Algebraic Structures (MST20010)

Problem sheet 3

1. Let G be a group and let $a \in G$.
   (a) Show that if $a^2 = e$ then $a^{-2} = e$.
   (b) More generally, show that if $k \in \mathbb{N}$ and $a^k = e$, then $a^{-k} = e$.
   (c) Deduce that $|a| = |a^{-1}|$, (Recall that $|a|$ denotes the order of a.)

2. Which of the following are groups, and why?

$$(\{-1, 1\}, \cdot), \text{ where } \cdot \text{ is the usual product of integers.}$$

$$(\mathbb{N} \cup \{0\}, +).$$

3. Write down all the elements of S_3 in the form

$$\begin{pmatrix} 1 & 2 & 3 \\ a & b & c \end{pmatrix}.$$

4. Consider the following elements of S_4:

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 1 & 4 \end{pmatrix} \text{ and } \beta = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix}.$$

   (a) Compute $\alpha\beta$ and $\beta\alpha$.
   (b) Find the smallest integer k such that $\alpha^k = \mathrm{Id}$.
   (c) Show that $\alpha^{k-1} = \alpha^{-1}$.
   (d) Determine β^{-1}.

5. We saw on an example that the product of two permutations in S_4 is in general not commutative.
   (a) Describe all the elements of S_2 and show that the product of elements of S_2 is always commutative (i.e. $\sigma\gamma = \gamma\sigma$ for every $\sigma, \gamma \in S_2$).
   (b) Show that the product of elements in S_n is in general not commutative (i.e. that there are least 2 elements α, β of S_n such that $\alpha\beta \neq \beta\alpha$) whenever $n \geq 3$.

6. Optional (but it is a good idea to try it):

   Write down your answer to 1(a) as if you were answering an exam question, swap this with someone else in the class, and give each other some critical feedback (i.e., not just "it's fine"). You can also try this with any other exercise that involves presenting a reasoning (it is also a good idea). If you want you can also show me your work (in person, or email me a picture).

   The objective is to have you practice writing down your ideas in a way that someone else can follow. In maths it is of course necessary to understand the topics and be able to answer the questions, but it is also very important to be able to write it in a correct way that is as clear as possible.