Problem sheet 10

1. Let R be a ring. Show that the following properties are equivalent
   (a) R is von Neumann regular (see problem sheet 9).
   (b) Every principal (=generated by one element) left ideal of R is generated by an idempotent (see problem sheet 9)
   (c) Every principal left ideal of R is a direct summand in R. Hint: Exercise 3.8 in the course note can give some inspiration.

   (These statements are also equivalent to: Every finitely generated left ideal is generated by an idempotent, but the proof of this is really a trick.)

2. (a) Show that every semisimple ring is von Neumann regular.
   (b) Show that if R is von Neumann regular, then $J(R) = \{0\}$. Hint: If $a \in J(R)$, consider the left ideal Ra.
   (c) Let R be a ring. Show that R is semisimple if and only if R is left Noetherian and von Neumann regular.

3. Let M be a semisimple R-module. Show that $\mathrm{End}_R(M)$ is von Neumann regular. Hint: If $f \in \mathrm{End}_R(M)$, write $M = N \oplus \ker f$ and use this to define g such that $fgf = f$.

4. The core of a left ideal L, denoted $\mathrm{Core}(L)$ is the sum of all (two-sided) ideals included in L (it is therefore the largest ideal included in L; it is obviously equal to L when L is an ideal). Show the following. Recall that R/L is an R-module.
   (a) Prove that $\mathrm{Ann}_R(R/L)$ is a 2-sided ideal of R, and that $\mathrm{Core}(L) \cdot R = \mathrm{Core}(L)$.
   (b) $\mathrm{Core}(L) = \mathrm{Ann}_R(R/L)$.
   (c) If L is a maximal left ideal of R, then $R/\mathrm{Core}(L)$ is a primitive[1] ring. Hint: Show that R/L is a simple R-module.
   (d) R is a primitive ring if and only if R has a maximal left ideal L such that $\mathrm{Core}(L) = \{0\}$. Hint: We have seen a description of simple modules.

[1] A ring S is called primitive if there is a simple S-module M such that $\mathrm{Ann}_S M = \{0\}$