Problem sheet 5

Notation (we will not use it in this exercise sheet, but is is convenient to introduce it at some point):

If I is a left ideal of R and N is a subset of an R-module M, the notation $I \cdot N$ (or simply IN) will mean

$$IN = \{\sum_{k=1}^{r} a_k n_k \mid r \in \mathbb{N},\ a_k \in I,\ n_k \in N\}.$$

This is similar to the notation for the product of ideals. The sum symbol is there so that the resulting set is closed under sums. It is easy to check that IN is a subomdule of M, since I is a left ideal of R.

1. Let M be an simple R-module. We will use the notion of annhilator, as seen in the previous exercise sheet.

   Let $m_0 \in M$ and let I be a left ideal of R such that $\mathrm{Ann}_R\{m_0\} \subsetneq I$.

   (a) Show that there is $i \in I$ such that $m_0 = im_0$. Hint: Consider $I \cdot m_0$, where
   $$I \cdot m_0 = \{im_0 \mid i \in I\}$$
   (it is consistent with the notation introduced at the start of this exercise sheet).

   (b) Deduce that $I = R$ and that $\mathrm{Ann}_R\{m_0\}$ is a maximal left ideal of R.

2. (a) Compute the annihilator of the $\mathbb{Z}$-module $\mathbb{Z}/8\mathbb{Z} \times \mathbb{Z}/12\mathbb{Z}$.
   (The product of an element $n \in \mathbb{Z}$ by an element $(a, b) \in \mathbb{Z}/8\mathbb{Z} \times \mathbb{Z}/12\mathbb{Z}$ is $n \cdot (a, b) = (na + nb)$.)

   Let M be an R-module and let I be an ideal of R such that $I \subseteq \mathrm{Ann}_R(M)$. Define a product of an element of R/I by an element of M as follows: $(r + I) \cdot m = rm$.

   (b) Show that this product is well-defined (i.e. if $(r + I) = (s + I)$ then $(r + I) \cdot m = (s + I) \cdot m$).

   (c) Show that this product, together with the sum of elements of M, turns M into and R/I-module.

3. (a) Let R be a ring without zero divisors. Assume that for every $a \in R$, $\{a^n \mid n \in \mathbb{N}\}$ is finite. Show that R is a division ring.

   (b) Let R be a ring and let I be a left ideal of R. Show that I is a maximal left ideal of R if and only if for every left ideal J of R, either $J \subseteq I$ or $I + J = R$.