

ALGEBRAIC STRUCTURES (MST20010)

Extra example on equivalence relations

In analysis, it is often convenient to identify functions that are not very different. There are several ways to do this, we look at one here:

We work with functions from $\mathbb{R}$ to $\mathbb{R}$, and denote by $F(\mathbb{R})$ the set of all such functions.

We define a relation on $F(\mathbb{R})$ by:

$$f \sim g \Leftrightarrow f \text{ and } g \text{ differ at most at a finite number of points.}$$

Another way to put it is that $f \sim g$ if there is a finite set A (possibly even empty, and it can depend on f and g) such that $f(x) = g(x)$ for every $x \in \mathbb{R} \setminus A$.

This is sometimes expressed as saying that f and g are equal almost everywhere.

1. Show that $\sim$ is an equivalence relation on $F(\mathbb{R})$.
2. Take $f \in F(\mathbb{R})$. What is $[f]$?

The solution is on the next page, but make a serious attempt at the questions first.

1. We need to check the three properties of the definition of equivalence relation, i.e., do we have, for every $f, g, h \in F(\mathbb{R})$:

- $f \sim f$?
- $f \sim g$ implies $g \sim f$?
- $(f \sim g \text{ and } g \sim h)$ implies $f \sim h$?

We do it in order:

- $f \sim f$ is clear since f is equal to f on the whole of $\mathbb{R}$.
- We assume that $f \sim g$, i.e., f and g differ at most at a finite number of points. Then clearly g and f differ at most at a finite number of points, i.e., $g \sim f$.
- We assume that $f \sim g$ and $g \sim h$, i.e., f and g differ at most at a finite number of points, and g and h differ at most at a finite number of points. Then clearly f and h differ at most at a finite number of points. If we want to be more precise:

f and g are equal on $\mathbb{R} \setminus A$ where A is a finite set (possibly empty), and g and h are equal on $\mathbb{R} \setminus B$ where B is a finite set (again possibly empty). Then f and h are equal on $\mathbb{R} \setminus (A \cup B)$, and thus $f \sim h$ since $A \cup B$ is a finite set.

2. $[f] = \{g \in F(\mathbb{R}) \mid f \sim g\}$ by definition of equivalence class. So $[f]$ is the set of all functions that only differ from f at most at a finite number of points (the set of functions that are equal “almost everywhere” to f).