Algebraic Structures (MST20010)

Problem sheet 8

1. We apply Proposition 5.10 from the notes in both cases. What does this proposition mean? It means that for H to be a subgroup, H be must be non-empty, and that when we compute products and inverses of elements of H, the result is always in H (we say that H is closed under products and inverses).

   By product we mean what we obtain when using the group operation, even in cases where it is not a product (as in the first case below).

   (a) $n\mathbb{Z}$ is non-empty. If $nx, ny \in n\mathbb{Z}$ then $nx + ny = n(x + y) \in n\mathbb{Z}$ and the inverse of nx in the group $(\mathbb{Z}, +)$ is $-nx = n(-x)$, which also belongs to $n\mathbb{Z}$. So $n\mathbb{Z}$ is a subgroup of $(\mathbb{Z}, +)$.

   (b) $\mathbb{N}$ is obviously non-empty, and the sum of two elements of $\mathbb{N}$ is again in $\mathbb{N}$. But, if $n \in \mathbb{N}$, the inverse of n (for the operation $+$) is $-n$, which is not in $\mathbb{N}$. So $\mathbb{N}$ is not a subgroup of $(\mathbb{Z}, +)$.

   (c) $\{-1, 1\}$ is non-empty. If $x, y \in \{-1, 1\}$ then $xy \in \{-1, 1\}$. If $x \in \{-1, 1\}$ then $x^{-1} (= x) \in \{-1, 1\}$. So $\{-1, 1\}$ is a subgroup of $(\mathbb{R} \setminus \{0\}, \cdot)$.

   (d) $H \cap K$ is non-empty: It contains e because both H and K contain e.
   If $x, y \in H \cap K$ then $xy \in H \cap K$: By hypothesis we have $x, y \in H$ and $x, y \in K$. Since H is a subgroup we get $xy \in H$. Similarly $xy \in K$. So $xy \in H \cap K$.
   If $x \in H \cap K$ then $x^{-1} \in H \cap K$: By hypothesis we have $x \in H$ and $x \in K$. Since H is a subgroup we have $x^{-1} \in H$. Similarly $x^{-1} \in K$. So $x^{-1} \in H \cap K$.

2. (Solution with way too much detail, since it is the first time we do this kind of thing.)

   The subgroup generated by (1 2) and (3 4) (denoted $\langle(1\ 2), (3\ 4)\rangle$), is by definition the set of all possible products of (1 2), (3 4) and their inverses (in any order, and as many of them as we want). But (1 2) is the inverse of (1 2) and (3 4) is the inverse of (3 4). So $\langle(1\ 2), (3\ 4)\rangle$ is the set of all possible products of (1 2) and (3 4). An element of it will look like

$$(1\ 2)^{r_1}(3\ 4)^{s_1}(1\ 2)^{r_2}(3\ 4)^{s_2}\cdots(1\ 2)^{r_n}(3\ 4)^{s_n},$$

for some $n \in \mathbb{N}$ and $r_i, s_i \in \mathbb{N} \cup \{0\}$.

But (1 2) and (3 4) are disjoint cycles, so (1 2)(3 4) = (3 4)(1 2), so we can put all the powers of (1 2) first and all the powers of (3 4) second in the above expression, and we obtain

$$(1\ 2)^r(3\ 4)^s$$

for some $r, s \in \mathbb{N} \cup \{0\}$.

Finally both (1 2) and (3 4) have order 2, so any power of (1 2) is either id or (1 2) and any power of (3 4) is either id or (3 4). So we are left with 4 possibilities:

$$\text{id (if } r = s = 0),\ (1\ 2), (3\ 4), (1\ 2)(3\ 4).$$

So the subgroup of S_4 generated by (1 2) and (3 4) is

$$\{\text{id}, (1\ 2), (3\ 4), (1\ 2)(3\ 4)\}.$$

3. $(x^r)^s = x^{rs} = e$ by hypothesis. Since the order of x^r is the smallest k such that $(x^r)^k = e$, we still have to show that if $(x^r)^t \neq e$ for any $t \in \mathbb{N}$, $t < s$. Suppose it is not the case, i.e. there is $t \in \mathbb{N}$, $t < s$ such that $(x^r)^t = e$. Then $x^{rt} = e$. But $rt < rs$, which is not possible since the order of x is rs.

4. We use proposition 5.10 (it is almost always this when you have to check that something is a subgroup).

   It is a non-empty subset of G (because H is non-empty). Let axa^{-1}, $aya^{-1} \in aHa^{-1}$ (so with $x, y \in H$). Then $axa^{-1}aya^{-1} = a(xy)a^{-1} \in aHa^{-1}$ since $xy \in H$ (because H is a subgroup, so the product of two elements of H is still in H), and $(axa^{-1})^{-1} = ax^{-1}a^{-1} \in aHa^{-1}$ since $x^{-1} \in H$ (again, because H is a subgroup, so the inverse of an element of H is in H).