Problem sheet 6 - Solution

1. (a) "$\Leftarrow$" If W is a subspace of V, then either $\dim W = 0$, in which case $W = \{0\}$, or $\dim W = 1$, in which case $W = V$.

   "$\Rightarrow$" If the only subspaces of V are $\{0\}$ and V, then V must have dimension 1 (it is easy to construct more subspaces if the dimension is greater than 1).

   (b) (i) $\Rightarrow$ (ii): Let $\{u_1, \ldots, u_n\}$ be a basis of V, and take $V_i = \text{Span}\{u_1, \ldots, u_i\}$.
   (ii) $\Rightarrow$ (i): By the observation above: $\dim V_{i+1}/V_i = 1$, i.e. $\dim V_{i+1} = \dim V_i + 1$. It follows that $\dim V = n$.

2. Let $\pi : M \to M/N$, $\pi(m) = m + N$.
   (a) $\Rightarrow$ (b): Since N is a proper submodule of M, we have $M/N \neq \{0\}$. Let P be a submodule of M/N. Then $P = \pi(L)$ for some submodule L of M such that $N \subseteq L$ (proposition 2.13). By hypothesis, $L = N$ or $L = M$, so $P = \pi(L) = \{0\}$ or M/N.
   (b) $\Rightarrow$ (a): Let P be a submodule of M such that $N \subseteq P$. Then $\pi(P)$ is a submodule of M/N so is equal to $\{0\}$ or M. In the first case $P = N$ and in the second $P = M$ (this case requires a little more details and uses that $N \subseteq P$. To get some practice, write in detail the proof of $M \subseteq P$).

3. We only have the left ideal I and we want to show $I = I_1 \times I_2$ for some left ideals I_1 of R_1 and I_2 of R_2. We first have to find I_1 and I_2. For this we take

$$\begin{aligned} I_1 &= \text{ all first coordinates of elements of } I \\ &= \{a \in R_1 \mid (a, y) \in I \text{ for some } y \in R_2\} \end{aligned}$$

and

$$\begin{aligned} I_2 &= \text{ all second coordinates of elements of } I \\ &= \{b \in R_2 \mid (x, b) \in I \text{ for some } x \in R_1\} \end{aligned}$$

   We first show that I_1 and I_2 are left ideals (I just do it for I_1):
   Let $a_1, a_2 \in I_1$, so there are $b_1, b_2 \in R_2$ such that $(a_1, b_1), (a_2, b_2) \in I$. Then $(a_1, b_1) + (a_2, b_2) = (a_1 + a_2, b_1 + b_2) \in I$ (since I is a left ideal), which shows that $a_1 + a_2 \in I_1$.
   If $r_1 \in R_1$ then $(r_1, 0) \in R$ and $(r_1, 0)(a_1, b_1) = (r_1 a_1, 0) \in I$ (since I is a left ideal), which shows that $r_1 a_1 \in I_1$.

   We now show that $I = I_1 \times I_2$. This is an equality between sets, so we have to show $I \subseteq I_1 \times I_2$ and $I_1 \times I_2 \subseteq I$.

"$I \subseteq I_1 \times I_2$": Let $(a, b) \in I$. Then by definition of I_1 and I_2 we have $a \in I_1$ and $b \in I_2$, so $(a, b) \in I_1 \times I_2$.
"$I_1 \times I_2 \subseteq I$": Let $(a, b) \in I_1 \times I_2$, i.e. $a \in I_1$ and $b \in I_2$. By definition of I_1 and I_2 this means that there are $u \in R_2$ and $v \in R_1$ such that $(a, u) \in I$ and $(v, b) \in I$. We want to show $(a, b) \in I$ out of that. We observe that

$$(a, b) = (a, 0) + (0, b) = (1, 0)(a, u) + (0, 1)(v, b)$$

which is a sum of elements of I since $(a, u), (v, b) \in I$ and I is a left ideal.

4. (a) Use the same proof we used in class, but apply Zorn's lemma to the set of proper ideals containing x. (This set is non-empty because it contains the ideal generated by x, which is proper since x is not invertible).

   (b) If x is not invertible by the previous exercise there is a maximal ideal L such that $x \in L$. Since there is only one maximal ideal in R, we have $L = M$ and $x \in M$, a contradiction.