Algebraic Structures (MST20010)

Two of the exercises from Chapter 4 (from the notes or the lecture)

1. (Example 4.4(2)) Let S the the set of 2×2 matrices. The relation defined on S by $A \sim B$ if there is an invertible matrix P such that $P^{-1}AP = B$, is an equivalence relation (exercise; and it is a useful relation in linear algebra).

   Observe that the definition says the following: Once you are given A and B, you look if there is a P such that $P^{-1}AP = B$. The P can depend on A and B, it may not always be the same P.

   We check the properties from the definition

   (a) Do we have $A \sim A$ for every 2×2 matrix A?
   Yes, because $I_2^{-1}AI_2 = I_2AI_2 = A$.

   (b) Do we have $A \sim B$ implies $B \sim A$, for every 2×2 matrices A and B?

   To prove this we assume that we have $A \sim B$ and out of this we try to see if we can prove $B \sim A$. If we can, the property is true. If we think that it is not true we need to find a counter-example: so two well-chosen matrices A and B such that $A \sim B$, but $B \not\sim A$ (you are free to choose them as you want, because the statement is that something is true for any A and B. To show that it is not the case, it suffices to produce one A and one B for which it is not true).

   So we assume that we have 2 matrices A and B such that $A \sim B$. By definition it means that $P^{-1}AP = B$ for some invertible matrix P. Multiplying both sides on the left by P and on the right by P^{-1} gives $A = PBP^{-1}$, i.e., $PBP^{-1} = A$. So $(P^{-1})^{-1}B(P^{-1}) = A$, which gives $B \sim A$ (it is the definition of $\sim$, where we use the invertible matrix P^{-1}).

   (c) Do we have ($A \sim B$ and $B \sim C$) implies $A \sim C$ for every 2×2 matrices A, B, C?

   We assume that $A \sim B$ and $B \sim C$. By definition it means that there are two invertible matrices P and Q (they don't need to be the same) such that $P^{-1}AP = B$ and $Q^{-1}BQ = C$. Replacing B

in the second equation by its value give in the first equation, we get $Q^{-1}P^{-1}APQ = C$, so $(PQ)^{-1}A(PQ) = C$. It gives $A \sim C$ (it is the definition of $\sim$ where we use the invertible matrix PQ).

2. (This one was mentioned during the lecture) Let S be the set of all lines in the plane. We define the relation $L_1 \sim L_2$ (where L_1, L_2 are lines) by "L_1 and L_2 are parallel". We check that it is an equivalence relation. We need to show, for any lines L_1, L_2, L_3:

   (a) $L_1 \sim L_1$? Yes, it is clearly true.

   (b) $L_1 \sim L_2$ implies $L_2 \sim L_1$? It is also true:
   If we know that L_1 and L_2 are parallel, then L_2 and L_1 are parallel.

   (c) ($L_1 \sim L_2$ and $L_2 \sim L_3$) implies $L_1 \sim L_3$? It is also true:
   If we know that L_1 and L_2 are parallel and L_2 and L_3 are parallel, then clearly L_1 and L_3 are parallel.