Problem sheet 10 - Solution

1. (1) $\Rightarrow$ (2): $(ba)^2 = baba = ba$. Let Ra be a principal left ideal. we show that $Ra = R(ba)$. Obviously $R(ba) \subseteq Ra$. Let now $ra \in Ra$. Then $ra = ra(ba) \in R(ba)$.

   (2) $\Rightarrow$ (1): Let $a \in R$. Then $Ra = Re$ for some e with $e^2 = e$. Then $a = ye$ and $e = xa$ for some $y, x \in R$. Then $axa = a$ (direct computation).

   (2) $\Rightarrow$ (3): Let I be a principal left ideal. Then $I = Ra$ for some idenpotent a. Show then that $R = Ra + R(1-a)$, cf. exercise 3.8 from the course notes.

   (3) $\Rightarrow$ (2): Let Ra be a principal left ideal. Then $R = Ra \oplus J$ with J left ideal. Then use exercise 3.8 from the course notes.

2. (a) By the previous exercise, since every principal left ideal will be direct summand of R (because of the semisimplicity).

   (b) $Ra = Re$ for some idempotent e. But then $e \in J(R)$, so $1-e$ is invertible. Since $e(1-e) = 0$ we get $e = 0$ and thus $a = 0$.

   (c) Let R be semisimple. Then R is left artinian and therefore left noetherian. It is also von neumann regular by (a).

   Let R be left noetherian and von neumann regular. By the noetherianity, every left ideal is finitely generated, so is a direct summand of R (see previous exercise sheet), so R is semisimple.

3. Since M is semisimple, there is an R-module N such that $M = N \oplus \ker f$. We want $g \in \operatorname{End}_R M$ such that $fgf = f$. Observe that if $x \in \ker f$, then any value for $g(x)$ will do, since $fgf(x) = f(g(0)) = 0 = f(x)$.

   Consider now $x \in N$. To have $fgf(x) = f(x)$, it would be nice to have $g = f^{-1}$ at least for these $x \in N$. So it would be nice to be able to invert f on N. So what cane we say about $f|_N$?
   $f|_N : N \to M$ is injective (since $\ker f \cap N = \{0\}$). Therefore, if we consider $f|_N$ as a map from N to $f(N)$ is is bijective, and thus as an inverse. This is what we want, except that we want a map defined on M, so we need to "extend" this inverse to M. Since M is semisimple, we can write $M = f(N) \oplus P$ for some submodule P, and we define

$$g : M = f(N) \oplus P \to M, \; g = \begin{cases} \text{the inverse of } f|_N & \text{on } f(N) \\ 0 & \text{on } P. \end{cases}$$

The map g is R-linear, so belongs to $\operatorname{End}_R M$, and, for $x \in N$ we have:

$$fgf(x) = f(g(f(x))) = f(x)$$

since $g(f(x)) = x$ by definition of g.

4. (a) It is a 2-sided ideal because R/L is an R-module (we saw such a result in class). Obviously $\operatorname{Core}(L) \cdot R \subseteq \operatorname{Core}(L)$ since $\operatorname{Core}(L)$ is a 2-sided ideal (sum of 2-sided ideals). The other inclusion is trivial.

   (b) $\operatorname{Ann}_R(R/L)$ is a 2-sided ideal contained in l, so $\operatorname{Ann}_R(R/L) \subseteq \operatorname{Core}(L)$ by definition of $\operatorname{Core}(L)$. For the other direction: we know that $\operatorname{Core}(L) \cdot R = \operatorname{Core}(L) \subseteq L$, from which follows that $\operatorname{Core}(L) \subseteq \operatorname{Ann}_R(R/L)$.

   (c) R/L is a simple R-module because L is a maximal left ideal (seen in class several times). It is also a simple $R/\operatorname{Core}(L)$-module and is a simple $R/\operatorname{Core}(L)$-module using that $\operatorname{Core}(L) = \operatorname{Ann}_R(R/L)$.

   (d) "$\Leftarrow$" Follows from the previous question since $R/\{0\} = R$.
   "$\Rightarrow$" Let M be a simple R-module such that $\operatorname{Ann}_R M = \{0\}$[1], i.e. $M \cong R/L$ for some maximal left ideal L of R. Then $\{0\} = \operatorname{Ann}_R(R/L) = \operatorname{Core}(L)$.

[1] The property $\operatorname{Ann}_R M = \{0\}$ is called "M is faithful"