\Phi'(\bar{z}, \bar{\delta}) = \exists \bar{u} \in G\ \vartheta(\bar{z}, \bar{u}\bar{\delta}) = \exists \bar{u} \in G\ \bigwedge_{i=1}^{k} \gamma_i(\bar{z}, \bar{u}\bar{\delta}).$$

We have for every reduced special group G and $\bar{g} \in G$:
$G[\Delta] \models \Phi(\bar{g}\bar{h})$ if and only if there exists $\bar{\delta} \in \Delta^n$ such that $G[\Delta] \models \Phi'(\bar{g}\bar{h}, \bar{\delta})$.
All we have to do is to find pp-formulas $\Phi_{\bar{\delta}}$ such that for every reduced special group G and every $\bar{g} \in G$, $G[\Delta] \models \Phi'(\bar{g}\bar{h}, \bar{\delta})$ if and only if $G \models \Phi_{\bar{\delta}}(\bar{g})$.
For this we consider an atomic formula $\gamma(\bar{z}, \bar{u}\bar{\delta}) =$ "$t(\bar{z}, \bar{u}\bar{\delta}) \in D\langle 1, s(\bar{z}, \bar{u}\bar{\delta})''$ from $\Phi'(\bar{z}, \bar{\delta})$, and we use the definition of representation for extensions. This leads us to consider several cases according to the coordinates on Δ of the different terms (remark that for $\bar{u}, \bar{z} \in G$ we have $\pi_H(t(\bar{z}\bar{h}, \bar{u}\bar{\delta})) = t(\bar{h}, \bar{\delta})$ and $\pi_H(s(\bar{z}\bar{h}, \bar{u}\bar{\delta})) = s(\bar{h}, \bar{\delta})$):

1. $t(\bar{h}, \bar{\delta}) = 1 = s(\bar{h}, \bar{\delta})$. In that case we have, for every reduced special group G and every $\bar{g} \in G$:

$$t(\bar{g}\bar{h}, \bar{u}\bar{\delta}) \in D\langle 1, s(\bar{g}\bar{h}, \bar{u}\bar{\delta})\rangle$$
$$\Leftrightarrow$$
$$\pi_G(t(\bar{g}\bar{h}, \bar{u}\bar{\delta})) \in D\langle 1, \pi_G(s(\bar{g}\bar{h}, \bar{u}\bar{\delta}))\rangle$$
$$\Leftrightarrow$$
$$t(\bar{g}, \bar{u}) \in D\langle 1, s(\bar{g}, \bar{u})\rangle$$

We define $\gamma_{\bar{\delta}}(\bar{g},\bar{u}) =$ "$t(\bar{g},\bar{u}) \in D\langle 1, s(\bar{g},\bar{u})\rangle$".

2. $t(\bar{h},\bar{\delta}) = 1 \neq s(\bar{h},\bar{\delta})$. Then for every reduced special group G and every $\bar{g} \in G$ we have $D\langle 1, s(\bar{g}\bar{h},\bar{u}\bar{\delta})\rangle = \{1, s(\bar{g}\bar{h},\bar{u}\bar{\delta})\}$ so:

$$t(\bar{g}\bar{h},\bar{u}\bar{\delta}) \in D\langle 1, s(\bar{g}\bar{h},\bar{u}\bar{\delta})\rangle \\ \Leftrightarrow \\ t(\bar{g}\bar{h},\bar{u}\bar{\delta}) = 1 \\ \Leftrightarrow \\ t(\bar{g},\bar{u}) = 1,$$

   and the last equivalence holds because $\pi_h(t(\bar{g}\bar{h},\bar{u}\bar{\delta})) = 1$, so $t(\bar{g}\bar{h},\bar{u}\bar{\delta}) = 1$ if and only if $\pi_G(t(\bar{g}\bar{h},\bar{u}\bar{\delta})) = 1$, i.e. $t(\bar{g},\bar{u}) = 1$.
   We define $\gamma_{\bar{\delta}}(\bar{g},\bar{u}) =$ "$t(\bar{g},\bar{u}) = 1$".

3. $t(\bar{h},\bar{\delta}) \neq 1 = s(\bar{h},\bar{\delta})$. Then $D\langle 1, s(\bar{g}\bar{h},\bar{u}\bar{\delta})\rangle$ is either included in G (if $s(\bar{g}\bar{h},\bar{u}\bar{\delta}) \neq -1$) or equal to $G[H]$ (if $s(\bar{g}\bar{h},\bar{u}\bar{\delta}) = -1$). The first case is impossible since $\pi_H(t(\bar{g}\bar{h},\bar{\delta})) = t(\bar{h},\bar{\delta}) \neq 1$, so for every reduced special group G and every $\bar{g} \in G$:

$$t(\bar{g}\bar{h},\bar{u}\bar{\delta}) \in D\langle 1, s(\bar{g}\bar{h},\bar{u}\bar{\delta})\rangle \\ \Leftrightarrow \\ s(\bar{g}\bar{h},\bar{u}\bar{\delta}) = -1 \\ \Leftrightarrow \\ s(\bar{g},\bar{u}) = -1.$$

   We define $\gamma_{\bar{\delta}}(\bar{g},\bar{u}) =$ "$s(\bar{g},\bar{u}) = -1$".

4. $t(\bar{h},\bar{\delta})$, 1 and $s(\bar{h},\bar{\delta})$ are all different. This is impossible, and we define $\gamma_{\bar{\delta}}(\bar{g},\bar{u}) =$ "$1 = -1$".

We then define $\Phi_{\bar{\delta}}(\bar{g})$ to be $\exists \bar{u}\ \bigwedge_{i=1}^{k} \gamma_{i\bar{\delta}}(\bar{g},\bar{u})$.
By construction of the $\gamma_{i\bar{\delta}}$ we have, for every reduced special group G and every $\bar{g} \in G$, $G[\Delta] \models \Phi(\bar{g}\bar{h})$ if and only if there exists $\bar{\delta} \in \Delta$ such that $G \models \Phi_{\bar{\delta}}(\bar{g})$. □

Lemma 4 *There is a natural number $B(m,n)$ such that for every reduced special group G and every pp-formula $\Phi(\bar{g})$ with parameters $\bar{g} \in G^n$ with at most m quantifiers the following holds.*

If Z is a finite subspace of X_G with $Z \not\models \Phi(\bar{g})$, then there is a subspace Y of X_G such that $|Y| \leq B(m,n)$ and $Y \not\models \Phi(\bar{g})$.

PROOF. We have $Z \not\models \Phi(\bar{g})$, i.e. $G_Z \not\models \Phi(\bar{g})$. If we show that there is a subspace Y of Z (so a subspace of X_G) such that $\operatorname{card} Y \le B(m,n)$ such that $Y \not\models \Phi(\bar{g})$, then the result will be proved.

Since G_Z is a finite reduced special group, it is built up from $\mathbb{Z}_2$ using a finite number of times the operations of product and extension (see [5] for the corresponding result for spaces of orderings), and we can proceed by induction on the construction of G_Z (we essentially follow the argument pp. 12-13 in [8]).

We first define $B(m,n)$ inductively by $B(m,0) = 1$ for every $m \in \mathbb{N}$ and $B(m,n) = 2^n 2^{2^{mnB(m,n-1)}}$ if $n \ge 1$. The induction on the construction of G_Z is:

- $G_Z = \mathbb{Z}_2$. Since 1 and -1 are in the language, we can assume $n = 0$, and we take for Y the whole space of orderings of G_Z, which consists of the identity from G_Z onto itself. The required properties are satisfied.

- $G_Z = G_1 \times G_2$ for G_1, G_2 reduced special groups. Since $G_Z \not\models \Phi(\bar{g})$ we have for example $G_1 \not\models \Phi(\pi_1(\bar{g}_1))$, where π_1 is the canonical projection from G_Z onto G_1. Then by induction we have a finite subspace Y of X_{G_1} such that $|Y| \le B(m,n)$ and $Y \not\models \Phi(\pi_1(\bar{g}))$. But Y can also be seen as a subspace of $Z = X_{G_Z}$, which gives $Y \not\models \Phi(\bar{g})$.

- $G_Z = G_1[H]$. We distinguish two cases:
  The first one is when $\bar{g} = \{g_1, \dots, g_n\} \subseteq G_1$. Then $G_1 \not\models \Phi(\bar{g})$ and by induction we have a subspace Y of X_{G_1} such that $|Y| \le B(m,n)$ such that $Y \not\models \Phi(\bar{g})$. Since Y can also be seen as a subspace of $Z = X_{G_Z}$, the result is proved in this case.

  If $\{g_1, \dots, g_n\} \subsetneq G_1$, say if $g_n \in G_1[H] \setminus G_1$. By lemma 2 there is an automorphism of special groups $\xi : G_Z \to G_Z$ such that $\xi(g_n) \in H$, so we can assume that $g_n \in H$. We write g_i as $a_i h_i$ with $a_i \in G_1$ and $h_i \in H$ (so $a_n = 1$). Let Δ be the subgroup of H generated by $\{h_1, \dots, h_n\}$.
  We have $G_1[\Delta] \not\models \Phi(\bar{g})$. Using 3, there are pp-formulas $\Phi_{\bar{\delta}}$ with m existential quantifiers, for $\bar{\delta} \in \Delta^m$, such that for every reduced special group K and every $\bar{k} \in K$:

  $$K[\Delta] \not\models \Phi(\bar{k}\bar{h}) \quad \text{if and only if} \quad \forall \bar{\delta} \in \Delta \quad K \not\models \Phi_{\bar{\delta}}(\bar{k}). \qquad (1)$$

  We first apply this for $K = G_1$ and $\bar{k} = \bar{a}$. Note that every formula $\Phi_\delta(\bar{a})$ has $n-1$ parameters since $a_n = 1$.
  By induction we then have, for every $\bar{\delta} \in \Delta$, a subspace $Y_{\bar{\delta}}$ of X_{G_1} such that $|Y_{\bar{\delta}}| \le B(m, n-1)$ and $Y_{\bar{\delta}} \not\models \Phi_{\bar{\delta}}(\bar{a})$.

Let Y_0 be the subspace of X_{G_1} generated by $\cup_{\bar{\delta}\in\Delta^m} Y_{\bar{\delta}}$. Then $|Y_0| \leq 2^{2^{mn}B(m,n-1)}$ since $|\cup_{\bar{\delta}\in\Delta} Y_{\bar{\delta}}| \leq 2^{mn}B(m,n-1)$.
We now take for Y the extension of Y_0 by Δ. Then by (1), $Y \not\models \Phi(\bar{g})$ and $|Y| \leq |Y_0|2^n = B(m,n)$. □

2.3 Satisfying a pp-formula on finite subspaces.

In this section we fix an n-tuple $\bar{y}$ of variables and a pp-formula $\Phi(\bar{y})$ with m quantifiers. Hence $\Phi(\bar{y}) = \exists x_1 \dots x_m\ \vartheta(\bar{y},\bar{x})$, where $\vartheta(\bar{y},\bar{x})$ is a conjunction of atomic L_{SG}-formulas.

Lemma 5 *Let $\Phi(\bar{y})$ be a pp-formula with m quantifiers and n free variables in the language of special groups. Let G and H be reduced special groups, let $\bar{g} \in G^n$ and let $\xi : G \to H$ be a morphism of special groups. If $Z \subseteq X_H$ is a finite subset such that $Z \not\models \Phi(\xi(\bar{g}))$, then there exists a subspace Y of X_G of cardinality at most $B(m,n)$ such that $Y \not\models \Phi(\bar{g})$.*

Proof. Take $W = \{\sigma \circ \xi \mid \sigma \in Z\}$. W is finite and $W \not\models \Phi(\bar{g})$, so is included in a finite subspace W' such that $W' \not\models \Phi(\bar{g})$. The conclusion follows by lemma 4. □

We come to our main result.

Theorem 2 *For every pp-formula $\Phi(\bar{y})$ with m quantifiers and n free variables $\bar{y} = (y_1,\dots,y_n)$ there is a set $\{\Phi_i(\bar{y}) \mid i \in I\}$ of pp-formulas such that for every reduced special group G and every n-tuple $\bar{g} \in G^n$ the following are equivalent:*

1. *For all finite subspaces Y of X_G we have $Y \models \Phi(\bar{g})$.*
2. *For all subspaces Y of X_G with card$Y \leq B(m,n)$ we have $Y \models \Phi(\bar{g})$. (the natural number $B(m,n)$ is defined in Lemma 4)*
3. *There is some $i \in I$ with $G \models \Phi_i(\bar{g})$.*
4. *If $f : G \to H$ is a morphism of special groups of G into an ultraproduct H of finite reduced special groups, then $H \models \Phi(f(\bar{g}))$.*
5. *Let H be $(\prod_{Y\in J} G_Y)/\mathcal{U}$, where J is the set of finite subspaces of X and $\mathcal{U}$ is any ultrafilter containing all the sets $\{Y \in J \mid Y \supseteq Z\}$ $(Z \in J)$ (here $G_Y := G/Y^\perp$ for $Y \in J$).*

   Then $H \models \Phi(f(\bar{g}))$, where $f : G \to H$ is the canonical morphism from G into H.

Proof. For a reduced special group G let P_G be the set of all $\bar{g} \in G^n$ such that there is a subspace Y of X_G with $|Y| \leq B(m,n)$ and $Y \not\models \Phi(\bar{g})$. We show that G and P_G fulfill the requirements of 1 for the set Γ of finite conjunctions of atomic L_{SG}-formulas. Condition 2 of 1 is fulfilled by 5. In order to see condition 1 of 1, let $\mathcal{U}$ be an ultrafilter on some set J, let G_j be a reduced special group and let $\bar{g}_j \in P_{G_j}$ for each $j \in J$. For each $j \in J$ let $Y_j \subseteq X_{G_j}$ be of size at most $B(m,n)$ such that $Y_j \not\models \Phi(\bar{g}_j)$. Let $Y_j = \{\sigma_{j\alpha} \mid \alpha \leq B(m,n)\}$. Then $Z := \{\prod \sigma_{j\alpha}/\mathcal{U} \mid \alpha \leq B(mn,)\}$ is of size at most $B(m,n)$ and $Z \not\models \Phi(\prod \bar{g}_j/\mathcal{U})$. We then get a subspace Y with the required properties by applying lemma 4.

We then take for the Φ_i the formulas given by 1. This gives in particular that 2.⇔3. holds.

2.⇔1. is lemma 4. and 4.⇒5. is obvious.

5.⇒1. Suppose $Y \not\models \Phi(\bar{g})$ for some finite subspace Y of X_G. Let J be the set of finite subspaces of X_G. We denote by G_Y the finite reduced special group corresponding to a finite subspace Y of X_G ($G_Y = G/Y^{\perp}$).
For $Z \in J$ let $S_Z = \{Y \in J \mid Y \supseteq Z\}$. The sets S_Z form a filter basis. Let $\mathcal{U}$ be an ultrafilter containing the S_Z, for $Z \in J$, and let $f : G \to H := (\prod_{Y \in J} G_Y)/\mathcal{U}$ be the natural map induced by the quotients $G \to G_Y$. Clearly f is a morphism of special groups.

It remains to show that $H \not\models \Phi(f(\bar{g}))$. If Z is a finite subspace of X_G containing Y then $Z \not\models \Phi(\bar{g})$. In particular, if $A = \{Z \in J \mid G_Z \not\models \Phi(\bar{g} \bmod Z^{\perp})\}$ then $A \supseteq S_Y$, so $A \in \mathcal{U}$. This proves $H \not\models \Phi(f(\bar{g}))$.

1.⇒4. Let $f : G \to H := \prod_{j \in J} G_j/\mathcal{U}$ be a morphism of special groups, where all G_j are finite reduced special groups, such that $H \not\models \Phi(f(\bar{g}))$. We have to find a finite subspace Y of X_G such that $Y \not\models \Phi(\bar{g})$. Let $f(\bar{g}) := (\bar{g}_j)/\mathcal{U}$. Then $A := \{j \in J \mid G_j \not\models \Phi_j(\bar{g}_j)\} \in \mathcal{U}$. By 4, for each $j \in J$ there is a subset Y_j of X_{G_j} of size at most $B(m,n)$ with $Y_j \not\models \Phi(\bar{g}_j)$. Let $Y_j = \{\sigma_{j\alpha} \mid \alpha \leq B(m,n)\}$. Then $Z := \{\prod \sigma_{j\alpha}/\mathcal{U} \mid \alpha \leq B(mn,)\}$ is of size at most $B(m,n)$ and $Z \not\models \Phi(\prod \bar{g}_j/\mathcal{U})$. We then get a subspace Y with the required properties by applying lemma 4. □

Remark: The equivalent conditions of the theorem imply that the property "there is a finite subspace Y with $Y \not\models \Phi(\bar{x})$" is axiomatized by the set of formulas $\{\neg\Phi_i(\bar{x}) \mid i \in I\}$.

3 Applications

Applying theorem 2 to (LG($\Phi, \bar{g}$)) gives us the following.

Theorem 1 *Let $\Phi(\bar{y})$ be a pp-formula with m quantifiers in n free variables $\bar{y} = (y_1, \ldots, y_n)$. Then there is a set $\{\Phi_i(\bar{y}) \mid i \in I\}$ of pp-formulas such that for every reduced special group G and every n-tuple $\bar{g} \in G^n$ the following are equivalent:*

1. *Either $G \models \Phi(\bar{g})$ or there is a finite subspace Y of X_G such that $Y \not\models \Phi(\bar{g})$.*
2. *Either $G \models \Phi(\bar{g})$ or there is a subspace Y of X_G of size at most $B(m,n)$ such that $Y \not\models \Phi(\bar{g})$.*
3. *For each $i \in I$, $G \models \Phi_i(\bar{g}) \to \Phi(\bar{g})$.*
4. *There is a morphism $f : G \to H$ of special groups where H is an ultraproduct of finite reduced special groups, such that $G \models \Phi(\bar{g})$ if $H \models \Phi(f(\bar{g}))$.*
5. *Let H be $(\prod_{Y \in J} G_Y)/\mathcal{U}$, where J is the set of finite subspaces of X and $\mathcal{U}$ is any ultrafilter containing all the sets $\{Y \in J \mid Y \supseteq Z\}$ $(Z \in J)$.*

   If $H \models \Phi(f(\bar{g}))$, then $G \models \Phi(\bar{g})$, where $f : G \to H$ is the canonical morphism from G into H.

Proof. This is an immediate consequence of theorem 2. □

Remark: The equivalence of 1. and 3. says that the property $(\mathrm{LG}(\Phi, \bar{g}))$ is axiomatized by the set of formulas $\{\Phi_i(\bar{g}) \to \Phi(\bar{g}) \mid i \in I\}$.

3.1 Pure morphisms of reduced special groups

We now consider how a positive answer to (LG) or to $(LG(\Phi, \bar{g}))$ is preserved under pure morphisms, and we start with a result showing that in presence of pure morphisms of reduced special groups, it is always possible to extend the language:

Corollary 3 *Let G and H be reduced special groups such that G is an L_{SG}-substructure of H, and the inclusion of G in H is pure.*
Then every $\sigma \in X_G$ can be extended to $\sigma^H \in X_H$ in such a way that the inclusion from G into H becomes pure in the language $L_{SG} \cup \{\sigma\}_{\sigma \in X_G}$.

Proof. We use proposition 2. Since G and H are reduced special groups, the structures induced by G on $\{1^G, -1^G\}$ and by H on $\{1^H, -1^H\}$ are both isometric to $\mathbb{Z}_2$. So by proposition 2, with $M = G$, $N = H$, $L = L_{SG}$ and $\mathcal{F} = \mathcal{X}_G$ we get the corollary. □

Proposition 3 *Let G and H be reduced special groups such that G is an L_{SG}-substructure of H and such that the inclusion $\xi : G \to H$ is pure. Let $\Phi(\bar{x})$ be a pp-formula. Then H satisfies* $(\mathrm{LG}(\Phi, \xi(\bar{g}))$ *if and only if G satisfies* $(\mathrm{LG}(\Phi, \bar{g}))$.

PROOF. This is a consequence of condition 4. in 1. □

Corollary 4 *Let G be a reduced special group. Then G satisfies* (LG) *for pp-formulas with only one parameter in G.*

PROOF. Let $\Phi(g)$ be a pp-formula with parameter g. Let $K = \{1, g, -g, -1\}$, equipped with the isometry induced by G. Then K is a special subgroup of G.

Let $\sigma_1, \sigma_2 \in X_G$ be such that $\sigma_1(g) = -1$ and $\sigma_2(-g) = -1$, and let $\Delta = ker(\sigma_1) \cap ker(\sigma_2)$. Δ is a saturated subgroup of G of index 4. Then G/Δ is a reduced special group with 4 elements and is equal to $G/\Delta = \{1/\Delta, g/\Delta, -g/\Delta, -1/\Delta\}$. Since there is only one 4-elements reduced special group up to isomorphism, we have an isomorphism:

$$\begin{array}{ccc} K & \to & G/\Delta \\ x & \longmapsto & x/\Delta \end{array} .$$

This shows that the projection $G \to G/\Delta$ is a retract of the inclusion of K in G, which is then pure. Since K is finite, it satisfies (LG) and we can apply proposition 3. □

3.2 Pseudo finite special groups

Let $\mathcal{F} = \{F_i \mid i \in \kappa\}$ be the set of all finite special groups. For $F_i \in \mathcal{F}$ let $Th(F_i)$ be the first-order theory of F_i (without parameters) in the language L_{SG}.

Definition 3 *A pseudo finite special group is a model of* $\bigcap_{i \in \kappa} Th(F_i)$.

Remark: Using lemma 1, we see that a reduced special group G is pseudo finite if and only if it is an elementary substructure of an ultraproduct of finite special groups.

Proposition 4 *Let G be a reduced special group. Then G satisfies* (LG) *if and only if G is a pure substructure of an ultraproduct of finite reduced special groups.*

In particular, pseudo finite reduced special groups satisfy (LG).

PROOF. This is 1. $\Leftrightarrow$ 4. $\Leftrightarrow$ 5. in 1. □

3.3 Profinite special groups

Profinite reduced special groups are inverse limits of finite reduced special groups and are dual to direct limits of finite spaces of orderings which have been studied in [4]. The following result is an easy adaptation of the proof of theorem 1.1 in [4].

Lemma 6 *Let $G = \varprojlim G_i$ be the inverse limit of an inverse system of finite reduced special groups $\{G_i\}_{i\in I}$. Then the canonical inclusion $\varinjlim G_i \to \prod_{i\in I} G_i$ is pure.*

Proof. Let $\bar{g} = (\bar{g}_i)_{i\in I} \in G$ and $\Phi(\bar{g})$ be a pp-formula such that $\prod_{i\in I} G_i \models \Phi(\bar{g})$. Assume that $\Phi(\bar{g})$ is of the form $\exists\bar{x}\ \vartheta(\bar{g},\bar{x})$ where ϑ is a conjunction of atomic formulas. Then there exists $\bar{a} \in \prod_{i\in I} G_i$ such that $\prod_{i\in I} G_i \models \vartheta(\bar{g},\bar{a})$, i.e. $G_i \models \vartheta(\bar{g}_i,\bar{a}_i)$ for every $i \in I$.
For $i \le j \in I$ let $V_{ij} = \{(a_i) \in \prod_{i\in I} G_i \mid \pi_{ji}(a_j) = a_i\}$. V_{ij} is closed in $\prod_{i\in I} G_i$ (equipped with the product topology), and $\varprojlim G_i = \bigcap_{i\le j\in I} V_{ij}$.
For $i \in I$ let $\vartheta(\bar{g}_i, G_i) = \{\bar{x} \in G_i \mid G_i \models \vartheta(\bar{g}_i,\bar{x})\}$ and $\vartheta(\bar{g}, \prod_{i\in I} G_i) = \{\bar{x} \in \prod_{i\in I} G_i \mid \prod_{i\in I} G_i \models \vartheta(\bar{g},\bar{x})\}$. $\vartheta(\bar{g}_i, G_i)$ is closed in G_i, so $\vartheta(\bar{g}, \prod_{i\in I} G_i)$ is closed in $\prod_{i\in I} G_i$.
For $i \le j \in I$ define $\bar{b} \in \prod_{i\in I} G_i$ by $b_k = a_k$ if $k \neq i$ and $b_i = \pi_{ji}(a_j)$. We have $G_k \models \vartheta_k(\bar{g}_k, \bar{b}_k)$ for every $k \in I$ (it is clear for $k \neq i$, and for $k = i$ it follows from $G_j \models \vartheta(\bar{g}_j,\bar{a}_j)$ since π_{ji} is a morphism of special groups and $\pi_{ji}(a_j) = a_i$). This shows that $\bar{b} \in \vartheta(\bar{g}, \prod_{i\in I} G_i) \cap V_{ji}$. We check similarly that if $i_1 \le j_1, \ldots, i_n \le j_n \in I$ then $\vartheta(\bar{g}, \prod_{i\in I} G_i) \cap V_{j_1 i_1} \cap \cdots \cap V_{j_n i_n} \neq \emptyset$. By compactness we then have $\vartheta(\bar{g}, \prod_{i\in I} G_i) \bigcap_{i\le j\in I} V_{ji} \neq \emptyset$, which shows $\varprojlim G_i \models \Phi(\bar{g})$. □

Corollary 5 *Let $G = \varprojlim G_i$ be the inverse limit of an inverse system of finite reduced special groups. Then G is pure in an ultraproduct of finite reduced special groups. In particular G satisfies* (LG).

Proof. For $i \in I$ let $S_i = \{j \in I \mid j \le i\}$. The sets S_i form a filter basis, and are then all included in some ultrafilter $\mathcal{U}$.
Let:

$$\begin{aligned} \xi : \varprojlim G_i &\to \textstyle\prod_{i\in I} G_i/\mathcal{U}. \\ (g_i)_{i\in I} &\mapsto (g_i)_{i\in I}/\mathcal{U} \end{aligned}$$

We show that ξ is pure, and the result will follow by proposition 3. Let $\Phi(\bar{g})$ be a pp-formula with parameters in G such that $\prod_{i\in I} G_i/\mathcal{U} \models \Phi(\xi(\bar{g}))$.

Suppose there exists $i \in I$ such that $G_i \not\models \Phi(\bar{g}_i)$. Let $j \in S_i$, i.e. $j \geq i$. We have a morphism of special groups $\pi_{ji} : G_j \to G_i$ such that $\pi_{ji}(g_j) = g_i$. This implies $G_j \not\models \Phi(\bar{g}_j)$ (otherwise we would get $G_i \models \Phi(\bar{g}_i)$ by applying π_{ji}). Thus $G_j \not\models \Phi(\bar{g}_j)$ for every $j \in S_i$, which gives $\prod_{i \in I} G_i/\mathcal{U} \not\models \Phi(\xi(\bar{g}))$, a contradiction.
So $G_i \models \Phi(\bar{g}_i)$ for every $i \in I$, that is $\prod_{i \in I} G_i \models \Phi(\bar{g}))$. By lemma 6 we get $\varprojlim G_i \models \Phi(\bar{g})$. □

Remark: It is also possible to get this result by considering (LG) as a kind of compactness property: since a finite subset of X_G generates a finite subspace of X_G, (LG) is true for G if and only if every positive-primitive formula $\Phi(\bar{g})$ with parameters in G which holds in every finite subset of X_G holds in X_G. Let $\Phi(\bar{g})$ be a pp-formula with parameters in G. $\Phi(\bar{g})$ has the form $\exists \bar{x}\, \vartheta(\bar{g}, \bar{x})$, where $\vartheta(\bar{g}, \bar{x})$ is a conjunction of atomic formulas.

The property $(\mathrm{LG}(\Phi, \bar{g}))$ is an implication. The hypothesis is that for every finite subset Y of X_G, there is some $\bar{b} \in G$ such that $\vartheta(\sigma(\bar{g}), \sigma(\bar{b}))$ is true in G for every $\sigma \in Y$. In other words, the set of formulas $\{\vartheta(\sigma(\bar{g}), \sigma(\bar{x})) \mid \sigma \in X_G\}$ is finitely satisfied in G (this is a set of formulas in the language $L_{SG} \cup \{\sigma\}_{\sigma \in X_G}$). The conclusion of (LG) gives some $\bar{b}$ in G such that $\vartheta(\sigma(\bar{g}), \sigma(\bar{b}))$ is true in G for every $\sigma \in X_G$. In other words, the set of formulas $\{\vartheta(\sigma(\bar{g}), \sigma(\bar{x})) \mid \sigma \in X_G\}$ has a solution in G.

It is easy to check that if G is a compact topological group in which $\ker(\sigma)$ is closed for every $\sigma \in X_G$, then the set of elements of G^n satisfying the formulas $\vartheta(\sigma_1(\bar{g}), \sigma_1(\bar{x})), \ldots, \vartheta(\sigma_k(\bar{g}), \sigma_k(\bar{x}))$ is closed, and we then conclude by compactness that the set of formulas $\{\vartheta(\sigma(\bar{g}), \sigma(\bar{x})) \mid \sigma \in X_G\}$ has a solution in G.

Since for a profinite reduced special group G the set of $\sigma \in X_G$ that are continuous is dense in X_G, a simple modification of the argument above shows that G satisfies (LG).

Using corollary 5 together with proposition 3, we get that a reduced special group G satisfies (LG) if G is pure in a profinite reduced special group. Hence

Proposition 5 *G satisfies* (LG) *if and only if G is pure (and also dense) in a profinite reduced special group.*

Proof. "⇒" The reduced special groups G_Y (for Y a finite subspace of X_G), together with the canonical projections $G_Y \to G_Z$ (if $Z \subseteq Y$), form an inverse system. Its limit $\varprojlim G_Y$ is a profinite special group in which G is pure (using that G satisfies (LG)) and dense.
"⇐" This is proposition 3. □

Corollary 6 *If G is a pseudo finite reduced special group then G is pure (and dense) in a a profinite reduced special group.*

Proof. By propositions 4 and 5 since pseudo finite reduced special groups satisfy (LG). $\square$

3.4 Further consequences

We start with an easy remark about the isotropy theorem, but which can be applied to any other such result:
Since the L_{SG}-formula expressing that a quadratic form p is isotropic is positive-primitive, the simple fact that the isotropy theorem is true, together with 1 gives a bound on the cardinality of the finite subspaces involved in the isotropy theorem (since the number of free variables and quantifiers in the formula "p is isotropic" only depends on the dimension of p). More precisely, there exists a map $K : \mathbb{N} \to \mathbb{N}$ such that for every reduced special group G and every anisotropic quadratic form f over G with $\dim f = n$, there exists a finite subspace Y of X_G such that $\operatorname{card} Y \leq K(n)$ and f is anisotropic over Y. Note that the general nature of 1 does not give as sharp a bound as the one already known for the isotropy theorem (see [1], theorem 6.4 p. 114).

Proposition 6 *Let $\Phi(\bar{g})$ be a pp-formula in the language $L_{SG}(\bar{g}) = L_{SG} \cup \{\bar{g}\}$.*

1. *The class of reduced special groups satisfying* (LG($\Phi, \bar{g}$)) *is closed by direct limits (the morphisms being $L_{SG}(\bar{g})$-morphisms).*

2. *If G is a reduced special group satisfying* (LG) *and Δ is a saturated subgroup of G, then the quotient G/Δ satisfies* (LG) *(in terms of spaces of orderings: the class of spaces of orderings satisfying* (LG) *is closed by subspaces).*

Proof. Item 1. is a consequence of point 4. in 1.

2. We first consider the case $\Delta = D_G(p)$, where $p = \langle\langle a_1, \ldots, a_n \rangle\rangle$ is a Pfister form with coefficients $\bar{a} \in G$.

The formula $x \in D(p)$ is a pp-formula, and for every atomic formula $a \in D\langle 1, b\rangle$ we have $G/\Delta \models a/\Delta \in D\langle 1, b/\Delta\rangle$ if and only if $G \models \exists x, y \;\; x, y \in Dp \wedge ax \in D\langle 1, by\rangle$, which is a pp-formula. Replacing each atomic formula in Φ by a pp-formula as above, we get a new pp-formula $\Phi'(\bar{a}, \bar{g})$ such that $G/\Delta \models \Phi(\bar{g}/\Delta)$ if and only if $G \models \Phi'(\bar{a}, \bar{g})$.

Assume now that for every finite subspace Y of $X_G(\bar{a})$, $Y \models \Phi(\bar{g}/\Delta)$. Let Z be a finite subspace of X_G. Then $Z(\bar{a})$ is a finite subspace of $X_G(\bar{a})$ so $Z(\bar{a}) \models \Phi(\bar{g}/\Delta)$. This gives $Z \models \Phi'(\bar{a}, \bar{g})$. Since this is the case for every finite subspace Z, we get $X_G \models \Phi'(\bar{a}, \bar{g})$ (recall that G satisfies (LG)), which yields $X_G(\bar{a}) \models \Phi(\bar{g}/\Delta)$.

We now consider the general case. We have

$$\Delta = \cup\{D_G\langle\langle a_1, \ldots, a_n\rangle\rangle \mid n \in \mathbb{N}, a_1, \ldots, a_n \in \Delta\}.$$

For $\bar{a}, \bar{b} \in \Delta$ with $\bar{a} \subseteq \bar{b}$ we have a canonical morphism of special groups $f_{\bar{a},\bar{b}} : G/D_G\langle\langle a_1, \ldots, a_n\rangle\rangle \to G/D_G\langle\langle b_1, \ldots, b_m\rangle\rangle$. If we take the direct limit of the system $\{G_{\bar{a}}, f_{\bar{a},\bar{b}} \mid \bar{a}, \bar{b} \in \Delta\}$ we get:

$$G/\Delta = \varinjlim\{G_{\bar{a}}, f_{\bar{a},\bar{b}} \mid \bar{a}, \bar{b} \in \Delta\},$$

as reduced special groups.
But every $G_{\bar{a}}$ satisfies (LG), and the result then follows from the first part of the proposition. □

3.5 An equivalent formulation of (LG)

Lemma 7 *Let $\Phi(\bar{g})$ be a pp-formula with parameters in G, and suppose that $X \not\models \Phi(\bar{g})$ (i.e. $G \not\models \Phi(\bar{g})$). Then there exists a subspace X_0 of X_G, minimal for inclusion, such that $X_0 \not\models \Phi(\bar{g})$.*

The proof is an application of Zorn's lemma. See [7], proposition 2.2.

We first remark that $(\mathrm{LG}(\Phi, \bar{g}))$ is only interesting for an infinite special group / space of orderings (otherwise the answer is obvious since X_G is a finite subspace of X_G).

Proposition 7 *Let $\Phi(\bar{x})$ be a pp-formula in the language L_{SG}. The following are equivalent.*

1. $(\mathrm{LG}(\Phi, \bar{g}))$ *is true for every infinite reduced special group G and every interpretation of $\bar{g}$ in G.*
2. *For every infinite reduced special group G and every interpretation of $\bar{g}$ in G, if $Y \models \Phi(\bar{g})$ for every proper subspace Y of X_G, then $G \models \Phi(\bar{g})$.*

Proof. 1.⇒2. Easy since a finite subspace of X_G (which is infinite) is necessarily proper.

2.⇐1. Let G be an infinite reduced special group, and let $\Phi(\bar{g})$ be a pp-formula with parameters in G such that $Y \models \Phi(\bar{g})$ for every finite subspace Y of X_G.

Suppose $G \not\models \Phi(\bar{g})$, i.e. with our notation, $X_G \not\models \Phi(\bar{g})$. Then by lemma 7, there exists a minimal subspace X_0 of X_G such that $X_0 \not\models \Phi(\bar{g})$. X_0 is not

empty because every finite subspace of X_G satisfies $\Phi(\bar{g})$.
Then every proper subspace of X_0 satisfies $\Phi(\bar{g})$, which implies (by assumption) that $X_0 \models \Phi(\bar{g})$, a contradiction. □

This reformulation has some interest also because the first part "$Y \models \Phi(\bar{g})$ for every proper subspace Y of X_G" can be expressed in a very natural way by the following first-order formula:

$$\forall a \ \ a \neq 1 \rightarrow \text{“}\Phi(\bar{g}) \text{ modulo } D\langle 1, a\rangle\text{”}.$$

This is first-order because $D\langle 1, a\rangle$ is definable.
In particular, the class of special groups for which this modified version of $(\mathrm{LG}(\Phi, \bar{g}))$ is true, is elementary. This result is in some ways weaker than 1 but gives a very explicit and natural axiomatization.

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