Realizing profinite reduced special groups

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1 Introduction

The theory of special groups is an axiomatization of the algebraic theory of quadratic forms, introduced by Dickmann and Miraglia (see [4]). The class of special groups, together with its morphisms, forms a category. As for other such axiomatisations, the main examples of special groups are provided by fields, in this case by applying the special group functor, which associates to each field $F$ a special group $SG(F)$ describing the theory of quadratic forms over $F$.

The category of special groups is equivalent to that of abstract Witt rings via covariant functors, while the category of reduced special groups is equivalent, via the restriction of the same covariant functors, to the category of reduced abstract Witt rings (see [4, 1.25 and 1.26]; recall that the special group of a field $F$ is reduced if and only if $F$ is formally real and Pythagorean). The category of reduced special groups is also equivalent, via contravariant functors, to the category of abstract spaces of orderings (see [4, Chapter 3]).

The question whether it is possible to realize every (reduced) special group as the special group of some (formally real, Pythagorean) field is still open, but the case of finite reduced special groups (actually of reduced special groups of finite chain length) has been positively answered by the combination of two results of Kula and Marshall: In [10], Kula showed that the product of two finite special groups of (formally real, Pythagorean) fields is still the special group of some (formally real, Pythagorean) field, and in [14], Marshall showed that every finite reduced special group can be constructed from the special group of any real closed field by applying a finite number of times the operations of product and extension (Marshall’s result is actually stated and proved for abstract spaces of orderings). Since the extension of the special group of a (formally real, Pythagorean) field is still the special group of a (formally real, Pythagorean) field, it shows that every finite reduced special group (or reduced special group of finite chain length) is realized as the special group of a field.
After finite reduced special groups, the simplest objects to consider are probably projective limits of finite reduced special groups, i.e. profinite reduced special groups, which have already been considered in several publications ([1], [11], [12], [13]).

In this paper, we show that every profinite reduced special group is the special group of some (necessarily formally real and Pythagorean) field.

2 Preliminaries

2.1 On special groups

We assume some familiarity with the theory of special groups, as presented in [4], and only introduce the following notation:

If \( G \) is a special group, \( \text{Ssat}(G) \) denotes the poset of saturated subgroups of \( G \), ordered by inclusion.

Definition 2.1 A profinite reduced special group is the projective limit of a projective system of finite reduced special groups.

If \( (G_i', f_{ij})_{i \leq j \in I} \) is a projective system of finite reduced special groups, where \( (I, \leq) \) is a downward directed poset, and if \( G \) is the projective limit of this system, the fact that \( G \) is indeed a special group (with the structure induced by its inclusion in the product \( \prod_{i \in I} G_i' \)) follows immediately from [5, Theorem 3.24]. Moreover, as proved in [12, Proposition 1.9.11], it is always possible to describe \( G \) as the projective limit of a projective system \( (G_i, f_{ij})_{i \leq j \in I} \) having the properties

1. For every \( i \in I \), \( G_i \) is \( G/\Delta_i \) with \( \Delta_i \) saturated subgroup of \( G \) of finite index;

2. For every \( i \leq j \in I \), \( \Delta_i \subseteq \Delta_j \) and \( f_{ij} \) is the canonical projection of special groups induced by this inclusion.

We briefly sketch the argument: Let \( \iota : G \hookrightarrow \prod_{j \in I} G_j' \) be the canonical embedding given by the definition of projective limit, and let \( \pi_i : \prod_{j \in I} G_j' \rightarrow G_i' \) be the canonical projection. We define \( \Delta_i := \ker(\pi_i \circ \iota) \), \( G_i := G/\Delta_i \) and, for \( i \leq j \in I \), \( f_{ij} \) to be the canonical projection induced by \( \Delta_i \subseteq \Delta_j \).

The system \( (G_i, f_{ij})_{i \leq j \in I} \) is a projective system, whose projective limit is isomorphic to \( G \), via the map \( g \in G \mapsto (g.\Delta_i)_{i \in I} \in \lim_{\longrightarrow}(G/\Delta_i, f_{ij})_{i \leq j \in I} \).

Remark 2.2 If \( \mathcal{M} = (M_i, f_{ij})_{i \leq j \in I} \) is any projective system, and if \( i^+ \in I \), by restricting this system to the set \( I' := \{i \in I \mid i \leq i^+\} \) we obtain a new system

\[
\mathcal{M}' := (M_i, f_{ij})_{i \leq j \in I'}.
\]
Since $I'$ is coinitial in $I$, $\mathcal{M}$ and $\mathcal{M}'$ have isomorphic projective limits, and $\mathcal{M}'$ possesses the following extra property

3. The index set of the projective system has a maximum element.

**Definition 2.3** We call adequate a projective system of special groups that satisfies conditions 1, 2 and 3 above.

Let $G_0, G_1$ be abstract groups and denote by $\pi_0 : G_0 \times G_1 \rightarrow G_0 : (g_0, g_1) \mapsto g_0$, $\pi_1 : G_0 \times G_1 \rightarrow G_1 : (g_0, g_1) \mapsto g_1$ the canonical projections and by $\iota_0 : G_0 \times G_1 \rightarrow G_0 \times G_1 : g_0 \mapsto (g_0, 1)$, $\iota_1 : G_1 \rightarrow G_0 \times G_1 : g_1 \mapsto (1, g_1)$ the canonical injections.

**Fact 2.4** Let $G_0, G_1$ be special groups. Then the canonical map

$$\psi : \text{Ssat}(G_0 \times G_1) \rightarrow \text{Ssat}(G_0) \times \text{Ssat}(G_1)$$

$$\Delta \mapsto (\iota_0^{-1}[\Delta], \iota_1^{-1}[\Delta]) = (\pi_0[\Delta], \pi_1[\Delta])$$

is an order-preserving bijection, whose inverse is $(\Delta_0, \Delta_1) \mapsto \Delta_0 \times \Delta_1$. In particular, if $\Delta \in \text{Ssat}(G_0 \times G_1)$ and $(\Delta_0, \Delta_1) := (\iota_0^{-1}[\Delta], \iota_1^{-1}[\Delta])$ then $\Delta = \Delta_0 \times \Delta_1$. Moreover:

- The canonical surjective morphism of special groups $G_0 \times G_1 \rightarrow G_0/\Delta_0 \times G_1/\Delta_1$ induces a natural isomorphism of (reduced) special groups $(G_0 \times G_1)/\Delta \cong G_0/\Delta_0 \times G_1/\Delta_1$;

- If $\Delta \subseteq \Delta' \in \text{Ssat}(G_0 \times G_1)$, then the projection $(G_0 \times G_1)/\Delta \rightarrow (G_0 \times G_1)/\Delta'$ is naturally identified with the (product) projection $G_0/\Delta_0 \times G_1/\Delta_1 \rightarrow G_0/\Delta_0' \times G_1/\Delta_1'$.

### 2.2 On projective systems of (valued) fields

Let $(I, \leq)$ be a poset. For each $i, j \in I$ such that $i \leq j$ we define $d(i, j) := \max\{\text{length of a chain from } i \text{ to } j\} \in \mathbb{N} \cup \{\infty\}$, and if $i \nleq j$ then we set $d(i, j) := -\infty$. Of course, in case $i$ and $j$ are comparable, we have $d(i, j) = d(j, i)$ if and only if $i = j$ if and only if $d(i, j) = 0$.

We will often consider $(I, \leq)$ as a directed graph whose vertices are the elements of $I$, and where there is an edge from $i$ to $j$ if and only if $i \leq j$ and $d(i, j) = 1$.

We first remark that it is possible describe some projective systems of fields as projective systems whose morphisms are all inclusions.
Remark 2.5 Let $\mathcal{F} := (F_i, f_{ij})_{i \leq j \in I}$ be a projective system of fields over a downward directed poset $(I, \leq)$ with maximum element $i^+ \in I$. Then there is an isomorphic projective system of fields $\mathcal{F}' := (F'_i, \iota_{ij})_{i \leq j \in I}$ such that: if $i \leq j \in I$, then $F'_i \subseteq F'_j$ and the morphism of fields $\iota_{ij} : F'_i \rightarrow F'_j$ is the inclusion.

We briefly sketch the argument. For each $i \in I$, we define $F'_i := f_{i+}[F_i] \subseteq F_{i+}$. Since for $i \leq j \in I$, $f_{j+} = f_{ji} \circ f_{ij}$, we obtain $F'_i \subseteq F'_j$, so we can define $\iota_{ij}$ to be this inclusion. It follows that $\mathcal{F}$ and $\mathcal{F}'$ are isomorphic via the morphisms $(f_{i+})_{i \in I}$.

The next results lead to corollary 2.9, which shows that any finite projective system of fields or of valued fields is denoted by $\mathcal{F}$.

If $(K, v)$ is a valued field, we denote by $Kv$ the residue field of $v$, by $vK$ its value group, by $O_K$ the valuation ring associated to $v$ and by $M_K$ its maximal ideal (if there is no ambiguity about the valuation $v$ under consideration). If $a \in O_K$, we denote by $av$ the class of $a$ in the residue field $Kv$. Finally, if $v$ has rank one, $K^v$ denotes a completion of $K$ with respect to $v$.

If $\mathcal{F} = (F_i, \xi_{ij})_{i \leq j \in I}$ is a projective system of fields, we denote by $SG(\mathcal{F})$ the system $(SG(F_i), SG(\xi_{ij}))_{i \leq j \in I}$ obtained from $\mathcal{F}$ by applying the $SG$ functor.

If $\mathcal{F} = ((F_i, v_i), \xi_{ij})_{i \leq j \in I}$ is a projective system of valued fields, we denote by $\text{res}\mathcal{F}$ or $\mathcal{F}v$ the induced residue projective system $(F_i v_i, (\xi v)_ij)_{i \leq j \in I}$, where the $(\xi v)_ij$ are the induced morphisms of fields.

If a projective system of fields or of valued fields is denoted by $(F_i)_{i \in I}$ or $(F_i, v_i)_{i \in I}$, i.e. without mention of the morphisms, it means that the morphisms are all inclusions (from a field within all fields with larger index).

Lemma 2.6 Let $(K, v)$ be a henselian valued field of residue characteristic zero and let $L$ be a subfield of $K$. Let $N$ be a subfield of $Kv$ such that $Lv \subseteq N \subseteq Kv$. Then there is a field $M$ such that $L \subseteq M \subseteq K$ and $Mv = N$. Moreover, if $[N : Lv]$ is algebraic, respectively finite, then $M$ can be chosen such that $[M : L]$ is algebraic, respectively finite.

Proof: Write $N = Lv(X)(\alpha_i, i \in \beta)$, where $X$ is a transcendence basis of $N$ over $Lv$ and $(\alpha_i, i \in \beta)$ is a (possibly infinite) tuple of elements that are algebraic over $Lv(X)$, indexed by an ordinal $\beta$. Let $Y$ be a set of transcendental elements over $L$ such that $Yv = X$. By [7, Corollary 2.2.2], the restriction of $v$ to $L(Y)$ is the Gauss extension of $v$ from $L$ to $L(Y)$. In particular $L(Y)v = Lv(X)$.

We now proceed by induction on $k \in \beta$ to find elements $a_i \in K$, $i < k$, such
that \( L(Y)(a_i, i < k)v = L(v)(X)(a_i, i < k) \).

If \( k = 0 \) there is nothing to prove since \( L(Y)v = L(v) \).

Assume we have found all \( a_i \) for \( i < k \). Let \( N_k = L(Y)(a_i, i < k) \) and \( M_k = L(v)(X)(a_i, i < k) \). By hypothesis we have \( N_kv = M_k \) Let \( P \in O_K[T] \) be a unitary polynomial such that \( Pv \) is the minimal polynomial of \( \alpha_k \) over \( Lv(X) \). Let \( a_k \) be a root of \( P \) in \( K \) such that \( a_kv = \alpha_k \) (it exists since \( (K,v) \) is henselian of residue characteristic zero). We have \( N_k(a_k)v = M_k(\alpha_k) \) and the fundamental inequality (see [7, Theorem 3.3.4]) tells us \( [N_k(a_k)v : N_kv] \leq [N_k(\alpha_k) : N_k](\leq \deg P) \). Since \([M_k(\alpha_k) : M_k] = \deg P\), it follows that \( N_k(a_k)v = M_k(\alpha_k) \), which is the desired result. \( \square \)

**Definition 2.7** Let \((K,v)\) be a valued field and let \((E_i)_{i=1,...,n}\) and \((F_i)_{i=1,...,n}\) be two sequences of fields of the same length. We say that \((F_i)_{i=1,...,n}\) is a good residue of \((E_i)_{i=1,...,n}\) in \((K,v)\) if

1. \( E_i \subseteq K \) and \( F_i \subseteq Kv \) for \( i = 1, \ldots, n \);

2. For every \( A \subseteq \{1, \ldots, n\} \) \( (E_i, i \in A)v = (F_i, i \in A) \) (where \((L_i, i \in A)\) denotes the compositum of the fields \( L_i \)).

**Lemma 2.8** Let \((K,v)\) be a henselian valued field of residue characteristic zero, and let \((E_i)_{i=1,...,n}\) and \((F_i)_{i=1,...,n}\) be two sequences of fields of length \( n \) such that \((F_i)_{i=1,...,n}\) is a good residue of \((E_i)_{i=1,...,n}\) in \((K,v)\). Let \((F'_i)_{i=1,...,m}\) be a sequence of subfields of \( Kv \), and let, for \( i \in \{1, \ldots m\} \)
\[
A_i = \{ j \in \{1, \ldots, n\} \mid F_j \subseteq F'_i \}.
\]

Then there is a sequence \((E'_i)_{i=1,...,m}\) of subfields of \( K \) such that

1. for every \( i \in \{1, \ldots, m\} \) and every \( j \in A_i \), \( E_j \subseteq E'_i \) and \( \trdeg E'_i \mid E_j = \trdeg F'_i \mid F_j \);

2. \((F'_i)_{i=1,...,m}\) is a good residue of \((E'_i)_{i=1,...,m}\).

**Proof:** We will use the following fact, which is a reformulation of [7, Remark 4.1.2 (3)]:

Fact 1: Let \((N,w)\) be a valued field and let \( P, Q \in O_N[T] \) and \( R \in N[T] \) be such that \( P = QR \). Assume that \( Q \) is primitive (i.e. \( w'(Q) = 0 \), where \( w' \) is the Gauss extension of \( w \) to \( N[T] \)), i.e., \( \min_{i=0,\ldots,k}w(a_i) = 0 \) if one writes \( Q = a_0 + \cdots + a_kT^k \). Then \( R \in O_N[T] \).

Proof of fact 1: Write \( P = aP_1 \) and \( R = cR_1 \) with \( a, c \in N \) and \( P_1, R_1 \in N[T] \) such that \( w'(P_1) = w'(R_1) = 0 \) (so \( P_1, R_1 \in O_N[T] \)). Then \( w(c) = w'(Q) + w'(cR_1) = w'(QR) = w'(P) \geq 0 \) since \( P \in O_N[T] \). This yields
\[ R = cR_1 \text{ with } w'(R) = w(c) + w'(R_1) = w(c) \geq 0, \text{ i.e. } R \in O_N[T]. \] End of proof of fact 1.

We need to fix some notation:

For \( A \subseteq \{1, \ldots, n\} \) we denote by \( F_A \) the field \( \langle F_i, i \in A \rangle \) and similarly by \( E_A \) the field \( \langle E_i, i \in A \rangle \).

For \( i = 1, \ldots, m \) let \( X_i = \{x_{i1}, \ldots, x_{ik_i}\} \) be a transcendence basis of \( F'_i \) over \( F_{A_i} = E_{A_i},v \), and let \( Y_i = \{y_{i1}, \ldots, y_{ik_i}\} \subseteq K \) be a set of transcendental elements over \( E_{A_i} \) such that \( Y_i v = X_i \). Note that by \([7, \text{ corollary 2.2.2}]\), it implies that the restriction of \( v \) to \( E_{A_i}(Y_i) \) is the Gauss extension of \( v \) from \( E_{A_i} \) to \( E_{A_i}(Y_i) \). In particular we have \( E_{A_i}(Y_i)v = E_{A_i}v(X_i) = F_{A_i}(X_i) \) (the last equality holds because \( \langle F_i \rangle_{i=1,\ldots,n} \) is a good residue of \( \langle E_i \rangle_{i=1,\ldots,n} \).

Write \( F'_i = F_{A_i}(X_i)(\alpha_i) \), where \( \alpha_i = (\alpha_{ij})_{j \in \beta_i} \) is a (possibly infinite) tuple of elements algebraic over \( F_{A_i}(X_i) \). For \( i \in \{1, \ldots, m\} \) and \( j \in \beta_i \) let \( P_{ij} \in O_{E_{A_i}(Y_i)}[T] \) be a unitary polynomial such that \( P_{ij}v \) is the minimal polynomial of \( \alpha_{ij} \) over \( E_{A_i}(Y_i)v = F_{A_i}(X_i) \), and let \( a_{ij} \in O_K \) be a root of \( P_{ij} \) with \( a_{ij}^{\alpha_{ij}} = \alpha_{ij} \) (\( a_{ij} \) exists since \( (K,v) \) is henselian of residue characteristic zero). We take for \( E'_i \) the field \( E_{A_i}(Y_i)(a_i) \), where \( a_i = (a_{ij})_{j \in \beta_i} \). The first conclusion of the lemma is obviously satisfied. Let \( A \subseteq \{1, \ldots, m\} \).

**Claim:** Let \( L \) be a subfield of \( K \) such that \( (L,v) \) is henselian, \( \langle E_j, j \in A_i, i \in A \rangle(Y_i, i \in A) \subseteq L \), and \( \langle F'_i, i \in A \rangle \subseteq Lv \). Then \( a_i \in L \) for every \( i \in A \), i.e. \( L \supseteq \langle F'_i, i \in A \rangle \).

**Proof of claim:** Let \( i \in A \) and \( j \in \beta_i \). Since \( \alpha_{ij} \in Lv \) and \( (L,v) \) is henselian (of residue characteristic zero), there is \( b_{ij} \in O_L \) such that \( b_{ij}v = \alpha_{ij} \) \( b_{ij} \) is a root of \( P_{ij} \). Assume \( b_{ij} \neq a_{ij} \). Then we can write \( P_{ij}(T) = (T - a_{ij})(T - b_{ij})R(T) \) in \( E_{A_i}(Y_i, a_{ij}, b_{ij}) \). But \( P_{ij}, (T - a_{ij}), (T - b_{ij}) \in O_{E_{A_i}(Y_i, a_{ij}, b_{ij})}[T] \) and \( (T - a_{ij})(T - b_{ij}) \) is primitive, so by Fact 1 we have \( R(T) \in \hat{O}_{E_{A_i}(Y_i, a_{ij}, b_{ij})}[T] \). Going to the residue field \( Kv \) we get \( P_{ij}v(T) = (T - \alpha_{ij})^2Rv(T) \), so \( \alpha_{ij} \) is a root of order at least 2 of \( P_{ij}v \), which is impossible since \( P_{ij}v \) is the minimal polynomial of \( \alpha_{ij} \) and \( \text{char } K v = 0 \). So \( a_{ij} = b_{ij} \in L \). End of proof of the claim.

We have \( E'_A = \langle E'_i, i \in A \rangle = \langle \langle E_j, j \in A_i \rangle(Y_i)(a_i), i \in A \rangle = \langle E_j, j \in A_i, i \in A \rangle(Y_i, i \in A)(a_i, i \in A) = \langle E_j, j \in A_i, i \in A \rangle v(X_i, i \in A) = \langle F_j, j \in A_i, i \in A \rangle(X_i, i \in A). \) Moreover \( \langle F'_i, i \in A \rangle = \langle \langle F_j, j \in A_i \rangle(X_i)(a_i), i \in A \rangle = \langle F_j, j \in A_i, i \in A \rangle(X_i, i \in A)(a_i, i \in A). \) So \( \langle F'_i, i \in A \rangle \) is an algebraic extension of \( \langle E_j, j \in A_i, i \in A \rangle(Y_i, i \in A) \). In particular (see lemma 2.6) there is an algebraic extension \( E''_v \) of \( \langle E_j, j \in A_i, i \in A \rangle(Y_i, i \in A) \) (inside \( K \)) such that \( E''_v = \langle F'_i, i \in A \rangle \). Let \( \hat{E} \) be the henselian closure of \( E''_v \) in \( (K,v) \). We have \( \hat{E}v = \langle F'_i, i \in A \rangle, E''_v \subseteq \hat{E} \). By the claim, since \( \hat{E} \) is henselian and \( \hat{E}v \supseteq \langle F'_i, i \in A \rangle \), we have \( a_i \in \hat{E} \) for every \( i \in A \). It implies \( E'_A \subseteq \hat{E} \), which gives, taking
residues $E'_{A}v \subseteq \bar{E}v = \langle F'_{i}, i \in A \rangle$. But by construction of the $E'_{i}$ we obviously have $E'_{A}v \supseteq \langle F'_{i}, i \in A \rangle$. It follows $E'_{A}v = \langle F'_{i}, i \in A \rangle$. \hfill \Box

**Corollary 2.9** Let $(F_{i})_{i \in I}$ be a finite projective system of fields of characteristic zero and let $i_{0}$ be the minimum of $I$. Assume that $(I, \leq)$ has a maximum $i^{+}$ and let $(E_{i_{0}}, v_{i_{0}})$ be a valued field such that $E_{i_{0}}v_{i_{0}} = F_{i_{0}}$. Then there is a projective system of valued fields $(E_{i}, v_{i})_{i \in I}$ such that $(F_{i})_{i \in I} \cong \text{res}(E_{i}, v_{i})_{i \in I}$ and, for every $i \in I$, $\text{trdeg} E_{i} \mid E_{i_{0}} = \text{trdeg} F_{i} \mid F_{i_{0}}$. Moreover

- we can assume that all $(E_{i}, v_{i}), i \in I$, are henselian;

- if $v_{i_{0}}$ has rank one, then we can choose the valuations $v_{i}, i \in I$, such that they all have rank one.

**Proof:** Claim: There is an extention $(K, v)$ of $(E_{i_{0}}, v_{i_{0}})$ such that $Kv = F_{i_{+}}$, and such that $v$ has rank one if $v_{i_{0}}$ has rank one. In particular $Kv \supseteq F_{i}$ for every $i \in I$.

Indeed, write $F_{i_{+}} = F_{i_{0}}(X)(\bar{a})$, where $X$ is a set of elements transcendental over $F_{i_{0}}$ and $\bar{a}$ is a sequence of elements algebraic over $F_{i_{0}}(X)$. Take $Y$ a set of indeterminates with the same cardinality as $X$ and consider the Gauss extension $w$ of $v_{i_{0}}$ to $E_{i_{0}}(Y)$. Then $E_{i_{0}}(Y)w = F_{i_{0}}(X)$. Note that $w$ has rank one if $v$ has rank one. Using now for instance [6, Satz 1], we find an (algebraic) extension $(K, v)$ of $(E_{i_{0}}(Y), w)$ such that $Kv = F_{i_{+}}$ (and with $v$ of rank one if $w$ has rank one). End of the proof of the claim.

We construct all the valued fields $(E_{i}, v_{i})$ (for $i \neq i_{0}$) as sub-valued fields of $(K, v)$. Since the valuation will always be $v$, we only look for the subfields $E_{i}$. Let $i_{0}$ be the minimum of $I$. We find the fields $E_{i}$ by induction on $d(i_{0}, i)$ (note that $d(i_{0}, i_{+}) = \max_{j \in I} d(i_{0}, j)$).

For $l \in \{0, \ldots, d(i_{0}, i_{+})\}$, let $D_{l} = \{i \in I \mid d(i_{0}, i) = l\}$. If $d(i_{0}, i) = 0$, we already have the field $E_{i_{0}}$. Note that since $D_{0} = \{i_{0}\}$ the sequence of fields $(F_{i})_{i \in D_{0}}$ is a good residue of $(E_{i})_{i \in D_{0}}$ in $(K, v)$.

Assume we have found a system of fields $(E_{i})_{i \in I, d(i_{0}, i) \leq l}$ such that $\text{res}(E_{i}, v \mid E_{i})_{i \in I, d(i_{0}, i) \leq l} = (F_{i})_{i \in I, d(i_{0}, i) \leq l}$ and $(F_{i})_{i \in D_{l}}$ is a good residue of $(E_{i})_{i \in D_{l}}$ in $(K, v)$. We write $D_{l+1} = \{i_{k} \mid 1 \leq k \leq m\}$, then we apply lemma 2.8 with $(F'_{k})_{k=1,\ldots,m} = (F_{i_{0}})_{k=1,\ldots,m}$, and obtain in this way a sequence $(E'_{k})_{k=1,\ldots,m}$. We define the fields $E_{i}$ for $i \in D_{l+1}$ by $(E_{i})_{k=1,\ldots,m} = (E'_{k})_{k=1,\ldots,m}$.

Finally, we can replace $(E_{i_{+}}, v_{i_{+}})$ by one of it henselian closures, and each $(E_{i}, v_{i})$ by its henselian closure inside $(E_{i_{+}}, v_{i_{+}})$. The residue system is still the same, which shows that we can assume that all $(E_{i}, v_{i})$ are henselian. \hfill \Box
3 Main results

Our main result, corollary 3.3, is a direct consequence of the following two

theorems, whose proofs are given in sections 4 and 5 respectively.

**Theorem 3.1** Let \( K := (K_i, f_{ij})_{i \leq j \in I} \) be a projective system of fields (respectively formally real Pythagorean fields) such that \( SG(K_i) \) is finite for every \( i \in I \). Let \( (G_i, \lambda_{ij})_{i \leq j \in I} = SG(K) \) and let \( G \) be the projective limit of this projective system of finite special groups. Then \( G \) is isomorphic to the special group of some field (respectively formally real Pythagorean field).

**Theorem 3.2** Let \( G := (G_i, \lambda_{ij})_{i \leq j \in I} \) be an adequate projective system of finite reduced special groups (see definition 2.3). Then there is a projective system \( K \) of formally real Pythagorean fields whose morphisms are inclusions, such that \( G \cong SG(K) \).

Consider now a profinite reduced special group \( G \). Say it is the projective limit of the system \( G = (G_i, f_{ij})_{i \leq j \in I} \) of finite reduced special groups. Let \( i^+ \) be any element in \( I \) and consider the system \( G' \) equal to \( G \) restricted to indices in \( I' := \{ i \in I \mid i \leq i^+ \} \). The special group \( G \) is the projective limit of the system \( G' \), whose index set \( I' \) has a maximum element. We can now use the strategy outlined after definition 2.1 to express \( G \) as an adequate projective system whose index set is \( I' \). Applying theorem 3.2 then theorem 3.1 now yields

**Corollary 3.3** Every profinite reduced special group is isomorphic to the special group of some Pythagorean field.

4 Proof of theorem 3.1

If \( (I, \leq) \) is a downward directed poset and \( i \in I \), then \( i^- \) denotes \( \{ j \in I \mid j \leq i \} \) and \( i^+ \) denotes \( \{ j \in I \mid j \geq i \} \).

First of all, we assume the following reductions:

1. \( I \) has a maximum \( i_+ (I = i^+_+) \);

2. all \( K_i, i \in I \), are subfields of the field \( M := K_{i^+} \), and the morphisms \( f_{ij} : K_i \to K_j \) are inclusions. In particular, the projective limit of the system \( K \) is isomorphic to the intersection of the fields \( K_i, i \in I \).

These assumptions can safely be made because for the original projective system of fields \( K := (K_j, f_{jk})_{j \leq k \in I} \) and for each \( i_+ \in I \) fixed:
(i) The set $i^+_-_\mathbb{I}$ is a cofinal subset of $I$;  
(ii) If $j \leq i_+ \in \mathbb{I}$, we can identify $K_j$ with the subfield $K'_j := f_{j_+}(K_j)$ of $K_{i_+}$, and the morphisms $f_{jk} : K_j \to K_k$ are naturally identified with inclusions $\iota_{jk} : K'_j \hookrightarrow K'_k$.

The assumptions above give us:
\[
\lim_{j \leq k \in \mathbb{I}} (K_j, f_{jk})_{j \leq k} \cong \lim_{i \in \mathbb{I}^-} (K_i, \iota_{jk})_{j \leq k} \cong \lim_{i \in \mathbb{I}^-} (K'_i)_{j \leq k} \\
\cong \bigcap_{j \in i^-} f_{j_+}[K_j] \subseteq K_{i_+};
\]

\[
G := \lim_{j \leq k \in \mathbb{I}} (SG(K_j), SG(f_{jk}))_{j \leq k} \cong \lim_{i \in \mathbb{I}^-} (SG(K_i), SG(f_{jk}))_{j \leq k}.
\]

Now consider the language $L = L_R \cup \{R_i\}_{i \in \mathbb{I}}$, where $L_R$ is the language of rings and the $R_i$ are unary relation symbols. We turn $M$ into an $L$-structure by interpreting each $R_i$ in $M$ by the subfield $K_i$.

Let $N$ be an $|I|^+$-saturated elementary extension of $M$ in the language $L$ (see [3, Chapter 5 and Lemma 5.1.2], or [9, p. 480 and Corollary 10.2.2] for the definition of saturated models and the existence result we just used. Note that this notion of saturation is not linked to the existing one for subgroups of special groups). Each $F_i := R_i^N$ is a field (a subfield of $N$), and the fields $L_i$ form a projective system of fields $\mathcal{F}$ (since for $k \leq i, j \in \mathbb{I}$ the sentence “$R_k \subseteq R_i \cap R_j$” is in the theory of $M$). Moreover, for every $i \in \mathbb{I}$, $K_i \hookrightarrow F_i$ is an $L_R$-elementary embedding and therefore induces an isomorphism of special groups $SG(K_i) \xrightarrow{\cong} SG(F_i)$ (since the special groups $SG(K_i)$, being finite, are described in the theory of $M$). More generally $SG(K) \cong SG(\mathcal{F})$, so $G \cong \lim_{i \in \mathbb{I}} SG(\mathcal{K}) \cong \lim_{i \in \mathbb{I}} SG(\mathcal{F})$.

Let $F := \bigcap_{i \in \mathbb{I}} F_i$ and define
\[
\xi : \quad SG(F) \quad \to \quad \lim_{i \in \mathbb{I}} SG(F) \\
\quad a \cdot F^2 \quad \mapsto \quad (a \cdot F^2_i)_{i \in \mathbb{I}}.
\]

We show that $\xi$ is an isomorphism of special groups, which yields $G \cong SG(F)$ as needed (in particular, if the fields $K_i$, $i \in \mathbb{I}$, are formally real Pythagorean, then $F = \bigcap_{i \in \mathbb{I}} F_i$ is formally real Pythagorean, since $SG(F)$ is a reduced special group).

1) It is clear that $\xi$ is well-defined and is a morphism of groups.

2) $\xi$ is a morphism of special groups:
Let $a \cdot F^2, b \cdot F^2 \in SG(F)$ be such that $a \cdot F^2 \in D_{SG(F)}(1, b \cdot F^2)$. There are
then \( c, d \in F \) such that, for all \( i \in I \), \( a = c^2 + bd \) in \( F_i \). Then \( a \cdot F^2_i \in D_{\text{SG}(F_i)}(1, b \cdot F^2_i) \) for every \( i \in I \), and therefore \( \xi(a \cdot F^2) \in D_{\text{G}}(1, \xi(b \cdot F^2)) \).

3) \( \xi \) is surjective: Let \( a = (a_i \cdot F^2_i)_{i \in I} \in \lim \text{SG}(F_i) \). So for all \( i \leq j \in I \), \( a_i \cdot F^2_i = a_j \cdot F^2_j \). We want \( x \in N \) satisfying the following set of formulas

\[
\Delta := \{ x \in F_i \}_{i \in I} \cup \{ x = a_i \mod F^2_i \}_{i \in I}.
\]

Every finite part of \( \Delta \) is satisfied in \( N \) since \( a = (a_i \cdot F^2_i)_{i \in I} \in \lim \text{SG}(F_i) \) (it suffices to take \( x = a_k \), where \( k \) is less than every one of the indices \( i \in I \) occuring in this finite part). By \(|I|^+\)-saturation, \( \Delta \) has a solution \( x \) in \( N \). Then \( \xi(x) = (a_i \cdot F^2_i) \).

The rest of the proof relies on the following lemma.

**Lemma 4.1** Let \( n \in \mathbb{N} \) and let \( P(X_1, \ldots, X_n) \in F[X_1, \ldots, X_n] \). Assume that \( P(X_1, \ldots, X_n) = 0 \) has a solution in every \( F_i \), \( i \in I \). Then the equation \( P(X_1, \ldots, X_n) = 0 \) has a solution in \( F \).

**Proof:** We are looking for \( \bar{x} \in N \) such that the set of formulas

\[
\sum := \{ P(\bar{x}) = 0 \} \cup \{ \bar{x} \in F_i \}_{i \in I}
\]

is satisfied in \( N \).

Since the \( F_i \), together with the inclusions between them, form a projective system, every finite part of \( \sum \) has a solution, and by the \(|I|^+\)-saturation of \( N \), \( \sum \) has a solution in \( N \). \( \square \)

We go back to proving that \( \xi \) is an isomorphism:

4) \( \xi \) is injective: Let \( a = a \cdot F^2 \in \text{SG}(F) \) be such that \( \xi(a) = 1 \), i.e. \( a \in F_i^2 \) for every \( i \in I \), i.e. the polynomial \( X^2 - a \) has a root in each \( F_i \), \( i \in I \). By lemma 4.1, \( X^2 - a \) has a root in \( F \), hence \( a \in F^2 \).

5) \( \xi \) is a monomorphism of special groups: Let \( a, b \in F \) be such that, for every \( i \in I \), \( a \cdot F^2_i \in D(1, b \cdot F^2_i) \). Let \( P(X, Y) = a - (X^2 + by^2) \in F[X, Y] \). By hypothesis, \( P(X, Y) = 0 \) has a solution in each \( F_i \), hence a solution in \( F \) by lemma 4.1, which means \( a \in D_{\text{SG}(F)}(1, b) \).

### 5 Proof of theorem 3.2

5.1 Reducing to a finite projective system

Since \( \mathcal{G} \) is adequate, the set \((I, \leq)\) has a maximum element \( i^+ \).
The purpose of this short section is to observe that it is enough to prove theorem 3.2 when 
\( \mathcal{G} := (G_i, f_{ij})_{i \leq j \in I} \) is a finite projective system of special 
groups such that \( I \) has a maximum (which we will also denote by \( i^{+} \)).

Let \( L \) be the language \( \{0, 1, +, \cdot\} \cup \{F_i\}_{i \in I} \). The projective system of 
fIELDS we are looking for is a model of the theory \( \Omega \) consisting in the following \( L \)-formulas:

- for every \( i \in I \):
  "(\( F_i, 0, 1, +, \cdot \)) is a field" (technically speaking, + and \( \cdot \) are functional 
symbols globally defined whose restrictions to \( F_i \) give internal operations on \( F_i \));

- for every \( i \in I \):
  "there is an isomorphism of special groups \( SG(F_i) \rightarrow G_i \)" (we denote 
it by \( \lambda_i \));

- for every \( i \leq j \in I \):
  "\( F_i \subseteq F_j \), and the morphism of special groups induces by this inclusion 
is naturally identified via the \((\lambda_i)_{i \in I}\) with \( f_{ij} \)."

(The statements between quotation marks can be expressed by first-order 
formulas in our language. This relies on the fact that the special groups \( G_i \) 
are all finite.)

Using the compactness theorem (see [3, Theorem 1.3.22] or [9, Theorem 
6.1.1]), to find a model of this theory, we only need to find a model of every 
finite part \( \Omega_0 \subseteq \Omega \). Let \( J \) be the set of elements of \( I \) occurring in this finite 
part \( \Omega_0 \), together with \( i^{+} \). Since \( I \) is downward directed, we can assume that 
\( J \) is also downward directed (taking a larger set \( J \) if necessary), that is \( J \) 
has a first element \( i_0 \). In particular \( J \) determines a finite projective system 
of special groups whose index set that has a maximum and a minimum.

### 5.2 Description of the proof by induction

We therefore assume from now on that the index poset \((I, \leq)\) is finite and 
that it has a minimum \( i_0 \) and a maximum \( i^{+} \). We find a finite projective 
system of fields \( \mathcal{K} \) such that \( \mathcal{G} \cong SG(\mathcal{K}) \) by induction on the construction of 
\( G_{i_0} \) by products and extensions.

Recall that since \( \mathcal{G} \) is an adequate projective system, the morphisms \( f_{ij}, 
\ i \leq j \in I \), are quotients by saturated subgroups (see the paragraph after 
definition 2.1).

If \( G_{i_0} \cong \mathbb{Z}_2 \), then all special groups in the system are (isomorphic to) \( \mathbb{Z}_2 \) 
and all morphisms are the identity. We can obviously realize such a system 
by taking \( F_i = F, i \in I \), where \( F \) is any fixed real closed field.
If $G_{i_0} \cong G'_{i_0} \times G''_{i_0}$. Since all morphisms and special groups in the systems are quotients of $G_{i_0}$ by (larger and larger) saturated subgroups, and using fact 2.4, the whole projective system $(G_i, f_{ij})_{i,j \in I}$ splits according to the product $G_{i_0} \cong G'_{i_0} \times G''_{i_0}$. By induction these systems are realized by two projective systems of formally real Pythagorean fields $F' = (F'_i)_{i \in I}$ and $F'' = (F''_i)_{i \in I}$ (where the morphisms are inclusions), so we just need to “glue” them together. For this we use results from [10], which describe how to realize a finite product of finite special groups when each one is already realized. This is achieved in section 5.3.

If $G_{i_0} \cong G'_{i_0}[H]$. As above, the morphisms of special groups in the projective system are quotients of $G_{i_0}$ by (larger and larger) saturated subgroups $\Delta_i$. This case is dealt with in section 5.4, using results from [2].

### 5.3 Gluing, the product case

Since we are dealing with reduced special groups, all fields here will be formally real thus, in particular, will have characteristic 0.

We begin with a reformulation of some results from [10].

**Definition 5.1** Let $F$ be a field equipped with $n$ mutually independent valuations of rank one $v_1, \ldots, v_n$, and let $f_i$ be an embedding of $F$ into $F^{v_i}$, a completion of $F$ with respect to $v_i$. We say that $(F, f_1, \ldots, f_n)$ fulfills the global squares property if for every $a \in F$:

$$a \in \hat{F}^2 \iff \forall i \in \{1, \ldots, n\} \ f_i(a) \in (\hat{F}^{v_i})^2.$$  

**Theorem 5.2 ([10], Corollary 2.5)** With notation as in definition 5.1, assume that $(F, f_1, \ldots, f_n)$ fulfills the global squares property. Then the map

$$\xi_F : SG(F) \to \prod_{i=1}^n SG(F^{v_i})$$  
$$a F^2 \mapsto (f_i(a)(F^{v_i}))_{i=1, \ldots, n}$$

is an isomorphism of special groups.

**Theorem 5.3 ([10], Theorem 2.6)** Let $(L_i, v_{i1}, \ldots, v_{im})_{i \in I}$ be a finite projective system of fields equipped with $n$ mutually independent valuations of rank one, and such that $I$ has a maximum element $i^+$. Then for every $i \in I$ there is an algebraic extension $E(L_i)$ of $L_i$ and a morphism of special groups $\eta_i : SG(E(L_i)) \to \prod_{k=1}^n SG((L_i)^{v_{ik}})$ such that

1. $E(L_i) \subseteq E(L_j)$ for every $j \in I, j \geq i$
2. $SG(\mathcal{E}(L_i)) \cong \prod_{k=1}^{n} SG((L_i)^{v_{i,k}})$; 

3. The morphism of special groups $\prod_{k=1}^{n} SG((L_i)^{v_{i,k}}) \to \prod_{k=1}^{n} SG((L_j)^{v_{j,k}})$, given by the product of the morphisms of special groups induced by $(L_i)^{v_{i,k}} \subseteq (L_j)^{v_{j,k}}$, is naturally identified via the $(\eta_{i})_{i \in I}$ with the morphism of special groups $SG(\mathcal{E}(L_i)) \to SG(\mathcal{E}(L_j))$ induced by $\mathcal{E}(L_i) \subseteq \mathcal{E}(L_j)$.

PROOF: Since a valuation $v_{i,k}$ is the restriction on $L_i$ of the valuation $v_{i+k}$, we drop the first index and simply denote it by $v_k$. For $k \in \{1, \ldots, n\}$ we fix a completion $L_i^k$ of $L_i$ with respect to $v_k$ and define, for $i \in I$, $L_i^k$ to be the completion of $L_i$ in $L_i^k$ with respect to $v_k$. The systems $(L_i^k)_{i \in I}$, for $k \in \{1, \ldots, n\}$, are all projective systems of fields, where the morphisms are the inclusions (since $L_i^k$ is simply the set of limits in $L_i^k$ of $v_k$-Cauchy sequences of elements of $L_i$).

Let $K_i$ be an algebraic closure of $L_i^k$. We define the set

$$\mathcal{L} := \{(E_i, \iota_{i1}, \ldots, \iota_{in})_{i \in I} \mid (E_i, \iota_{i1}, \ldots, \iota_{in})_{i \in I} \text{ projective system of fields,}$$

$$L_i \subseteq E_i \subseteq K_i, \text{ with } E_i \mid L_i \text{ algebraic, and equipped with}$$

$$\text{the } L_i\text{-embeddings of fields } \iota_{ik} : E_i \to L_i^k \text{ for } k = 1, \ldots, n\},$$

(note that the condition that $(E_i, \iota_{i1}, \ldots, \iota_{in})_{i \in I}$ is a projective system implies $\iota_{ik} \subseteq \iota_{jk}$ for $i \leq j \in I$ and $k \in \{1, \ldots, n\}$, which is possible since $L_i^k \subseteq L_j^k$).

We equip $\mathcal{L}$ with the partial ordering:

$$(E_i, \iota_{i1}, \ldots, \iota_{in})_{i \in I} \leq (F_i, \kappa_{i1}, \ldots, \kappa_{in})_{i \in I}$$

if and only if

for every $i \in I$ and $k \in \{1, \ldots, n\}$, $E_i \subseteq F_i$ and $\iota_{ik} \subseteq \kappa_{ik}$.

By Zorn’s lemma, $\mathcal{L}$ has a maximal element $(M_i, f_{i1}, \ldots, f_{in})_{i \in I}$. We show that, for $j \in I$, $(M_j, f_{j1}, \ldots, f_{jn})$ fulfills the global squares property. Let $j \in I$ and let $a \in M_j \setminus \{0\}$ be such that $f_{jk}(a) \in (L_j^k)^{\times 2}$, for $k = 1, \ldots, n$. Assume $\sqrt{a} \not\in M_j$. Fix a square root $\sqrt{a}$ of $a$ and $\alpha_k \in L_j^k$ such that $\alpha_k^2 = f_{jk}(a)$.

Then each morphism $f_{jk}$ can be (properly) extended to $M_j' := M_j(\sqrt{a})$ by sending $\sqrt{a}$ to $\alpha_k$. Moreover, with $A_j := \{r \in I \mid r \geq j\}$, and since for $r \in A_j$ we have $L_j^k \subseteq L_r^k$, the same reasoning tells us that, for each $r \in A_j$ and $k \in \{1, \ldots, n\}$, each morphism $f_{rk}$ can be extended to $M_r' := M_r(\sqrt{a})$ by sending $\sqrt{a}$ to $\alpha_k$ (since $\alpha_k \in L_r^k$). If $r \in I \setminus A_j$, we take $M_r' := M_r$. We obtain in this way $(M_j', f_{j1}', \ldots, f_{jn}')_{i \in I}$, a projective system of fields equipped with $n$ morphisms of fields that is (strictly) larger than $(M_i, f_{i1}, \ldots, f_{in})_{i \in I}$, a contradiction. It follows that $\sqrt{a} \in M_j$ and thus that $(M_j, f_{j1}, \ldots, f_{jn})$,
for \( j \in I \), fulfills the global squares property. If we take \( \mathcal{E}(L_i) = M_i \) for \( i \in I \), the first conclusion of the theorem then holds, and the second follows by theorem 5.2 ([10, Corollary 2.5]), with \( \eta_i : SG(M_i) \xrightarrow{\cong} \prod_{i=1}^n SG((L_i)^k) \) : \( aM_i^2 \mapsto (f_i(a) \cdot (L_i^k)^2)_{i=1,\ldots,n} \), for \( i \in I \). The third conclusion is proved in the next lemma.

**Lemma 5.4** With notation as in theorem 5.3 and its proof:
Let \( (K,v_1,\ldots,v_n) \subseteq (L,v_1,\ldots,v_n) \) be two fields equipped with \( n \) mutually independent valuations of rank one. For \( k = 1,\ldots,n \) let
- \( K^k \) be a completion of \( K \) with respect to \( v_k \) and let \( L^k \) be a completion of \( L \) with respect to \( v_k \) such that \( K^k \subseteq L^k \),
- \( f_k \) be an embedding of \( K \) into \( K^k \) and \( g_k \) be an embedding of \( L \) into \( L^k \) extending \( f_k \).

Assume that \( (K,f_1,\ldots,f_n) \) and \( (L,g_1,\ldots,g_n) \) satisfy the global squares property.

Let \( \lambda : \prod_{k=1}^n SG(K^k) \rightarrow \prod_{k=1}^n SG(L^k) \) be the product of the morphisms of special groups induced by the inclusions \( K^k \subseteq L^k \) for \( k = 1,\ldots,n \), and let \( \mu : SG(K) \rightarrow SG(L) \) be the morphism of special groups induced by \( M \subseteq L \).

Then \( \lambda \cong \mu \).

**Proof:** By theorem 5.2 the isomorphism between \( SG(K) \) and \( SG(K^1) \times \cdots \times SG(K^n) \) is

\[
\xi_K : SG(K) \rightarrow SG(K^1) \times \cdots \times SG(K^n)
\]

\[
x \cdot \hat{K}^2 \mapsto (f_1(x) \cdot (K^1)^2,\ldots,f_n(x) \cdot (K^n)^2).
\]

Similarly, the isomorphism between \( SG(L) \) and \( SG(L^1) \times \cdots \times SG(L^n) \) is

\[
\xi_L : SG(L) \rightarrow SG(L^1) \times \cdots \times SG(L^n)
\]

\[
x \cdot \hat{L}^2 \mapsto (g_1(x) \cdot (L^1)^2,\ldots,g_n(x) \cdot (L^n)^2).
\]

Then \( \lambda = \xi_L \circ \mu \circ \xi_K^{-1} \) since, for \( k = 1,\ldots,n \), \( g_k \mid_K = f_k \).

We now turn our attention to the two finite projective systems of formally real Pythagorean fields \( \mathcal{F}' = (F'_i)_{i \in I} \) and \( \mathcal{F}'' = (F''_i)_{i \in I} \) obtained in section 5.2. Since \( I \) is finite, it has a minimum \( i_0 \). We first show that we can assume that the fields in \( \mathcal{F}' \) and \( \mathcal{F}'' \) are at most countable and of finite transcendence degree over \( Q \). This is achieved by the following proposition.

**Proposition 5.5 ([10], Proposition 3.1)** Let \( \mathcal{L} := (L_i)_{i \in I} \) be a finite projective system of fields such that \( SG(\mathcal{L}) \) is a finite projective system of finite special groups. There is a map \( \mathcal{F} \), defined on \( \{L_i\}_{i \in I} \), such that, for every \( i \leq j \in I \):

\[ 14 \]
1. $F(L_i)$ is a countable subfield of $L_i$ with finite transcendence degree over $Q$;

2. If $\varphi_i : F(L_i) \hookrightarrow L_i$ is the inclusion map, then $SG(\varphi_i) : SG(F(L_i)) \to SG(L_i)$ is an isomorphism of special groups;

3. $F(L_i) \subseteq F(L_j)$;

4. If $\lambda_{ij} : SG(L_i) \to SG(L_j)$ is the morphism of special groups induced by $L_i \subseteq L_j$, then the morphism of special groups $SG(F(L_i)) \to SG(F(L_j))$ induced by $F(L_i) \subseteq F(L_j)$ is naturally identified with $\lambda_{ij}$.

**Proof:** The proof is a trivial extension of Kula’s. If $L$ is a field with a finite number of square classes, a representative system of $SG(L)$ is a finite subset $R(L) = A \cup B$ of $L$ such that

- $A \subseteq \hat{L}$ and $\hat{L}/\hat{L}^2 = A/\hat{L}^2$;

- For every $a_1, a_2 \in A$ with $a_1 \in D_L(1, a_2)$, there are $b_1, b_2 \in B$ such that $a_1 = b_1^2 + a_2b_2^2$.

Claim: There is, for every $i \in I$, a representative system $R(L_i)$ of $L_i$ such that $R(L_i) \subseteq R(L_j)$ whenever $i \leq j$.

Proof of the claim: Direct by induction on $d(i_0, i)$ (just take a system of representatives of $L_i$ and add to it all the $R(L_j)$ for $i_0 \leq j < i$).

Then, just as Kula, take for $F(L_i)$ the algebraic closure of $Q(R(L_i))$ in $L_i$.\]

The following two propositions show that we can assume that atd($F'_i$) = atd($F''_i$) for every $i \in I$, where atd denotes the absolute transcendence degree, i.e. the transcendence degree over $Q$.

**Proposition 5.6 ([10], Lemma 3.2)** Let $(L_i)_{i \in I}$ be a finite projective system of countable fields of finite absolute transcendence degree. There is a map $T$ defined on $(L_i)_{i \in I}$ such that, for every $i \leq j \in I$:

1. $T(L_i)$ is a countable field that is an extension of $L_i$;

2. atd($T(L_i)$) = atd($L_i$) + 1;

3. If $\tau_i : L_i \hookrightarrow T(L_i)$ is the inclusion map, then $SG(\tau_i) : SG(L_i) \to SG(T(L_i))$ is an isomorphism of special groups;

4. $T(L_i) \subseteq T(L_j)$;
5. If \( \lambda_{ij} : SG(L_i) \to SG(L_j) \) is the morphism of special groups induced by \( L_i \subseteq L_j \), then the morphism of special groups \( SG(T(L_i)) \to SG(T(L_j)) \) induced by \( T(L_i) \subseteq T(L_j) \) is naturally identified with \( \lambda_{ij} \) (via \( SG(\tau_i) \) and \( SG(\tau_j) \)).

**Proof:** For \( i \in I \) let \( K_i := L_i(x)\left( \sqrt[n]{x} \right)_{n \in \mathbb{N}} \) (\( x \) is an indeterminate), and consider on \( K_i \) the unique extension \( v_i \) of the valuation on \( L_i(x) \) determined by the irreducible polynomial \( K \). The \( K_i \), together with their inclusions, form a projective system, and the sets \( \Phi_i := \{ v_i \} \) satisfy the hypothesis of theorem 5.3. We now apply the map \( E \) defined in theorem 5.3 to the projective system of the \( K_i \) and get the projective system of the \( T(L_i) \). Since \( L_i \) is countable, \( K_i \) and \( T(L_i) = E(K_i) \) are countable. Kula’s proof of [10, lemma 3.2] shows that the second and third claims of the proposition hold, and the last two hold by theorem 5.3. \( \Box \)

**Proposition 5.7** There exist two finite projective systems of fields \( K' = (K'_i)_{i \in I} \) and \( K'' = (K''_i)_{i \in I} \) such that

1. \( SG(K') \cong SG(F') \) and \( SG(K'') \cong SG(F'') \);

2. For every \( i \in I \), \( \text{atd}(K'_i) = \text{atd}(K''_i) < \infty \).

**Proof:** We assume there is some \( i \in I \) such that \( \text{atd}(F'_i) \neq \text{atd}(F''_i) \) and we proceed by induction on \( d(i_0, i) \), the maximal length of a chain from \( i_0 \) to \( i \).

- \( d(i_0, i) = 0 \), i.e. \( i = i_0 \). Let \( t := \text{max}\{ \text{atd}(F'_j), \text{atd}(F''_j) \} \}. \) We then apply proposition 5.6 as many times as necessary to the system \( F' \) or \( F'' \) (the one that does not realize the maximum), and we obtain two new systems \( F'_0 \) and \( F''_0 \) indexed by \( I \), whose fields of index \( i_0 \) have same (finite) absolute transcendence degree \( t \).

- \( d(i_0, i) = n > 0 \). We now proceed by induction on the number of \( i \)'s with \( d(i_0, i) = n \) and \( \text{atd}(F'_j) \neq \text{atd}(F''_j) \). We fix one of them: \( i_1 \). By induction we can assume that the projective systems \( F' \) and \( F'' \) satisfy \( \text{atd}(F'_j) = \text{atd}(F''_j) \) for every \( j \in I \), \( d(i_0, j) < n \). We consider the systems \( F' \mid i_1 \) and \( F'' \mid i_1 \). By applying proposition 5.6, we get two new systems \( T' \) and \( T'' \), indexed by \( i_1 \) whose fields indexed by \( i_1 \) have same absolute transcendence degree. We replace, in \( F' \), respectively \( F'' \), the subsystem \( F' \mid i_1 \) by \( T' \), respectively \( F'' \mid i_1 \) by \( T'' \) and we write \( F'_1, F''_1 \) for the new sets of fields. Since every field has been replaced by a field extension, we still get projective systems of fields and, moreover, \( SG(F'_1) \cong SG(F') \) and \( SG(F''_1) \cong SG(F'') \). Now \( \text{atd}(F'_{1i_1}) = \text{atd}(F''_{1i_1}) < \infty \), and we proceed by induction. \( \Box \)
So from now on, we assume that our two finite projective systems of fields $\mathcal{F}'$ and $\mathcal{F}''$ consist of countable fields having the same finite transcendence degree over $\mathbb{Q}$ at each index.

**Remark 5.8** Let $K$ be a field equipped with two independent valuations $v_1$ and $v_2$ and let $(L, w_1, w_2)$ be an extension of $(K, v_1, v_2)$. Then $w_1$ and $w_2$ are independent. Indeed, if it were not the case, then $w_1$ and $w_2$ would define the same topology on $L$ (see [7, Theorem 2.3.4]), and therefore the same induced topologies on $K$, which coincide with the topologies defined by $v_1$ and $v_2$. It shows that $v_1$ and $v_2$ define the same topology on $K$, a contradiction since they are independent (again by [7, theorem 2.3.4]).

**Lemma 5.9** There are two henselian valued fields $(E'_{i_0}, v')$ and $(E''_{i_0}, v'')$ both containing $\mathbb{Q}(X)$, such that

1. $v'$ and $v''$ are of rank one;
2. $E'_{i_0} v' \cong F'_{i_0}$ and $E''_{i_0} v'' \cong F''_{i_0}$;
3. $\text{atd } E'_{i_0} = \text{atd } F'_{i_0} + 1 = \text{atd } F''_{i_0} + 1 = \text{atd } E''_{i_0}$;
4. $v'E'_{i_0}$ and $v'E''_{i_0}$ are divisible (two-divisible is actually enough for our purposes);
5. The restrictions of $v'$ and $v''$ to $\mathbb{Q}(X)$ are independent.

**Proof:** Let $\{y_1, \ldots, y_k\}$ be a finite transcendence basis of $F_{i_0}$ over $\mathbb{Q}$, and let $E$ be $\mathbb{Q}(y_1, \ldots, y_k)(X)$, equipped with the valuation $v$ determined by the irreducible polynomial $X \in \mathbb{Q}(y_1, \ldots, y_k)[X]$. Then $Ev \cong \mathbb{Q}(y_1, \ldots, y_k)$, $vE = \mathbb{Z}$ and $F_{i_0}$ is isomorphic to an algebraic extension of $Ev$. By [6, Satz 1], there is an algebraic extension $E'_{i_0}$ of $E$ and an extension $v'$ of $v$ to $E'_{i_0}$ such that $E'_{i_0} = F_{i_0}$ and $v'E'_{i_0}$ is divisible of rank one.

To construct $(E''_{i_0}, v'')$, we proceed as above but start with the valuation on $\mathbb{Q}(y_1, \ldots, y_k)(X)$ associated to the irreducible polynomial $X - 1$. Obviously, $v' \nmid \mathbb{Q}(X)$ and $v' \nmid \mathbb{Q}(X)$ are independent over $\mathbb{Q}(X)$. \hfill $\square$

We now apply corollary 2.9 twice (with the valued fields $(E'_{i_0}, v')$ and $(E''_{i_0}, v'')$ given by lemma 5.9), and get two projective systems of henselian valued fields $\mathcal{E}' = (E'_i, v'_i)_{i \in I}$ and $\mathcal{E}'' = (E''_i, v''_i)_{i \in I}$ equipped with valuations of rank one, such that $\text{res}(\mathcal{E}') \cong \mathcal{F}'$ and $\text{res}(\mathcal{E}'') \cong \mathcal{F}''$. Up to renaming the transcendental elements, we can assume that for every $i \in I$ there is a finite set $X_i$ of transcendental elements over $\mathbb{Q}$ and an algebraic closure $Q_i$ of $\mathbb{Q}(X_i)$ such that $E'_i, E''_i \subseteq Q_i$, and such that, for every $i \leq j \in I$ $X_i \subseteq X_j$ and $Q_i \subseteq Q_j$. 

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Since, for \( i \in I \), \( E'_i \) and \( E''_i \) are both subfields of \( Q_i \), we can consider the projective system of valued fields \((E'_i \cap E''_i, v'_i, v''_i)\) \( i \in I \). Note that \( v'_i \) and \( v''_i \) are independent by remark 5.8 and lemma 5.9 (5). We recall now the following special case of a result due to Heinemann [8]:

**Theorem 5.10** Let \( K \) be a field equipped with two independent valuations \( v_1 \) and \( v_2 \). Fix an algebraic closure \( \tilde{K} \) of \( K \). Let \((H_i, v_i)\) be a henselian extension of \((K, v_i)\) for \( i = 1, 2 \), such that \( H_1, H_2 \subseteq \tilde{K} \) and \( K = H_1 \cap H_2 \). Then \((H_i, v_i)\) is a henselization of \((K, v_i)\), for \( i = 1, 2 \).

Applying this result, we obtain that, for every \( i \in I \), \((E'_i \cap E''_i, v'_i)\) is a henselization of \((E'_i \cap E''_i, v'_i)\) and \((E''_i, v''_i)\) is a henselization of \((E'_i \cap E''_i, v''_i)\). In particular:

1. \( v'(E'_i \cap E''_i) \) and \( v''(E'_i \cap E''_i) \) are two-divisible;
2. \( \text{res}(E'_i \cap E''_i, v'_i) \approx \mathcal{F}' \) and \( \text{res}(E'_i \cap E''_i, v''_i) \approx \mathcal{F}'' \);
3. \( v' \) and \( v'' \) are independent on \( E'_i \cap E''_i \) (by lemma 5.9.(5) and remark 5.8).

We now apply theorem 5.3 to the system \((E'_i \cap E''_i, v'_i, v''_i)_{i \in I}\) and get the system \((\mathcal{E}(E'_i \cap E''_i))_{i \in I}\), which satisfies \( \text{SG}((\mathcal{E}(E'_i \cap E''_i))_{i \in I}) \approx \text{SG}((E'_i \cap E''_i)^{v'_i} \times \text{SG}((E'_i \cap E''_i)^{v''_i}, g'_{ij} \times g''_{ij})_{i \leq j \in I}) \), where \( g'_{ij} \) and \( g''_{ij} \), respectively, is the map induced by \( (E'_i \cap E''_i)^{v'_i} \subseteq (E'_j \cap E''_j)^{v'_j} \), respectively \( (E'_j \cap E''_j)^{v''_j} \subseteq (E'_j \cap E''_j)^{v''_j} \).

We claim that this last projective system of (formally real Pythagorean) fields is isomorphic to \((\text{SG}(F'_i) \times \text{SG}(F''_i), f'_{ij} \times f''_{ij})_{i \leq j \in I}\). It suffices to check that, for instance, the projective system \((\text{SG}((E'_i \cap E''_i)^{v'_i}), g'_{ij})_{i \leq j \in I}\) is isomorphic to \((\text{SG}(F'_i), f'_{ij})_{i \leq j \in I}\). This is the content of the remainder of this section.

Since \((E'_i \cap E''_i)^{v'_i} \times \text{SG}((E'_i \cap E''_i)^{v''_i}, v'_i)_{i \in I} \approx \mathcal{F}' \), so \( \text{SG}((E'_i \cap E''_i)^{v'_i}, v'_i)_{i \in I} \approx \text{SG}(\mathcal{F}') = (\text{SG}(F'_i), f'_{ij})_{i \leq j \in I} \) and it suffices to show that \((\text{SG}((E'_i \cap E''_i)^{v''_i}, g''_{ij})_{i \leq j \in I} \approx \text{SG}(\text{res}((E'_i \cap E''_i)^{v'_i}, v'_i)_{i \in I})) \).

By [10, Theorem 2.7], and since \( v'(E'_i \cap E''_i)^{v'_i} \) is two-divisible (so the group \( T_v \) in Kula’s theorem is trivial), the following map is an isomorphism of special groups (if \( R \) is a field and \( x \in \tilde{R} \), we write \( x \) instead of \( x : \tilde{R}^2 \) to avoid unnecessary notational complications):

\[
\tau'_i: \text{SG}((E'_i \cap E''_i)^{v'_i}) \to \text{SG}((E'_i \cap E''_i)^{v'_i})^{v'_i} \quad a \mapsto \left( \frac{1}{r_i \circ v'_i(a)} a \right)^{v'_i}
\]

where \( r_i \) is any fixed map from \( 2v'_i(E'_i \cap E''_i)^{v'_i} \) to \((E'_i \cap E''_i)^{v'_i}) \times \mathbb{Z}^2 \) such that \( r_i(0) = 1 \) and \( r_i \circ v'_i = \text{Id} \) (a quick comment to help the reader identify the relevant bits in Kula’s proof: Our map \( \tau'_i \) is the first component of his map \( \mu \).
Note that the inverse map can be written

\[ \tau_i'^{-1} : SG((E'_i \cap E''_i)^{v'_i}) \rightarrow SG((E'_i \cap E''_i)^{v'_i}) \]

\[ x \mapsto x, \]

(this is well-defined by a simple use of Hensel’s lemma) with the abuse of notation that \( x \in (E'_i \cap E''_i)^{v'_i} \) can be identified with an element of the valuation ring of \( v'_i \) in \( (E'_i \cap E''_i)^{v'_i} \) (it is then easily seen that \( \tau_i'(x) \) is \( x \)).

We then have the following diagram, for \( i \leq j \in I \)

\[ \begin{array}{ccc}
SG((E'_i \cap E''_i)^{v'_i}) & \xrightarrow{\tau_i'} & SG((E'_i \cap E''_i)^{v'_i}) \\
\downarrow & & \downarrow \\
SG((E'_j \cap E''_j)^{v'_j}) & \xrightarrow{\tau_j'} & SG((E'_j \cap E''_j)^{v'_j})
\end{array} \]

and it remains to check that it is commutative. But this is equivalent to checking that the following diagram is commutative, which is now obvious:

\[ \begin{array}{ccc}
SG((E'_i \cap E''_i)^{v'_i}) & \xrightarrow{\tau_i'^{-1}} & SG((E'_i \cap E''_i)^{v'_i}) \\
\downarrow & & \downarrow \\
SG((E'_j \cap E''_j)^{v'_j}) & \xrightarrow{\tau_j'^{-1}} & SG((E'_j \cap E''_j)^{v'_j})
\end{array} \]

### 5.4 Gluing, the extension case

Recall that \( i_0 \) is the minimum of \( (I, \leq) \) and \( i^+ \) is the maximum of \( (I, \leq) \).

We have, for \( i \in I \), \( G_i \cong G'_{i_0}[H]/\Delta_i \), where \( H \) is a fixed group of exponent 2, \( \Delta_i \) is a proper saturated subgroup of \( G'_{i_0}[H] \) and \( \Delta_{i_0} = \{1\} \). Furthermore, if \( i \leq j \in I \) we have \( \Delta_i \subseteq \Delta_j \) and \( f_{ij} \) is naturally identified with the canonical projection from \( G'_{i_0}[H]/\Delta_i \) onto \( G'_{i_0}[H]/\Delta_j \). As \( H \cong H_1 \times H_2 \) entails \( G'_{i_0}[H] \cong (G'_{i_0}[H_1])[H_2] \), we may assume \( \dim_{F_2} H = 1 \), i.e. \( H = \{1, h\} \) with \( h^2 = 1 \) and \( h \neq 1 \).

We define, for \( i \in I \) and \( i \leq j \in I \):

\[ \Omega_i := \Delta_i \cap G''_{i_0}, \quad G''_i := G''_{i_0}/\Omega_i \]

(note that \( \Omega_i \subseteq \Omega_j \))

\[ q_{ij} : G''_i \rightarrow G''_j \]

the canonical projection.
and
\[ \Theta_i := \{(g \cdot \Omega_i).w \in G''_i[H] \mid g.w \in \Delta_i\}. \]

The following fact is then easily checked

**Fact 5.11**
1. \( \Omega_i \) is a saturated subgroup of \( G_{i0}' \);

2. \( \Theta_i \) is a saturated subgroup of \( G''_i[H] \) with \( G''_i \cap \Theta_i = \{1\} \);

3. The morphism of special groups \( q_{ij} \times \text{Id} : G''_i[H] \to G''_j[H] \) is such that \( (q_{ij} \times \text{Id})(\Theta_i) \subseteq \Theta_j \) and \( (q_{ij} \times \text{Id}) \mid \Theta_i : \Theta_i \to \Theta_j \) is injective;

4. The map
\[
\omega_i : G_{i0}[H]/\Delta_i \to (G_{i0}/\Omega_i)[H]/\Theta_i
\]
\[
(g \cdot h) \cdot \Delta_i \mapsto (g \cdot \Omega_i) \cdot h \cdot \Theta_i
\]
is an isomorphism of special groups;

5. The following diagram commutes
\[
\begin{array}{ccc}
G_{i0}[H]/\Delta_i & \xrightarrow{f_{ij}} & G_{i0}[H]/\Delta_j \\
\omega_i & \downarrow & \omega_j \\
G''_i[H]/\Theta_i & \xrightarrow{q_{ij} \times \text{Id}} & G''_j[H]/\Theta_j
\end{array}
\]
where \( q_{ij} \times \text{Id} \) is the canonical map induced on the quotients.

To finish the proof, we have to represent the projective system of special groups \( (G''_i[H]/\Theta_i, q_{ij} \times \text{Id})_{i \leq j \in I} \) by some projective system of formally real Pythagorean fields. By induction on \( G''_{i0} \), the projective system \( (G''_i, q_{ij})_{i \leq j \in I} \) is realized by a system of formally real Pythagorean fields \( (K_i)_{i \in I} \), i.e., if we denote by \( \gamma_{ij} \) the morphism of special groups induced by \( K_i \subseteq K_j \):

\[
(SG(K_i), \gamma_{ij})_{i \leq j \in I} \cong (G''_i, q_{ij})_{i \leq j \in I}.
\]

We define \( M_i = K_i((t)) \) for every \( i \in I \) and record the following well known result:

**Lemma 5.12**
\( \hat{M}_i/\hat{M}_i^2 = \{at^k \cdot \hat{M}_i^2 \mid a \in \hat{K}_i, k \in \{0,1\}\} \), and the isomorphism of special groups from \( SG(M_i) \) to \( SG(K_i)[H] \) is
\[
\lambda_i : SG(M_i) \to SG(K_i)[H]
\]
\[
at^k \cdot \hat{M}_i^2 \mapsto (a \cdot \hat{K}_i^2)h^k.
\]
Let $b$ map for every $j$. Proof: Lemma 5.13 There is a projective system of fields realizing the system $(SG(K_i)((t))), \gamma_{ij} \times Id)_{i \leq j \in I}$, and if the isomorphism is given at the beginning of the proof on page 28. This last isomorphism yields

\[
(SG(K_i((t))), /, \gamma_{ij} \times Id)_{i \leq j \in I} \cong (G''[H], \Theta_i, q_{ij} \times Id)_{i \leq j \in I}.
\]

This last isomorphism yields

\[
(SG(K_i((t))) / \Gamma_i, \gamma_{ij} \times Id)_{i \leq j \in I} \cong (G''[H] / \Theta_i, q_{ij} \times Id)_{i \leq j \in I}
\]

(where $\gamma_{ij} \times Id$ denotes the induced map on the quotients), which in turn shows that we only have to find a projective system of fields realizing the system $(SG(K_i((t))) / \Gamma_i, \gamma_{ij} \times Id)_{i \leq j \in I}$. Therefore, according to (1), and to keep notation simple, we may assume $SG(K_i)[H] = G''[H], \Gamma_i = \Theta_i, \gamma_{ij} \times Id = q_{ij} \times Id,$ and $q_{ij} \times Id$ is the map from $SG(K_i)[H] / \Theta_i \rightarrow SG(K_j)[H] / \Theta_j$ induced by the $\gamma_{ij} \times Id = q_{ij} \times Id$.

So for every $i \leq j \in I$ we have $SG(M_i) = G''[H], SG(M_i \subseteq M_j) = q_{ij} \times Id$, and the following diagram is commutative:

\[
\begin{array}{ccc}
M_i & \longrightarrow & M_j \\
p_i & & \downarrow p_j \\
SG(M_i) \times Id & \longrightarrow & SG(M_j)
\end{array}
\]

(where $p_i$ and $p_j$ denote the canonical maps).

Define, for $i \in I, n_i := \dim_{F_i} \Theta_i$. Note that $\dim_{F_i} \Theta_i \leq \dim_{F_i} H = 1$, so $n_i \in \{0, 1\}$. Since $(q_{ij} \times Id) | \Theta_i : \Theta_i \rightarrow \Theta_j$ is injective, we have $n_i \leq n_j$ whenever $i \leq j \in I$. If $n_i = 1$, write $\Theta_i = \{1, a_i h\},$ with $a_i \in G''$. In this case, and if $i \leq j \in I$, we have $(q_{ij} \times Id)(\Theta_i) = \Theta_j$, so $s_{ij}(a_i) = a_j$.

**Lemma 5.13** There is $b \in M_{i_0}$ such that, for every $i \in I, \Theta_i \subseteq \{1, p_i(b)\}$.

Proof: For every $i \leq j \in I$, the map $q_{ij}$ is surjective. In particular the map $q_{i(i+)}$ is surjective and, by diagram (2) above, $p_i(M_{i_0}) = \text{Im}(q_{i(i+)} \times Id)$. Let $b \in M_{i_0}$ be such that $\{1, p_i(b)\} = \Theta_i$. Let now $i \in I$ and let $x \in \Theta_i$. Then $(q_{i(i+)} \times Id)(x) \in \Theta_i = \{1, p_i(b)\}$. If $(q_{i(i+)} \times Id)(x) = 1$, we get $x = 1 \in \{1, p_i(b)\}$, because $(q_{ij} \times Id) | : \Theta_i \rightarrow \Theta_j$ is an injective group homomorphism. If $(q_{i(i+)} \times Id)(x) = p_i(b)$, since diagram (2) is commutative,
we get $p_i(b) = (q_{ii} \times \text{Id})(p_i(b))$, so $(q_{ii} \times \text{Id})(x) = (q_{ii} \times \text{Id})(p_i(b))$ and we conclude that $x = p_i(b)$. \hfill \Box$

We assume from now on that there is $i \in I$ such that $n_i = 1$ (or equivalently $n_{i+} = 1$). Otherwise $\Theta_j = \{1\}$ for every $j \in I$, and the projective system of fields $(M_i)_{i \in I}$ realizes the projective system of special groups $(G_i[H]/\Theta_i, q_{ij} \times \text{Id})_{i \leq j \in I}$.

Recall that an element $a$ of a special group $T$ is called rigid when $a \neq 1$ and $D_T(1,a) = \{1, a\}$ and an element $b$ of $T$ is birigid when $b$ and $-b$ are rigid. If $T = G[H]$, then every element in $G[H] \setminus G$ is birigid (this is essentially the only way to obtain birigid elements in a special group; see [4, Berman’s Theorem on p. 12]) and if $\Theta$ is a subgroup of $G[H]$ such that $\Theta \cap G = \{1\}$ then $\Theta$ is a saturated subgroup of $T$.

Since we assume that $n_i = 1$ for some $i \in I$ (in other words $n_{i+} = 1$), it follows that the element $b$ produced in lemma 5.13 is birigid in $M_i$ for every $i \in I$.

The next proposition is [2, Proposition 8.2] and uses the following notation: If $K$ is a field and $a \in K$ then $K(\sqrt[n]{a})$ stands for $K(\sqrt[n]{a}, n \in \mathbb{N})$.

**Proposition 5.14** Let $F$ be a field, let $a$ be a birigid element in $F$ (i.e., $a \in \hat{F}$ and $a.\hat{F}^2$ is birigid in $SG(F)$) and let $\varphi$ be a quadratic form over $F$. Let $L := F(\sqrt[n]{a})$. Then

1. $\hat{L} = \hat{F}.\hat{L}^2$ and $\hat{F} \cap \hat{L}^2 = \hat{F}^2 \cup a.\hat{F}^2$;

2. $\varphi$ is isotropic over $L$ if and only if $\varphi \oplus a.\varphi$ is isotropic over $F$.

We define, for $i \in I$:

$$L_i = \begin{cases} M_i & \text{if } n_i = 0 \\ M_i(\sqrt[n]{b}) & \text{if } n_i = 1 \end{cases}$$

Since $n_i = 1$ implies $n_j = 1$ whenever $i \leq j \in I$, the system $(L_i)_{i \in I}$ is a projective system of fields. Note that the following diagram of fields is obviously commutative (with the natural inclusions as morphisms)

$$\begin{array}{ccc} M_i & \longrightarrow & M_j \\ \downarrow & & \downarrow \\ L_i & \longrightarrow & L_j \end{array}$$

which implies that the induced diagram of special groups is also commutative

$$\begin{array}{ccc} SG(M_i) & \xrightarrow{q_{ij} \times \text{Id}} & SG(M_j) \\ \mu_i \downarrow & & \downarrow \mu_j \\ SG(L_i) & \xrightarrow{\tau_{ij}} & SG(L_j) \end{array} \quad (3)$$

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where \( \mu_i : SG(M_i) \to SG(L_i) \) is the map induced by \( M_i \subseteq L_i \) and \( \tau_{ij} \) is the map induced by \( L_i \subseteq L_j \).

**Lemma 5.15** For \( i \in I \), let \( \pi_i : SG(K_i)[H] \to SG(K_i)[H]/\Theta_i \) be the canonical projection. Then

1. \( \mu_i \) is surjective;
2. There is a unique isomorphism of special groups

\[ \xi_i : SG(L_i) \to SG(K_i)[H]/\Theta_i \]

such that the following diagram is commutative.

\[
\begin{array}{ccc}
SG(M_i) & \xrightarrow{\mu_i} & SG(L_i) \\
\downarrow{\lambda_i} & & \downarrow{\xi_i} \\
SG(K_i)[H] & \xrightarrow{\pi_i} & SG(K_i)[H]/\Theta_i \\
\end{array}
\]

In particular \( L_i \) is formally real and Pythagorean.

**Proof:** The case \( n_i = 0 \) is trivial, so we assume \( n_i = 1 \). To avoid unnecessary notational complications, if \( K \) is a field and \( x \in \bar{K} \), we simply write \( x \) for the class of \( x \) in \( \bar{K}/\bar{K}^2 \). By proposition 5.14 1 we know that \( \mu_i \) is surjective and that \( \ker(\mu_i \circ \lambda_i^{-1}) = \{1, \lambda_i(b)\} = D_{SG(K_i)[H]}\{1, \lambda_i(b)\} \). In particular, there is a unique isomorphism of groups \( \xi_i : SG(L_i) \to SG(K_i)[H]/\{1, \lambda_i(b)\} \) such that the following diagram commutes.

\[
\begin{array}{ccc}
SG(K_i)[H] & \xrightarrow{\lambda_i^{-1}} & SG(M_i) & \xrightarrow{\mu_i} & SG(L_i) \\
\downarrow{\pi_i} & & \downarrow{\xi_i} & & \\
SG(K_i)[H]/\{1, \lambda_i(b)\} & & & & \\
\end{array}
\]

We show that \( \xi_i \) is an isomorphism of special groups. The image of \(-1\) is
clearly \(-1\). Let \(\mu_i(c), \mu_i(d) \in SG(L_i)\), where \(c, d \in \tilde{M}_i\). We have

\[
\mu_i(c) \in D_{SG(L_i)}(1, \mu_i(d)) \iff c \in D_{L_i}(1, d)
\]

\(\Leftrightarrow \langle \langle -c, d \rangle \rangle\) isotropic over \(L_i\)

\(\Leftrightarrow \langle \langle -c, d \rangle \rangle \oplus b(\langle -c, d \rangle)\) isotropic over \(M_i\)

(by proposition 5.14 2)

\(\Leftrightarrow \langle 1, b \rangle \otimes \langle -c, d \rangle\) isotropic over \(L_i\)

which shows that \(\xi\) is an isomorphism of special groups. Since \(\{1, \lambda_i(b)\}\) is a proper saturated subgroup of \(SG(K_i)[H]\), then \(SG(L_i)\), being isomorphic to \(SG(K_i)[H]/\{1, \lambda_i(b)\}\), is a reduced special group, which entails that \(L_i\) is a formally real Pythagorean field. \(\square\)

Recall that, using the identifications made after equation (1), we have, for \(a \in \tilde{K}_i\) and \(k \in \{0, 1\}:

\[
\hat{q}_{ij} \times \text{Id}((a\hat{K}_i^2)h^k \cdot \Theta_i) = (a\hat{K}_j^2)h^k \cdot \Theta_j.
\]

**Proposition 5.16** The following diagram commutes

\[
\begin{array}{ccc}
SG(L_i) & \xrightarrow{\tau_{ij}} & SG(L_j) \\
\xi_i \downarrow & & \xi_j \\
SG(K_i)[H]/\Theta_i & \xrightarrow{\hat{q}_{ij} \times \text{Id}} & SG(K_j)[H]/\Theta_j
\end{array}
\]

In particular \(SG((L_i)_{i \in I}) \cong \lim \left(G_i[H]/\Theta_i, q_{ij} \times \text{Id}\right)_{i \leq j \in I}G, \text{ and } \lim SG((L_i)_{i \in I}) \cong \lim \left(G_i, f_{ij}\right)_{i \leq j \in I}\).

**Proof:** Since \(\mu_i\) and \(\mu_j\) are surjective (see lemma 5.15 1), the commutative diagram in lemma 5.15 2 completely determines \(\xi_i\) and \(\xi_j\). Let \(z = \ldots\)
\( \mu_i(at^k M_i^2) \in SG(L_i) \) (with \( a \in K_i \) and \( k \in \{0, 1\} \)). Then

\[
\xi_i(z) = \xi_i \circ \mu_i(at^k M_i^2) \\
= \pi_i \circ \lambda_i(at^k M_i^2) \text{ by lemma 5.15 2} \\
= \pi_i((aK_i^2)h^k) \\
= ((aK_i^2)h^k) \cdot \Theta_i.
\] (4)

Applying this, we obtain

\[
f_{ij} \circ \xi_i(z) = f_{ij}(((aK_i^2)h^k) \cdot \Theta_i) \\
= (aK_j^2)h^k \cdot \Theta_j,
\]

and

\[
\xi_j \circ \tau_{ij}(z) = \xi_j \circ \tau_{ij} \circ \mu_i(at^k M_i^2) \\
= \xi_j \circ \mu_j \circ (q_{ij} \times \text{Id})(at^k M_i^2) \text{ by diagram (3)} \\
= \xi_j \circ \mu_j(at^k M_j^2) \text{ since } q_{ij} \times \text{Id} \text{ is induced by } M_i \subseteq M_j \\
= (aK_j^2)h^k \cdot \Theta_j \text{ by (4)},
\]

which finishes the proof. \( \square \)

**References**


