

ALGEBRAIC STRUCTURES (MST20010)

Problem sheet 5

- The permutation is equal to $(1\ 2)(1\ 5)(1\ 4)(1\ 7)(2\ 6)(2\ 7)(2\ 4)$, which is odd.
 - The permutation is equal to $(1\ 3\ 7)(2\ 6) = (1\ 7)(1\ 3)(2\ 6)$, which is odd.

You can observe that a cycle of odd length is even, and a cycle of even length is odd.

- Let σ be an even permutation, so $\sigma = \tau_1 \cdots \tau_{2k}$ where τ_1, \dots, τ_{2k} are transpositions different from the identity. We know that $\sigma^{-1} = \tau_{2k}^{-1} \cdots \tau_1^{-1}$, and that $\tau_i^{-1} = \tau_i$. Therefore $\sigma^{-1} = \tau_{2k} \cdots \tau_1$ is even.

Let σ be an odd permutation and γ be an even permutation. Then $\sigma = \tau_1 \cdots \tau_{2n+1}$, $\gamma = \pi_1 \cdots \pi_{2k}$, where the τ_i and π_i are transpositions. Then $\sigma\gamma = \tau_1 \cdots \tau_{2n+1}\pi_1 \cdots \pi_{2k}$, and the number of transpositions appearing in this product is odd.

- $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} = (1\ 2\ 3)$ is a cycle of odd length so is an even permutation (a cycle of odd length is always even, a cycle of even length is always odd, see the observation before Theorem 3.27). If we want σ to be a cycle of length 4, σ will be odd, and the left-hand side of the equation will be odd, so cannot be equal to τ if τ is even.

- Let S be the set of all UCD students and B the relation “having the same birthday”, i.e. if x and y are students, xBy means that x and y have the same birthday. Show that B is an equivalence relation on S .

Reflexive: x has always the same birthday as x , so xBx .

Symmetric: If xBy , i.e. x has the same birthday as y , then y has the same birthday as x , i.e. yBx .

Transitive: If xBy and yBz , i.e. x has the same birthday as y and y has the same birthday as z , then x has the same birthday as z , i.e. xBz .

- Let $f : A \rightarrow B$ be a function. We define a relation R on A by

$$xRy \Leftrightarrow f(x) = f(y).$$

Show that R is an equivalence relation on A .

The proof is the same as the previous one, just replace each occurrence of “ u has the same birthday as v ” by $f(u) = f(v)$.

5. We define a relation on \mathbb{Z} by: xRy if and only if x and y have a common divisor greater than 1. Show that R is not an equivalence relation.

It is not reflexive: 1 is not in relation with 1 (since no divisor of 1 is greater than 1).

It is enough to show that it is not an equivalence relation, but we could also check that it is not transitive ($2R6$ and $6R3$, but $2R3$ is false). It is symmetric though.