Problem sheet 5 - Solution

1. (a) $I \cdot m_0$ is a submodule of M, which is simple. So $I \cdot m_0 = \{0\}$ or M. But $I \cdot m_0 = \{0\}$ implies that $I \subseteq \mathrm{Ann}_R\{m_0\}$, which is not the case by hypothesis. So $I \cdot m_0 = M$ and in particular there is $i \in I$ such that $im_0 = m_0$.

   (b) From $im_0 = m_0$ we deduce $(1-i)m_0 = 0$, so $1 - i \in \mathrm{Ann}_R\{m_0\} \subseteq I$. Since $i \in I$ we get $1 \in I$ and thus $I = R$.

   We check that $\mathrm{Ann}_R\{m_0\}$ is a maximal left ideal of R: Let I be a left ideal of R such that $\mathrm{Ann}_R\{m_0\} \subsetneqq I$. Then by the previous argument $I = R$. Done.

2. (a) Let $M = \mathbb{Z}/8Z \times Z/12\mathbb{Z}$.

$$\begin{aligned} n \in \mathrm{Ann}_{\mathbb{Z}}(M) &\Leftrightarrow \forall a, b \in \mathbb{Z}\ na = 0 \text{ in } \mathbb{Z}/8\mathbb{Z} \text{ and } nb = 0 \text{ in } \mathbb{Z}/12\mathbb{Z} \\ &\Leftrightarrow \forall a, b \in \mathbb{Z}\ 8|na \text{ and } 12 \mid nb \\ &\Leftrightarrow 8 \mid n \text{ and } 12 \mid n (\text{take } a = b = 1) \\ &\Leftrightarrow 24 \mid n \end{aligned}$$

   Therefore $\mathrm{Ann}_{\mathbb{Z}}(M) = 24\mathbb{Z}$.

   (b) $r + I = s + I$ implies $r = s + i$ for some $i \in I$. Therefore, and using that $im = 0$ since $i \in I \subseteq \mathrm{Ann}_R(M)$, we obtain $(r+I) \cdot m = rm = (s+i)m = sm + im = sm = (s+I)m$.

   (c) We check the properties for R/I-module. First of all M is non-empty because it is an R-module. Let $r, s \in R$ and $a, b \in M$.

      i. $(r+I)(a+b) = r(a+b) = ra + rb = (r+I)a + (r+I)b$.
      ii. $((r+I)+(s+I))a = ((r+s)+I)a = (r+s)a = ra+sa = (r+I)a+(s+I)a$.
      iii. $(r+I)((s+I)a) = (r+I)(sa) = r(sa) = (rs)a = (rs+I)a = ((r+I)(s+I))a$.
      iv. $(1+I)a = 1 \cdot a = a$.

3. (a) Let $a \in R \setminus \{0\}$. By hypothesis there are $n, k \in \mathbb{N}$ such that $a^n = a^{n+k}$. So $a^n(1-a^k) = 0$. Since R has no zero divisors, $a = 0$ (not the case) or $1 - a^k = 0$. So $a^k = 1$. If $k = 1$ then $a = 1$ is its own inverse, if $k > 1$ then a^{k-1} is the inverse of a.

   (b) "$\Rightarrow$" The left ideal $I + J$ contains I, so $I + J = I$ or $I + J = R$. If $I + J = I$ then $J \subseteq I$.

   "$\Leftarrow$" Let K be a left ideal such that $I \subseteq K$. Then either $K \subseteq I$ (in which case $K = I$, or $I + K = R$. In the later case $I + K = K$, so $K = R$.