Extra exercise, totally optional

We have seen that not every module has a basis, but that every module over a division ring as a basis. The purpose of this exercise is to show that the "converse" is true, i.e.

If R is a ring such that every R-module is free (it means "has a basis"), then R is a division ring.

In particular it tells you that if R is not a division ring, then you will have R-modules that do not have a basis.

Let R be a ring. The midterm exercises can trivially be modified to obtain:
If the only left ideals of R are $\{0\}$ and R, then R is a division ring.

Assume now that every R-module is free. Let L be a maximal left ideal of R.

3. Show that if B is a basis of the R-module R/L, then B has only one element.
4. Show that the R-modules R and R/L are isomorphic.
5. Show that R is a simple R-module.
6. Show that the only left ideals of R are $\{0\}$ and R.
7. Conclude.