Problem sheet 3

We recall the following:

- Submodules are what correspond to subspaces in vector spaces.
- If M is a module and N is a subset of M, the way to check that N is a submodule is exactly how you would check that a subset of a vector space is a subspace. So you check:
  1. N is not empty;
  2. For every $x, y \in N$ and every $r \in R$: $x + y \in N$ and $rx \in N$.
- Direct sums of submodules are defined exactly as direct sums of subspaces in vector spaces.

1. Let $n \in \mathbb{N}$. We can define as usual the product of an element of $M_n(\mathbb{R})$ and an element of $\mathbb{R}^n$ (the result is an element of $\mathbb{R}^n$). With this product and the usual sum of vectors, $\mathbb{R}^n$ is an $M_n(\mathbb{R})$-module (you do not have to check this).
   (a) Show that if $u \in \mathbb{R}^n$ then there is $A \in M_n(\mathbb{R}) \setminus \{0\}$ such that $Au = 0$.
   (b) Show that every non-empty set of elements of $\mathbb{R}^n$ is linearly dependent (over $M_n(R)$). In particular $\mathbb{R}^n$ does not have a basis as $M_n(\mathbb{R})$-module.
   (c) An R-module M is called simple if its only submodules are $\{0\}$ and M. Show that $\mathbb{R}^n$ is a simple $M_n(\mathbb{R})$-module. (A first step can be to show that for every $u \in \mathbb{R}^n \setminus \{0\}$ and every $v \in \mathbb{R}^n$, there is $A \in M_n(R)$ such that $Au = v$.)

2. The objective of this exercise is to show that if M is a finitely generated R-module and N is a submodule of M, N may not be finitely generated. This is something that cannot happen in vector spaces.

   Let $R = \mathbb{R}[X_1, X_2, X_3, \ldots]$ be the rings of polynomials with real coefficients in the indeterminates X_i for $i \in \mathbb{N}$. We know that R is an R-module, and we call M this R-module. Let $N = \text{Span}\{X_1, X_2, X_3, \ldots\}$, so that N is the submodule consisting of all polynomials with 0 constant term.

   (a) Observe that $M = \text{Span}\{1\}$. So in particular M is finitely generated.

   Assume that N is finitely generated: $N = \text{Span}\{P_1, \ldots, P_k\}$.

   (b) Show that for every $r \in \{1, \ldots, k\}$ there is a finite subset I_r of $\mathbb{N}$ such that $P_r \in \text{Span}\{X_i \mid i \in I_r\}$.
   (c) Deduce that there is $N \in \mathbb{N}$ such that $\{X_i \mid i \in \mathbb{N}\} \subseteq \text{Span}\{X_1, \ldots, X_N\}$.
   (d) Deduce a contradiction. For this you can use that if a_i (for $i \in \mathbb{N}$) are real numbers, then the map $f : R \to \mathbb{R}$, $f(P(X_{i_1}, \ldots, X_{i_s})) = P(a_{i_1}, \ldots, a_{i_s})$ is a morphism of rings (it is easy to check, just tedious, so we skip it).

3. Let R be a ring and let $e \in R$ be such that $e^2 = e$ (such an element is called an idempotent) and $er = re$ for every $r \in R$ (i.e. $e \in C(R)$; we say that e is a central idempotent). Let M be an R-module.

(a) Show that $eM = \{ex \mid x \in M\}$ is a submodule of M.
Similarly, $1 - e$ is also a central idempotent, and $(1 - e)M$ is also a submodule of M.

(b) Show that $M = eM \oplus (1 - e)M$ (the direct sum of modules is defined exactly as the direct sum of vector spaces). Hint: You can use that $1 = e + (1 - e)$ to show that $M = eM + (1 - e)M$.