Midterm exam - Solution

1. $f^{-1}(I)$ contains 0 so is non-empty. Let $x, y \in f^{-1}(I)$ and $r \in R$. Then $x + y \in f^{-1}(I)$ since $f(x + y) = f(x) + f(y) \in I$. Furthermore $rx \in f^{-1}(I)$ since $f(rx) = f(r)f(x) \in I$ and similarly $xr \in f^{-1}(I)$.

2. (a) Because R has no zero divisors.

   (b) Ri_0^2 is a left ideal, and is included in I. It is non-zero since $i_0^2 \neq 0$. So $Ri_0^2 = I$.

   (c) Since $i_0 \in I = Ri_0^2$, then $i_0 = ri_0^2$ for some $r \in R$. Then $(1 - ri_0)i_0 = 0$. Since R has no zero divisors and $i_0 \neq 0$ we get $1 - ri_0 = 0$, so $ri_0 = 1$. But $ri_0 \in I$, so $I_0 = R$.

3. (a) "$\Rightarrow$" Let $m \in M \setminus \{0\}$. Then $\text{Span}\{m\} \neq \{0\}$ and is a submodule of M. So $\text{Span}\{m\} = M$ since M is simple.

   "$\Leftarrow$" Let N be a submodule of M and assume that $N \neq \{0\}$. Take $m \in N \setminus \{0\}$. By hypothesis $\text{Span}\{m\} = M$. But $\text{Span}\{m\} \subseteq N$, so $N = M$.

   (b) Assume we have an infinite strictly decreasing chain of submodules of M/N:

   $$P_1 \supsetneqq P_2 \supsetneqq P_3 \supsetneqq \cdots$$

   To get a contradiction, we need to "bring" things back to M. One way to do this is to use the map $\pi : M \to M/N$, $\pi(m) = m + N$. We get a sequence of submodules of M:

   $$\pi^{-1}(P_1) \supseteq \pi^{-1}(P_2) \supseteq \pi^{-1}(P_3) \cdots$$

   Since π is surjective these inclusions are proper (for the first one: take $x + m \in P_1 \setminus P_2$. Then $x \in \pi^{-1}(P_1)$ but $x \notin \pi^{-1}(P_2)$. This is impossible since M is Artinian