Algebraic Structures (MST20010)

Problem sheet 10

1. (a) We consider the element $\sigma = (1\ 4\ 3)$ in S_5. Determine $\langle \sigma \rangle$, the subgroup generated by σ.

   (b) Let H be the subgroup of S_3 generated by $(1\ 2)$ and $(1\ 2\ 3)$.

      i. Show that 2 and 3 divide $|H|$. (There is a very simple argument involving what we saw at the end of Chapter 6.)

      ii. Deduce that $H = S_3$ (no heavy computations involved).

   Remark: The subgroup of S_3 generated by $(1\ 2)$ and $(1\ 2\ 3)$ is the set of all elements of S_3 that you can obtain by using these two elements and their inverses, and computing all possible products. So you could check "by hand" that this subgroup is S_3 itself, but it would require doing a lot of computations.

2. Let G be an infinite cyclic group, and let a be a generator of G, in other words $G = \{a^n \mid n \in \mathbb{Z}\}$ and $a^n \neq a^m$ if $n \neq m$.

   (a) Find a justification for the statement $a^n \neq a^m$ if $n \neq m$ (with $n, m \in \mathbb{Z}$) made above.

   (b) Show that a and a^{-1} are the only generators of G.

3. Let G be a group and let $x, y \in G$.

   (a) Show that, for $k \in \mathbb{N}$, $x^k = e$ if and only if $(yxy^{-1})^k = e$.

   (b) Deduce that x and yxy^{-1} have the same order.

   (c) Deduce that, for $a, b \in G$, ab and ba have the same order.

4. (Very easy) This exercise shows that if two elements in a group have finite order, their product may not have finite order.
   Let $GL_2(\mathbb{R})$ be the group of invertible 2×2 matrices with coefficients in $\mathbb{R}$ (the operation is the product of matrices).

   (a) Explain why the matrix $I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is the identity element of this group.

   Let $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}$.

   (a) Check that A and B have finite order (more precisely $A^4 = I_2$ and $B^6 = I_2$).

   (b) Show that AB does not have finite order i.e., $(AB)^n \neq I_2$ for every $n \in \mathbb{N}$.