

ALGEBRAIC STRUCTURES (MST20010)

Problem sheet 6

1. Determine the following element of S_7 : $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 5 & 3 & 7 & 4 & 6 & 1 & 2 \end{pmatrix}^{-1} (3\ 7\ 2)$, calculate its order, and determine if it is odd or even.
2. Let \sim be the relation on \mathbb{R}^2 defined by $(x, y) \sim (a, b)$ if and only if there is $\alpha \in \mathbb{R}^{>0}$ such that $\alpha \cdot (x, y) = (a, b)$ (recall that $\alpha \cdot (x, y) = (\alpha x, \alpha y)$). Show that \sim is an equivalence relation on \mathbb{R}^2 and represent graphically all its equivalence classes.
3. Let G be a group.

(a) Suppose that for every $a, b, c \in G$ we have

$$ac = cb \text{ implies } a = b.$$

Show that G is Abelian i.e., for every $x, y \in G$ $xy = yx$.

(b) Show that if every element of G has order at most 2 (i.e., $x^2 = e$ for every $x \in G$), then G is Abelian. Hint: Consider the element xy and use that $x^2 = e$, $y^2 = e$.

4. Consider an equilateral triangle with vertices A, B, C . Determine all symmetries of the triangle and write down the multiplication table of the operation \circ between these isometries (the composition of maps). We saw (or will shortly see) in class that this set, together with this operation, is a group denoted D_6 , called the dihedral group of order 6.
5. We saw in exercise sheet 5 that if $f : A \rightarrow B$ is a function, then the relation on A defined by $x \sim y$ iff $f(x) = f(y)$ is an equivalence relation. This exercise shows that every equivalence relation can be obtained in this way.

Let \sim be an equivalence relation on a set S . Let $\{A_i\}_{i \in I}$ be the collection of all different equivalence classes of \sim (so $A_i \cap A_j = \emptyset$ if $i \neq j$). Observe that every element of S belongs to an equivalence class (indeed $x \in [x]$, and $[x]$ is one of the A_i). Define the map $f : S \rightarrow I$, $f(x) =$ the unique i such that x belongs to A_i .

Show that $x \sim y$ if and only if $f(x) = f(y)$.