Problem sheet 7 - Solution

1. (a) Because R has no zero divisors.
   (b) Ri_0^2 is a left ideal, and is included in I. It is non-zero since $i_0^2 \neq 0$. So $Ri_0^2 = I$.
   (c) Since $i_0 \in I = Ri_0^2$, then $i_0 = ri_0^2$ for some $r \in R$. Then $(1 - ri_0)i_0 = 0$. Since R has no zero divisors and $i_0 \neq 0$ we get $1 - ri_0 = 0$, so $ri_0 = 1$. But $ri_0 \in I$, so $I_0 = R$.
   (d) Ra is included in I is is non-zero. So $Ra = I = R$. In particular there is $r \in R$ such that $ra = 1$, so a is left invertible.

   If a is also right invertible, it will be invertible and its inverse will be r. So we want to show that $ar = 1$. We obtain it using R has no zero divisors. If we want to use this property to show that $ar = 1$, we need to rephrase it as "something= 0". It is rather easy, it is equivalent to $ar - 1 = 0$. Using now that R has no zero divisors, it is equivalent to $(ar - 1)a = 0$. Developping, this is now equivalent to $ara - a = 0$, which is true since $ra = 1$.

2. '⇒" Clearly N is Artinian since it is a submodule of M. Assume now that we have an infinite strictly decreasing chain of submodules of M/N:

$$P_1 \supsetneq P_2 \supsetneq P_3 \supsetneq \cdots$$

To get a contradiction, we need to "bring" things back to M. One way to do this is to use the map $\pi : M \to M/N$, $\pi(m) = m + N$. We get a sequence of submodules of M:

$$\pi^{-1}(P_1) \supseteq \pi^{-1}(P_2) \supseteq \pi^{-1}(P_3) \cdots$$

Since π is surjective these inclusions are proper (for the first one: take $x + m \in P_1 \setminus P_2$. Then $x \in \pi^{-1}(P_1)$ but $x \notin \pi^{-1}(P_2)$. This is impossible since M is Artinian

"⇐" Let $M_1 \supseteq M_2 \supseteq \cdots$ be an descending chain of submodules of M. Let $\pi : M \to M/N$ be the canonical projection. Since M/N and N are Noetherian, there is $k \in \mathbb{N}$ such that $\pi(M_k) = \pi(M_r)$ and $N \cap M_k = N \cap M_r$, for every $r \geq k$. Let $r \geq k$. We show that $M_k = M_r$. By hypothesis we know that $M_k \supseteq M_r$, so we just have to show the other inclusion. Let $x \in M_k$. Since $\pi(M_k) = \pi(M_r)$, there is $y \in M_r$ such that $y + N = x + N$, i.e. there is $n \in N$ such that $x = y + n$. In this case $n = x - y \in M_k \cap N$. Since $M_r \cap N = M_k \cap N$ we obtain that $n \in M_r$, from which follows $x = y + r \in M_r$.

3. Let $M = \bigoplus_{i\in I} M_i$ with all M_i semisimple. Observe first that M is of finite length if and only if I is finite.
"$\Rightarrow$" Let $a_i \in M_i \setminus \{0\}$ for every $i \in I$ (finite). Since M_i is simple, M_i is generated by a_i and so M is generated by $\{a_i\}_{i\in I}$.
"$\Leftarrow$" Assume that M is generated by $b_1, \ldots, b_k \in \bigoplus_{i\in I} M_i$. Since each b_j is the sum of finite number of elements of the M_i, there is a finite subset J of I such that $b_1, \ldots, b_k \in \bigoplus_{j\in J} M_j$. So $M = \mathrm{Span}(b_1, \ldots, b_k) \subseteq \bigoplus_{j\in J} M_j$, and therefore $M = \bigoplus_{j\in J} M_j$.

4. (a) U is a submodule of M semisimple, so U is semisimple, and therefore any submodule of U is a direct summand in U. The result follows since $U \cap V$ is a submodule of U. Same reasonning for the part about V.

   (b) We have two things to show:
   (i) $M = N + (U \cap V) + P$.
   (ii) $N \cap ((U\cap V)+P) = \{0\}$, $(U\cap V)\cap(N+P) = \{0\}$ and $P\cap(N+(U\cap V)) = \{0\}$.

   We start with (i). Let $m \in M$. Since $M = U + V$ there are $a \in U$ and $b \in V$ such that $m = a + b$. Since $U = (U \cap V) + P$ and $V = (U \cap V) + N$ we have $a = x + p$ and $b = y + n$ where $x, y \in U \cap V$, $p \in P$ and $n \in N$. Therefore $m = (x + y) + n + p$ with $x + y \in U \cap V$, $n \in N$ and $p \in P$.
   For (ii) I only show the first one, the others are similar. Let $n \in N \cap ((U \cap V)+P)$, so $n \in N$ and $n = x+p$ with $x \in U\cap V$ and $p \in P \subseteq V$. Since $N \subseteq U$ we have $p = n - x \in U$, and thus $P \in U \cap V$. So $n \in N \cap (U \cap V) = \{0\}$.

   (c) Let $r = \ell(W_1)$ and $s = \ell(W_2)$. Let

$$\{0\} = A_0 \subsetneq A_1 \subsetneq \cdots \subsetneq A_r = W_1$$

   and

$$\{0\} = B_0 \subsetneq B_1 \subsetneq \cdots \subsetneq B_s = W_2$$

   be composition series for W_1 and W_2. Then

$$\{0\} = A_0 \subsetneq \cdots \subsetneq A_r = W_1 \subsetneq W_1 + B_1 \subsetneq \cdots \subsetneq W_1 + B_s = W_1 + W_2$$

   is a composition series for $W_1 + W_2$, proving the result.

   (d) Since $M = N \oplus (U \cap V) \oplus N$ we have $\ell(M) = \ell(N) + \ell(U \cap V) + \ell(P)$. Now $\ell(U) = \ell(U \cap V) + \ell(N)$ and $\ell(V) = \ell(U \cap V) + \ell(P)$ to get the result.