Problem sheet 2 - Solution

1. (a) λ is linear: Direct verification. Injective: Assume $Ax = AY$. Then $BAX = BAY$ i.e., $X = Y$. An injective linear map from a finite-dimensional vector space to a vector space of the same dimension is surjective.

   (b) Since f is surjective there is $C \in M_n(\mathbb{R})$ such that $f(C) = I_n$ i.e., $AC = I_n$. Since the left and right inverses are the same $AB = I_n$.

2. Let
$$I = \{\sum_{i=1}^{n} r_i x_i s_i \mid n \in \mathbb{N},\ x_i \in X,\ r_i, s_i \in R\}.$$
   We need to show that I is the smallest ideal containing X, so we have two things to show:

   (1) That I is an ideal. (2) That for every ideal J containing X, we have $I \subseteq J$.

   (1): I is clearly nonempty and closed under $+$. Let $a = \sum_{i=1}^{n} r_i x_i s_i \in I$ (with notation as above) and $r \in R$. Then
$$ra = \sum_{i=1}^{n} (rr_i) x_i s_i \in I, \text{ and}$$
$$ar = \sum_{i=1}^{n} r_i x_i (s_i r) \in I,$$
   so I is an ideal of R.

   (2): Let J be an ideal of R containing X. Then, for every $r, s \in R$ and $x \in X$, we have $rxs \in J$. Since J is closed under sum, we get that the elements of I are all in J.

3. (a) If a does not divide x, it means that $r \neq 0$. But then $r = x - qa \in I$ (since $x, a \in I$), which is impossible because r would then be a positive element of I that is smaller than a.

   (b) It follows that $I \subseteq a\mathbb{Z}$. But since $a \in I$, we have $a\mathbb{Z} \subseteq I$.

4. We just need to show that I is non-empty, that the sum of two elements of I is still in I, and that is $A \in R$ and $X \in I$, then $AX \in I$. All three are clear.

   It shows that the same statement as proposition 1.20, written for left ideals, is not correct (I is a left ideal of $M_n(R)$, but is not of the form $M_n(J)$ for J left ideal of R).

5. Let $x = a_0 + a_1 i + a_2 j + a_3 k \in C(\mathbb{H})$. Writing $xi = ix$, we get $2a_3 j - 2a_2 k = 0$, which gives $a_3 = a_2 = 0$. So we have $x = a_0 + a_1 i$. Writing $xj = jx$ we get $2a_1 k = 0$, i.e. $a_1 = 0$. So $x \in \mathbb{R}$. And by the definition of quaternions, we know that the elements of $\mathbb{R}$ commute with every quaternion. This proves $C(\mathbb{H}) = \mathbb{R}$.

6. (a) Let $a \in R \setminus \{0\}$. Consider Ra the left ideal generated by a. Since $Ra \neq \{0\}$, we get $Ra = R$, and since $1 \in R$, there is $r \in R$ such that $ra = 1$. So a has a left inverse. Similarly, considering the right ideal generated by a, we obtain that a has a right inverse. So a is invertible.

   (b) We know that $C(R)$ is a ring, and it is commutative by definition. So we only have to show that every non-zero element in $C(R)$ is invertible (left or right, it does not matter since $C(R)$ is commutative). Let $a \in C(R) \setminus \{0\}$. Consider Ra ($= aR$ since $a \in C(R)$) the ideal generated by a. It is non-zero, so $Ra = R$ (since R is simple). In particular there is $r \in R$ such that $ra = 1$, i.e. a has an inverse in R. We still have to check that this element $r = x^{-1}$ is in $C(R)$. But

$$\begin{aligned} x^{-1} \in C(R) &\Leftrightarrow \forall z \in R \; x^{-1}z = zx^{-1} \\ &\Leftrightarrow \forall z \in R \; z = xzx^{-1} \\ &\Leftrightarrow \forall z \in R \; zx = xz \end{aligned}$$

   and this last statement is true since $x \in C(R)$.