Algebraic Structures (MST20010)

Problem sheet 7

1. We use Proposition 5.10 in both cases.

   (a) The set is clearly non-empty. The sum of two elements of this form is still of this form (the operation is the sum from $\mathbb{R}$): $(a + b\sqrt{2}) + (c + d\sqrt{2}) = (a + c) + (b + d)\sqrt{2}$ with $a + c, b + d \in \mathbb{Q}$. The inverse of $a + b\sqrt{2}$ is $-a - b\sqrt{2}$ (since their sum is 0, the identity element), which is also in the set. Therefore this set, with operation the sum, is a subgroup of $(\mathbb{R}, +)$ (and, in particular, is a group).

   (b) Let us call this set H. Clearly H is non-empty. We check that the composition of any two elements of H is again in H: It is clear is one of the two elements if id. So we only have to check $(1\ 2\ 3)(1\ 3\ 2)$ and $(1\ 3\ 2)(1\ 2\ 3)$. The result is id in both cases, which is in H. Finally we need to check that the inverse of any element of H is again in H: $\mathrm{id}^{-1} = \mathrm{id} \in H$, and $(1\ 2\ 3)^{-1} = (1\ 3\ 2) \in H$, $(1\ 3\ 2)^{-1} = (1\ 2\ 3) \in H$.

2. Consider the row correspoding to the element a. It contains all the elements of the form ab for $b \in G$ (and such a result appears only once for each element b). If $c \in G$, then $c = a(a^{-1}c)$, so c appears in the row. If c were to appear twice, it would mean $c = ae = af$ for some $e \neq f$. But $ae = af$ implies $e = f$, so it is not possible.

3. The first row tells us $ac = a$, so $c = e$ the identity element. It allows us to fill the column under c and the row right of c. Now the row right of b contains every group element except c, so we know by the previous exercise that we need to put c in the empty slot. It gives us $ba = c$. Since c is the identity, it means that b is the inverse of a (and a the inverse of b), so $ab = c$, we can put that in the table. The second row gives us $bd = a$, therefore $abd = a^2$ and thus $d = a^2$ (since a is the inverse of b), we can put this in the table. Now ompleting the rest is easy using the previous exercise.

   The group is Abelian because it is symmetric across the diagonal (since Abelian means that we have $xy = yx$ for every x, y).

   There are probably plenty of different ways to proceed to fill in this table.

4. (a) We consider the powers of a: $e, a, a^2, a^3, \ldots$. They are all in H by hypothesis. Since H is finite they cannot be all different, so there are $i, j \in \mathbb{N}$ such that $a^{i+j} = a^j$. Therefore $a^i = e$ (multiply both sides by a^{-j}) and thus $a^{-1} = a^{i-1} \in H$ (since it is a non-negative power of a).

(b) By hypothesis, H satisfies conditions (a) and (b) from Proposition 5.10. And it satisfies condition (c) by the previous question.