Problem sheet 6

1. Let V be a vector space over a field K.

   (a) Show that V is simple iff $\dim V = 1$.

   (b) Show that the following are equivalent:

      i. $\dim V = n$;

      ii. There is a sequence of subspaces $\{0\} = V_0 \subseteq V_1 \subseteq \cdots \subseteq V_n = V$ such that V_{i+1}/V_i is simple (as a K-module) for $i = 0, \ldots, n-1$.

2. Let M be an R-module and let N be a proper submodule of M. Show that the following two statements are equivalent.

   (a) If P is a submodule of M such that $N \subseteq P \subseteq M$, then $P = N$ or $P = M$.

   (b) The R-module M/N is simple.

3. Let R_1 and R_2 be rings and let I be a left ideal of $R_1 \times R_2$. Show that $I = I_1 \times I_2$ for some left ideals I_1 of R_1 and I_2 of R_2.

   Hints: (a) Find a way to define/describe I_1 and I_2 out of I;
   (b) If $a = (a_1, a_2) \in I$, consider $(a_1, 0)$ and $(0, a_2)$.

   This can easily be generalised to a product $R_1 \times \cdots \times R_n$. It is exactly the same argument, just longer to write.

4. Let R be a commutative ring.

   (a) Let $x \in R$, x not invertible. Show that there is a maximal ideal M of R such that $x \in M$ (use Zorn's lemma).

   (b) Let R be a commutative ring such that R has only one maximal ideal M (such a ring is called a local ring). Show that $R \backslash M$ is the set of invertible elements of R.