Algebraic Structures (MST20010)

Problem sheet 4

1. Express each of the following elements of S_9 in the usual form as a table with 2 rows.

   (a) $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 3 & 1 & 4 & 2 & 7 & 6 & 9 & 8 & 5 \end{pmatrix}^{-1}$.

   (b) $(1\ 2\ 3\ 5\ 7)(2\ 4\ 7\ 6)$.

2. Let $\sigma = (1\ 2\ 3\ 4) \in S_5$. Determine all the permutations σ^n for $n \in \mathbb{Z}$.

3. Let G be a group and let $g \in G$ be such that $|g| = k$ (i.e. $g^k = e$ and $g^t \neq e$ for every $1 \leq t \leq k-1$). Let now $m \in \mathbb{N}$ be such that $g^m = \text{id}$.

   (a) Recall why there are two integers q and r such that $m = qk + r$, with $0 \leq r \leq k-1$.

   (b) Show that $g^{kq} = \text{id}$ and deduce that $g^r = \text{id}$.

   (c) Deduce that $r = 0$ (hint: what is the definition of $k = |g|$?).

   (d) Deduce the following very important result (**and remember it!!**):

   If g is an element of a group such that g has finite order, and if m is an integer such that $g^m = e$, then m is a multiple of the order of g.

4. This exercise is written in such a way that if you cannot do a question, you can always skip it and simply use its result to answer the following ones. In general, the result obtained at each question is useful in the following questions.

   Let $\sigma = \sigma_1 \sigma_2 \cdots \sigma_k$ where $\sigma_1, \ldots, \sigma_k$ are disjoint cycles. The objective of this exercise is to show that

   $$|\sigma| = \text{lcm}(|\sigma_1|, \ldots, |\sigma_k|).$$

   Recall that lcm means "least common multiple". For instance $\text{lcm}(4, 6) = 12$.

   Let $t = |\sigma|$, $t_i = |\sigma_i|$ and $N = \text{lcm}(|\sigma_1|, \ldots, |\sigma_k|)$.

   (a) Show that $\sigma^N = \text{id}$. Hint: Exercise 4 in problem sheet 2 can be useful.

   Therefore $|\sigma| \leq N$ by definition of the order of an element.

   (b) Show that if $\sigma_1^{r_1} = \sigma_2^{r_2} \cdots \sigma_k^{r_k}$ for some $r_1, \ldots, r_k \in \mathbb{Z}$ then $\sigma_1^{r_1} = \text{id}$. Hint: Show that $\sigma_1^{r_1}(x) = x$ for every $x \in \{1, \ldots, n\}$ by looking at what elements are "moved" by the different σ_i.

   (c) We know by definition of t that $\sigma^t = \text{id}$. Show that $\sigma_1^t = \text{id}$. Observe that similarly $\sigma_2^t = \text{id}$, ..., $\sigma_k^t = \text{id}$.

   (d) Show that t is a multiple of $t_1, \ldots, t_k$. Hint: Use exercise 3.

   (e) Deduce the result.