Problem sheet 8

1. Let M be an R-module.

   (a) Let N, P be submodules of M such that $M = N \oplus P$. Show that $M/N \cong P$.

   (b) Show that M is Noetherian (=satisfies the ascending chain condition on submodules) if and only if every submodule of M is finitely generated.

2. Let M be a D-module, with D a division ring, and such that $\dim_D M$ is infinite. Let $I_0 = \{f \in \operatorname{End}_D M \mid \dim_D \operatorname{Im}(f) \text{ is finite}\}$.

   (a) Show that I_0 is a nonzero proper ideal of $\operatorname{End}_D M$ (which is therefore not simple; recall that if $\dim_D M = n$ finite, then $\operatorname{End}_D M \cong M_n(D^{op})$ is simple).

   (b) Prove that $\operatorname{End}_D M$ is neither left Artinian nor left Noetherian. Hint: Show first that if N is a submodule of M, then $\{f \in \operatorname{End}_D M \mid f(N) = \{0\}\}$ is a left ideal of $\operatorname{End}_D M$.

3. Let R be a ring and let M be an R-module. Let $f \in \operatorname{End}_R M$ be injective.

   (a) Show that if N_1 and N_2 are submodules of M such that $N_1 \subsetneqq N_2$, then $f(N_1) \subsetneqq f(N_2)$.

   (b) Show that if M is Artinian, then f is surjective. Hint: By contradiction, using (a).

4. Let R be the ring of all continuous functions from $[0, 1]$ to $\mathbb{R}$ (equipped with the usual sum and product of real-valued functions). Show that R is not semisimple.

   Hint: Consider the sets

$$I_n = \{f \in R \mid f = 0 \text{ on } [0, 1/n]\}.$$