Problem sheet 9

1. Let R be a ring.

   (a) Let I be an ideal contained in $J(R)$. Show that $J(R/I) = J(R)/I$.

   (b) Let $(R_i)_{i\in I}$ be a family of rings. Show that $J(\prod_{i\in I} R_i) = \prod_{i\in I} J(R_i)$.

   (c) Let $f : R \to S$ be a surjective morphism of rings. Show that $f(J(R)) \subseteq J(S)$.

2. Let R be a ring and let $a \in R$ be such that $a + J(R)$ is invertible in $R/J(R)$. Show that a is invertible in R.

3. Let $R_n = \mathbb{Z}/2^n\mathbb{Z}$ for $n \in \mathbb{N}$ and let $R = \prod_{n\in\mathbb{N}} R_n$. Recall that R is a ring, with sum and product defined coordinate by coordinate. Define:

$$I = \bigoplus_{n\in\mathbb{N}} 2.R_n.$$

   In other words, I is the subset of $\prod_{n\in\mathbb{N}} 2.R_n$ consisting of elements having only a finite number of nonzero coordinates.
   Show that:

   (a) I is an ideal of R.

   (b) I is a nil ideal.

   (c) I is not a nilpotent ideal.

   Deduce that R is not left Artinian (this is not the most intelligent way to prove this: show that R is neither Artinian nor Noetherian by explicitly constructing strictly descending/ascending chains of left ideals).

4. Let R be a ring. We say that R is von Neumann regular if for every $x \in R$ there is $y \in R$ such that $x = xyx$.

   (a) Show that a division ring is von Neumann regular.

   (b) Let $R = \prod_{i\in I} D_i$ where the D_i are division rings. Show that R is von Neumann regular.

   (c) Let S be the ring of all functions from $\mathbb{R}$ to $\mathbb{R}$. Show that S is von Neumann regular.

   (d) Assume that R is von Neumann regular. Show that every principal (=generated by one element) left ideal of R is generated by an idempotent. Hint: If $a \in R$, and $aba = a$, show that ba is idempotent.